A PERSPECTIVE ON DERIVED NON-COMMUTATIVE GEOMETRY

This talk is based on a work in progress, so not everything has its final form, probably.

1. HISTORICAL INTRODUCTION:

This talk is about dualities between geometric spaces and algebraic structures.
This is a cornerstone of modern geometry, and the first instance is probably Stone's Theorem

Thm (Stone, 1936)
Boolean algebras are in duality with profinite topological spaces.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ‖
Totally disconnected compact Hausdorff
‖
Stone spaces.

Duality means contravariant equivalence of categories.

Stone's theorem has inspired others to find similar dualities.

The most famous one is

<u>Thm</u> (Gelfand, 1943)

The category of commutative C^*-algebras is in duality with compact Hausdorff spaces.

And then literature is full of such kind of results:

$$\{\text{Affine schemes}\} \longleftrightarrow \{\text{commutative rings}\}$$

$$\{\text{Smooth manifolds}\} \longleftrightarrow \{\text{Smooth algebras, } C^\infty(M)\}$$

$$\{\text{measure spaces}\} \longleftrightarrow \{\text{Commutative Von Neuman algebras}\}$$

$$\{\text{Stein analytic spaces}\} \longleftrightarrow \{\text{Stein algebras}\}$$

$$\{\text{Affinoid algebras}\} \longleftrightarrow \{\text{Affinoid rigid/non-archimedean spaces}\},$$

and many more. This inspired people to "create" dualities for defining new kind of geometries. At first this was mainly used in the non-commutative setting

$$\{\text{NC Topological spaces?}\} \longleftrightarrow \{C^*\text{-algebras}\}$$

$$\{\text{NC affine schemes?}\} \longleftrightarrow \{\text{Rings}\}$$

and here the problem is that we have a poor understanding of what is on the geometric side.

This line of thought has been used also for more exotic approaches like

$$\{\text{Affine schemes} / \mathbb{F}_1\} \longleftrightarrow \{\text{Commutative monoids, or similar}\}$$

More generally one has the pattern

$\mathcal{C}$ = category of some algebraic structures of interest

$\mathcal{C}^{op}$ = should be some form of affine models of a geometry determined by $\mathcal{C}$.

Sometimes this approach works and we understand $\mathcal{C}^{op}$ as some form of geometric spaces, possibly with some extra structure.

Recently, a new direction has been proved very powerful that is the direction of derived geometry. The pattern here is

$$\left\{\begin{array}{l}\infty\text{-category of commutative rings} \\ \simeq \\ \text{Homotopy category of simplicial} \\ \text{commutative rings}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Affine derived} \\ \text{schemes}\end{array}\right\}$$

$$\left\{E^{\infty}\text{-ring spectra}\right\} \longleftrightarrow \left\{\text{Affine spectral schemes}\right\}$$

and again other examples are possible.

What I would like to sketch in these lectures is a possible approach to an understanding of the duality

$$\left\{\infty\text{-category of rings}\right\} \longleftrightarrow \left\{\text{Derived affine NC schemes}\right\}.$$

2. Pointless Topology

In order to achieve the goal we would like to assign a spectrum to any homotopical ring. For doing this it will be useful to recall another direction of generalization of Stone's Theorem.

Stone's duality can be seen as a part of a more general

Stone's duality can be seen as a part of a more general construction that is a functor

$$\begin{aligned} S : \mathrm{Bool} &\longrightarrow \mathrm{Top} \\ A &\longmapsto S(A) \end{aligned}$$

that to each Boolean algebra associates its spectrum that is the set of ultrafilters of A, with base for the topology

$$U_b = \{ \mathcal{F} \in S(A) \mid b \in \mathcal{F} \}.$$

On the other hand, for any topological space X the family

$$Cl(X) = \{ U \subseteq X \mid U \text{ is clopen} \}$$

forms a boolean algebra. So we have a contravariant adjunction

$$Cl : \mathrm{Top} \rightleftarrows \mathrm{Bool} : S$$

and Stone's theorem says that this adjunction identify Bool^{op} with profinite spaces. The composition $S \circ Cl$ is often called <u>Banaschewski compactification</u> of X.

[illegible] $Cl(X) \subseteq Ouv(X) = \{$lattice of open subsets of $X\}$.

Notice that $\mathcal{Cl}(X) \subseteq \mathcal{O}uv(X) = \{$lattice of open subsets of $X\}$.

It is possible to generalize Stone duality using the full lattice of open subsets of X.

The family $\mathcal{O}uv(X)$ form a special kind of lattice, called frame: a <u>frame</u> F is a lattice such that

- F is distributive,
- F has all meets (equiv. all joins),
- the infinite distributivity $x \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} x \wedge s$, $\forall x \in F$, $S \subseteq F$, holds.

To any frame it is possible to associate a topological space

$$S(F) = \{\text{completely prime ultrafilter + Stone-like topology}\}.$$

We thus have another contravariant adjunction

$$\mathcal{O}uv : \mathrm{Top} \rightleftarrows \mathrm{Frm} : S$$

It restricts to an equivalence between two suitable subcategories on each side:

The adjunction $(\mathcal{O}_{uv}, S)$ restricts to a duality between the category of sober topological spaces and spatial frames.

Remark:
The category Frm^{op} is called the category of locales.

There is another relevant class of topological spaces to which the above adjunction restricts to an equivalence

$$\{\text{spectral spaces}\} \longleftrightarrow \{\text{bounded distributive lattices}\}$$

$$\text{spectral spaces} = \mathrm{Spec}(R),\ R \text{ commutative ring.} \qquad \text{bounded distributive lattices} \cong \text{Coherent locales}$$

The interest about these constructions is that it permits to reconstruct a topological space X from a different set of data. We will use this to define spectra of rings.

The main basic idea is that we would like to define a functor

$$\begin{tikzcd} \mathrm{Spec}: \mathrm{Rings} \arrow[r] & \mathrm{Top} \\ \mathrm{CRings} \arrow[u, "i"] \arrow[ur] & \end{tikzcd}$$

Likings

such that $\mathrm{Spec} \circ \iota$ is the classical spectrum functor, and we also would like this functor to satisfy the basic requirement that $\mathrm{Spec}(R) \neq \emptyset \quad \forall R \neq 0$.

People struggled a lot in trying to construct such a functor until:

<u>Thm</u> (Reyes, 2012)

No such functor exists.

So we will try our best in this impossible task.

3. <u>Topos Theory</u>

The constructions of last section are useful because there are sources of distributive lattices in nature.

<u>Example</u>

Let $(\mathcal{C}, \otimes)$ be a small tensor triangulated category, then the family of thick $\otimes$-ideals of $\mathcal{C}$ form a distributive lattice.

Sadly, not all lattices are distributive and it can happen

Sadly, not all lattices are [illegible] ..
To find them in applications. So what to do in that case?

Let us take a more general perspective.

There is another way of generalizing the notion of topological space, and it is the notion of a site. The lattice $\mathcal{O}uv(X)$ is in particular a poset, and all poset can be seen as categories canonically.

Definition

A Grothendieck topology on a category $\mathcal{C}$ is the data of a family of covers $\{U_i \to U\}_{i \in I}$ for any $U \in \mathcal{C}$ such that

1. $\{V \xrightarrow{\cong} U\}$ is a cover $\forall$ iso,
2. $\forall\, V \to U$ and cover $\{U_i \to U\}$, the family $\{U_i \times_U V \to V\}$ is a cover,
3. If $\{U_i \to U\}$ is a cover and $\forall_i \{V_{j_i} \to U_i\}$ is a cover, then $\{V_{j_i} \to U\}$ is a cover.

Example

1. $Ouv(X)$ is a category and $\{U_i \to U\}$ such that $\coprod U_i \to U$ is surjective is a Grothendieck Topology.
2. If D is a distributive lattice, the finite families $\{x_i \to x\}$ such that $\bigvee x_i = x$ form a Grothendieck Topology.
3. Algebraic geometry is a source of a lot of Grothendieck Topologies. The most basic is the Zariski Topology. A morphism of commutative rings $f: A \to B$ is said Zariski localization if the corresponding morphism $Spec(B) \to Spec(A)$ is an open immersion. There is the characterization

$$f \text{ Zariski localization} \Leftrightarrow f \text{ flat epimorphism of finite presentation.}$$

A finite family of Zariski localizations $\{f_i: A \to B_i\}$ determines a cover if $\coprod Spec(B_i) \to Spec(A)$ is surjective.

When we have a site, we would like to understand its geometry and we do it by probing it using sheaves (i.e. generalizations of and we do it by probing it using sheaves (i.e. generalizations of fiber bundles). If $\mathcal{C}$ is a site, a <u>sheaf of sets</u> on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \mathrm{Sets}$$

such that $\forall$ covers $\{ U_i \to U \}$ in $\mathcal{C}$ we have

$$\mathcal{F}(U) \cong \mathrm{Ker}\left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j) \right)$$

where the morphisms are given by the two morphisms $U_i \times_U U_j \rightrightarrows U_j, U_i$.

<u>Definition</u>

The category of sheaves over a small site is the full subcategory of $\mathrm{Func}(\mathcal{C}^{op}, \mathrm{Sets})$ determined by sheaves and such categories are called <u>Topos</u>.

We say that two small sites are <u>equivalent</u> if their associated Topoi are equivalent categories.

<u>Remark</u>

Very different looking sites can be equivalent.

The next theorem shows the previous remark in action.

Thm

Let L be a poset and τ a Grothendieck topology on L. Suppose that τ is coherent (i.e. all covers are finite), then (L, τ) is equivalent to the topos of a spectral topological space.

The topological space can be constructed esplicitly via a specific family of filters.

Example

Let $A \in \text{CRings}$, the family $\left\{ f : A \to B \mid f \text{ is a Zariski localization} \right\}$ is essentially small and form a lattice that is not distributive in general. The Zariski topology form a topology on the dual lattice and its associated spectral topological space is precisely $\text{Spec}(A)$.

We have given a different definition of $\text{Spec}(A)$.

Moreover the resulting spectrum is functorial, in the sense that

Moreover the resulting spectrum is functorial, in the sense that every morphism of commutative rings $f: A \to B$ induces a continuous map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Now we want to give a different one using derived geometry.