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6

THE DISCRETE FOURIER TRANSFORM

Normally a discussion of the discrete Fourier transform is based on an initial definition of the finite length discrete transform; from this assumed axiom those properties of the transform implied by this definition are derived. This approach is unrewarding in that at its conclusion there is always the unanswered question, "How does the discrete Fourier transform relate to the continuous Fourier transform?" To answer this question we find it preferable to derive the discrete Fourier transform as a special case of continuous Fourier transform theory.

In this chapter, we develop a special case of the continuous Fourier transform which is amenable to machine computation. The approach will be to develop the discrete Fourier transform from a graphical derivation based on continuous Fourier transform theory. These graphical arguments are then substantiated by a theoretical development. Both approaches emphasize the modifications of continuous Fourier transform theory which are necessary to define a computer-oriented transform pair.

6-1 A GRAPHICAL DEVELOPMENT

Consider the example function $h(t)$ and its Fourier transform $H(f)$ illustrated in Fig. 6-1(a). It is desired to modify this Fourier transform pair in such a manner that the pair is amenable to digital computer computation. This modified pair, termed the discrete Fourier transform is to approximate as closely as possibly the continuous Fourier transform.

To determine the Fourier transform of $h(t)$ by means of digital analysis

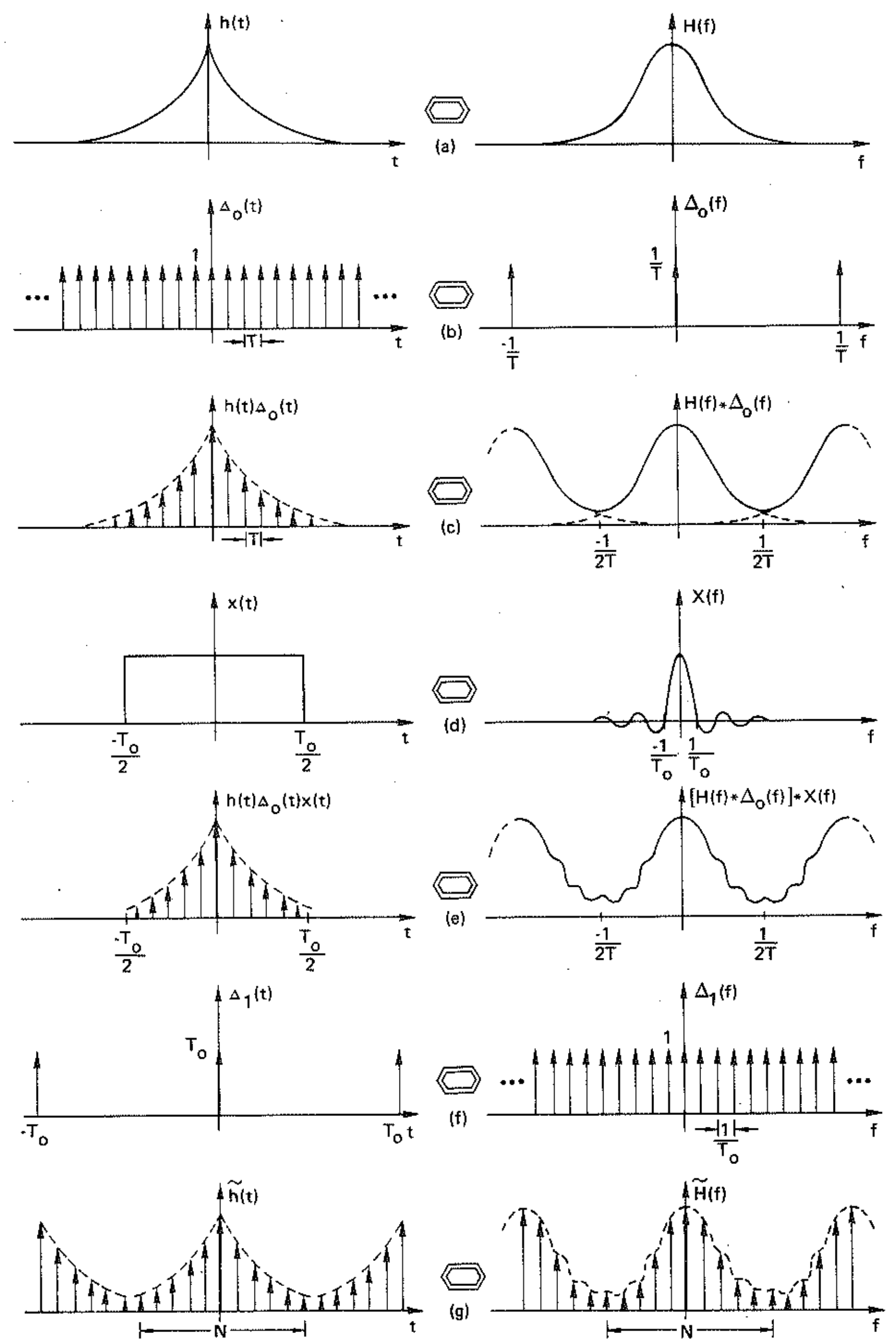

Figure 6-1. Graphical development of the discrete Fourier transform.

techniques, it is necessary to sample $h(t)$ as described in Chapter 5. Sampling is accomplished by multiplying $h(t)$ by the sampling function illustrated in Fig. 6-1(b). The sample interval is T. Sampled function $\hat{h}(t)$ and its Fourier transform are illustrated in Fig. 6-1(c). This Fourier transform pair represents the first modification to the original pair which is necessary in defining a discrete transform pair. Note that to this point the modified transform pair differs from the original transform pair only by the aliasing effect which result from sampling. As discussed in Sec. 5-3, if the waveform $h(t)$ is sampled at a frequency of at least twice the largest frequency component of $h(t)$, there is no loss of information as a result of sampling. If the function $h(t)$ is not band-limited; i.e., $H(f) \neq 0$ for some $|f| > f_c$, then sampling will introduce aliasing as illustrated in Fig. 6-1(c). To reduce this error we have only one recourse, and that is to sample faster; that is, choose T smaller. Sampled function

The Fourier transform pair in Fig. 6-1(c) is not suitable for machine computation because an infinity of samples of $h(t)$ is considered; it is necessary to truncate the sampled function $h(t)$ so that only a finite number of points, say N, are considered. The rectangular or truncation function and its Fourier transform are illustrated in Fig. 6-1(d). The product of the infinite sequence of impulse functions representing $h(t)$ and the truncation function yields the finite length time function illustrated in Fig. 6-1(e). Truncation introduces the second modification of the original Fourier transform pair; this effect is to convolve the aliased frequency transform of Fig. 6-1(c) with the Fourier transform of the truncation function [Fig. 6-1(d)]. As shown in Fig. 6-1(e), the frequency transform now has a *ripple* to it; this effect has been accentuated in the illustration for emphasis. To reduce this effect, recall the inverse relation that exists between the *width* of a time function and its Fourier transform (Sec. 3-3). Hence, if the truncation (rectangular) function is increased in length, then the sin f/f function will approach an impulse; the more closely the sin f/f function approximates an impulse, the less ripple or error will be introduced by the convolution which results from truncation. Therefore, it is desirable to choose the length of the truncation function as long as possible. We will investigate in detail in Sec. 6-4 the effect of truncation. Truncated function long truncation for low ripple

The modified transform pair of Fig. 6-1(e) is still not an acceptable discrete Fourier transform pair because the frequency transform is a continuous function. For machine computation, only sample values of the frequency function can be computed; it is necessary to modify the frequency transform by the frequency sampling function illustrated in Fig. 6-1(f). The frequency sampling interval is $1/T_0$.

The discrete Fourier transform pair of Fig. 6-1(g) is acceptable for the purposes of digital machine computation since both the time and frequency domains are represented by discrete values. As illustrated in Fig. 6-1(g), the original time function is approximated by N samples; the original Fourier transform $H(f)$ is also approximated by N samples. These N samples define the discrete Fourier transform pair and approximate the original Fourier DFT transform pair. Note that sampling in the time domain resulted in a periodic function of frequency; sampling in the frequency domain resulted in a periodic function of time. Hence, the discrete Fourier transform requires that both the original time and frequency functions be modified such that they become periodic functions. N time samples and N frequency values represent one period of the time and frequency domain waveforms, respectively. Since the N values of time and frequency are related by the continuous Fourier transform, then a discrete relationship can be derived.

must be periodic

6-2 THEORETICAL DEVELOPMENT

The preceding graphical development illustrates the point that if a continuous Fourier transform pair is suitably modified, then the modified pair is acceptable for computation on a digital computer. Thus, to develop this discrete Fourier transform pair, it is only necessary to derive the mathematical relationships which result from each of the required modifications: time domain sampling, truncation, and frequency domain sampling.

Sampling

Consider the Fourier transform pair illustrated in Fig. 6-2(a). To discretize this transform pair it is first necessary to sample the waveform $h(t)$; the sampled waveform can be written as $h(t)\ \Delta_0(t)$ where $\Delta_0(t)$ is the time domain sampling function illustrated in Fig. 6-2(b). The sampling interval is T. From Eq. (5-20) the sampled function can be written as

$$h(t)\Delta_0(t) = h(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$= \sum_{k=-\infty}^{\infty} h(kT)\, \delta(t - kT) \qquad (6\text{-}1)$$

The result of this multiplication is illustrated in Fig. 6-2(c). Note the aliasing effect which results from the choice of T.

Next, the sampled function is truncated by multiplication with the rectangular function $x(t)$ illustrated in Fig. 6-2(d):

$$x(t) = 1 \qquad -\frac{T}{2} < t < T_0 - \frac{T}{2}$$

$$= 0 \qquad \text{otherwise} \qquad (6\text{-}2)$$

where T_0 is the duration of the truncation function. An obvious question at this point is, "Why is the rectangular function $x(t)$ not centered at zero or $T_0/2$?" Centering of $x(t)$ at zero is avoided to alleviate notation problems. The reason for not centering the rectangular function at $T_0/2$ will become obvious later in the development.

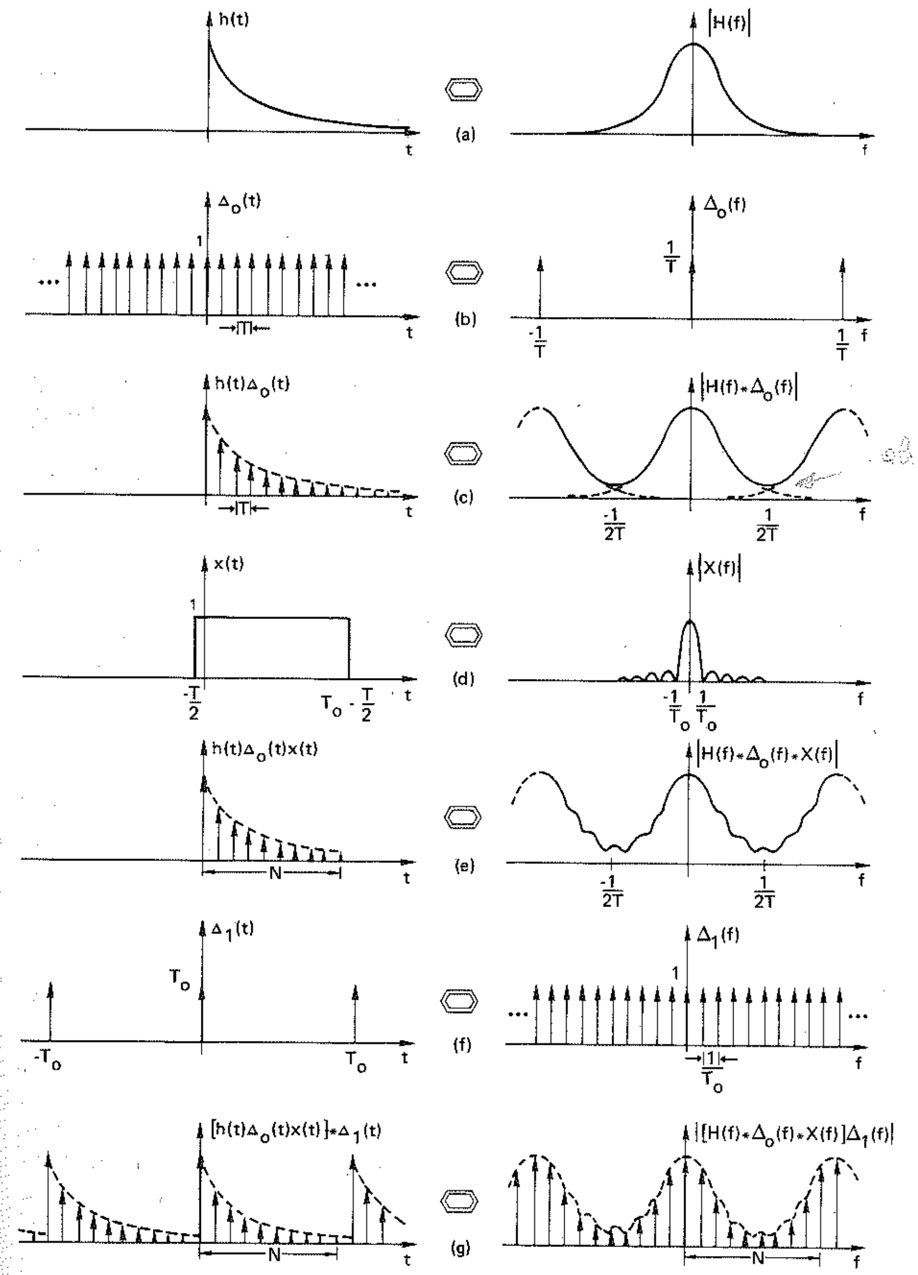

Figure 6-2. Graphical derivation of the discrete Fourier transform pair.

Truncation yields

$$h(t)\Delta_0(t)x(t) = \left[\sum_{k=-\infty}^{\infty} h(kT)\,\delta(t-kT)\right]x(t)$$
$$= \sum_{k=0}^{N-1} h(kT)\,\delta(t-kT) \qquad (6\text{-}3)$$

$N = T_0/T$

where it has been assumed that there are N equidistant impulse functions lying within the truncation interval; that is, $N = T_0/T$. The sampled truncated waveform and its Fourier transform are illustrated in Fig. 6-2(e). As in the previous example, truncation in the time domain results in *rippling* in the frequency domain.

frequency sampling

The final step in modifying the original Fourier transform pair to a discrete Fourier transform pair is to sample the Fourier transform of Eq. (6-3). In the time domain this product is equivalent to convolving the sampled truncated waveform (6-3) and the time function $\Delta_1(t)$, illustrated in Fig. 6-2(*f*). Function $\Delta_1(t)$ is given by Fourier transform pair (2-40) as

$$\Delta_1(t) = T_0 \sum_{r=-\infty}^{\infty} \delta(t - rT_0) \qquad (6\text{-}4)$$

The desired relationship is $[h(t)\Delta_0(t)x(t)] * \Delta_1(t)$; hence

$$[h(t)\Delta_0(t)x(t)] * \Delta_1(t) = \left[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)\right] * \left[T_0 \sum_{r=-\infty}^{\infty} \delta(t-rT_0)\right]$$
$$= \cdots + T_0 \sum_{k=0}^{N-1} h(kT)\,\delta(t+T_0-kT)$$
$$+ T_0 \sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)$$
$$+ T_0 \sum_{k=0}^{N-1} h(kT)\,\delta(t-T_0-kT) + \cdots \qquad (6\text{-}5)$$

Note that (6-5) is periodic with period T_0; in compact notation form the equation can be written as

$$\tilde{h}(t) = T_0 \sum_{r=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT-rT_0)\right] \qquad (6\text{-}6)$$

We choose the notation $\tilde{h}(t)$ to imply that $\tilde{h}(t)$ is an approximation to the function $h(t)$.

Choice of the rectangular function $x(t)$ as described by Eq. (6-2) can now be explained. Note that the convolution result of Eq. (6-6) is a periodic function with period T_0 which consists of N samples. If the rectangular function had been chosen such that a sample value coincided with each end point of the rectangular function, the convolution of the rectangular function with impulses spaced at intervals of T_0 would result in time domain aliasing. That is, the Nth point of one period would coincide with (and add to) the first point of the next period. To insure that time domain aliasing does not occur,

it is necessary to choose the truncation interval as illustrated in Fig. 6-2(d). (The truncation function may also be chosen as illustrated in Fig. 6-1(d), but note that the end points of the truncation function lie at the mid-point of two adjacent sample values to avoid time domain aliasing.)

To develop the Fourier transform of Eq. (6-6), recall from the discussion on Fourier series, Sec. 5-1, that the Fourier transform of a periodic function $h(t)$ is a sequence of equidistant impulses

$$\tilde{H}\left(\frac{n}{T_0}\right) = \sum_{n=-\infty}^{\infty} \alpha_n\,\delta(f-nf_0) \qquad f_0 = \frac{1}{T_0} \qquad (6\text{-}7)$$

where

$$\alpha_n = \frac{1}{T_0}\int_{-T/2}^{T_0-T/2} \tilde{h}(t)\,e^{-j2\pi nt/T_0}\,dt \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (6\text{-}8)$$

Substitution of (6-6) in (6-8) yields

$$\alpha_n = \frac{1}{T_0}\int_{-T/2}^{T_0-T/2} T_0 \sum_{r=-\infty}^{\infty}\sum_{k=0}^{N-1} h(kT)\,\delta(t-kT-rT_0)\,e^{-j2\pi nt/T_0}\,dt$$

Integration is only over one period, hence

$$\alpha_n = \int_{-T/2}^{T_0-T/2} \sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)\,e^{-j2\pi nt/T_0}\,dt$$
$$= \sum_{k=0}^{N-1} h(kT) \int_{-T/2}^{T_0-T/2} e^{-j2\pi nt/T_0}\,\delta(t-kT)\,dt$$
$$= \sum_{k=0}^{N-1} h(kT)\,e^{-j2\pi knT/T_0} \qquad (6\text{-}9)$$

Since $T_0 = NT$, Eq. (6-9) can be rewritten as

$$\alpha_n = \sum_{k=0}^{N-1} h(kT)\,e^{-j2\pi kn/N} \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (6\text{-}10)$$

and the Fourier transform of Eq. (6-6) is

$$\tilde{H}\left(\frac{n}{NT}\right) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{N-1} h(kT)\,e^{-j2\pi kn/N} \qquad (6\text{-}11)$$

From a cursory evaluation of (6-11), it is not obvious that the Fourier transform $\tilde{H}(n/NT)$ is periodic as illustrated in Fig. 6-2(g). However, there are only N distinct complex values computable from Eq. (6-11). To establish this fact let $n = r$ where r is an arbitrary integer; Eq. (6-11) becomes

$$\tilde{H}\left(\frac{r}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\,e^{-j2\pi kr/N} \qquad (6\text{-}12)$$

Now let $n = r + N$; note that

$$e^{-j2\pi k(r+N)/N} = e^{-j2\pi kr/N}\,e^{-j2\pi k}$$
$$= e^{-j2\pi kr/N} \qquad (6\text{-}13)$$

since $e^{-j2\pi k} = \cos(2\pi k) - j\sin(2\pi k) = 1$ for k integer valued. Thus for $n = r + N$

DFT periodic

$$\tilde{H}\left(\frac{r+N}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi k(r+N)/N} = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi kr/N} = \tilde{H}\left(\frac{r}{NT}\right) \tag{6-14}$$

Therefore, there are only N distinct values for which Eq. (6-11) can be evaluated; $\tilde{H}(n/NT)$ is periodic with a period of N samples. Fourier transform (6-11) can be expressed equivalently as

Transform over a period

$$\tilde{H}\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\, e^{-j2\pi nk/N} \qquad n = 0, 1, \ldots, N-1 \tag{6-15}$$

Eq. (6-15) is the desired discrete Fourier transform; the expression relates N samples of time and N samples of frequency by means of the continuous Fourier transform. The discrete Fourier transform is then a special case of the continuous Fourier transform. If it is assumed that the N samples of the original function $h(t)$ are one period of a periodic waveform, the Fourier transform of this periodic function is given by the N samples as computed by Eq. (6-15). Notation $\tilde{H}(n/NT)$ is used to indicate that the discrete Fourier transform is an approximation to the continuous Fourier transform. Normally, Eq. (6-15) is written as

$$G\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} g(kT)\, e^{-j2\pi nk/N} \qquad n = 0, 1, \ldots, N-1 \tag{6-16}$$

since the Fourier transform of the sampled periodic function $g(kT)$ is identically $G(n/NT)$.

6-3 DISCRETE INVERSE FOURIER TRANSFORM

The discrete inverse Fourier transform is given by

$$g(kT) = \frac{1}{N}\sum_{n=0}^{N-1} G\left(\frac{n}{NT}\right) e^{j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{6-17}$$

To prove that (6-17) and the transform relation (6-16) form a discrete Fourier transform pair, substitute (6-17) into Eq. (6-16).

$$G\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1}\left[\frac{1}{N}\sum_{r=0}^{N-1} G\left(\frac{r}{NT}\right) e^{j2\pi rk/N}\right] e^{-j2\pi nk/N} = \frac{1}{N}\sum_{r=0}^{N-1} G\left(\frac{r}{NT}\right)\left[\sum_{k=0}^{N-1} e^{j2\pi rk/N}\, e^{-j2\pi nk/N}\right] = G\left(\frac{n}{NT}\right) \tag{6-18}$$

Identity (6-18) follows from the orthogonality relationship

$$\sum_{k=0}^{N-1} e^{j2\pi rk/N}\, e^{-j2\pi nk/N} = \begin{cases} N & \text{if } r = n \\ 0 & \text{otherwise} \end{cases} \tag{6-19}$$

The discrete inversion formula (6-17) exhibits periodicity in the same manner as the discrete transform; the period is defined by N samples of $g(kT)$. This property results from the periodic nature of $e^{j2\pi nk/N}$. Hence, $g(kT)$ is actually defined on the complete set of integers $k = 0, \pm1, \pm2, \ldots$ and is constrained by the identity

$$g(kT) = g[(rN + k)T] \qquad r = 0, \pm1, \pm2, \ldots \tag{6-20}$$

In summary, the discrete Fourier transform pair is given by

$$g(kT) = \frac{1}{N}\sum_{n=0}^{N-1} G\left(\frac{n}{NT}\right) e^{j2\pi nk/N} \quad \Longleftrightarrow \quad G\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} g(kT)\, e^{-j2\pi nk/N} \tag{6-21}$$

It is important to remember that the pair (6-21) requires both the time and frequency domain functions to be periodic;

$$G\left(\frac{n}{NT}\right) = G\left[\frac{(rN+n)}{NT}\right] \qquad r = 0, \pm1, \pm2, \ldots \tag{6-22}$$

$$g(kT) = g[(rN + k)T] \qquad r = 0, \pm1, \pm2, \ldots \tag{6-23}$$

6-4 RELATIONSHIP BETWEEN THE DISCRETE AND CONTINUOUS FOURIER TRANSFORM

The discrete Fourier transform is of interest primarily because it approximates the continuous Fourier transform. Validity of this approximation is strictly a function of the waveform being analyzed. In this section we use graphical analysis to indicate for general classes of functions the degree of equivalence between the discrete and continuous transform. As will be stressed, differences in the two transforms arise because of the discrete transform requirement for sampling and truncation.

Band-Limited Periodic Waveforms: Truncation Internal Equal to Period

Consider the function $h(t)$ and its Fourier transform illustrated in Fig. 6-3(a). We wish to sample $h(t)$, truncate the sampled function to N samples, and apply the discrete Fourier transform Eq. (6-16). Rather than applying this equation directly, we will develop its application graphically. Waveform $h(t)$ is sampled by multiplication with the sampling function illustrated in Fig. 6-3(b). Sampled waveform $h(kT)$ and its Fourier transform are illustrated in Fig. 6-3(c). Note that for this example there is no aliasing. Also

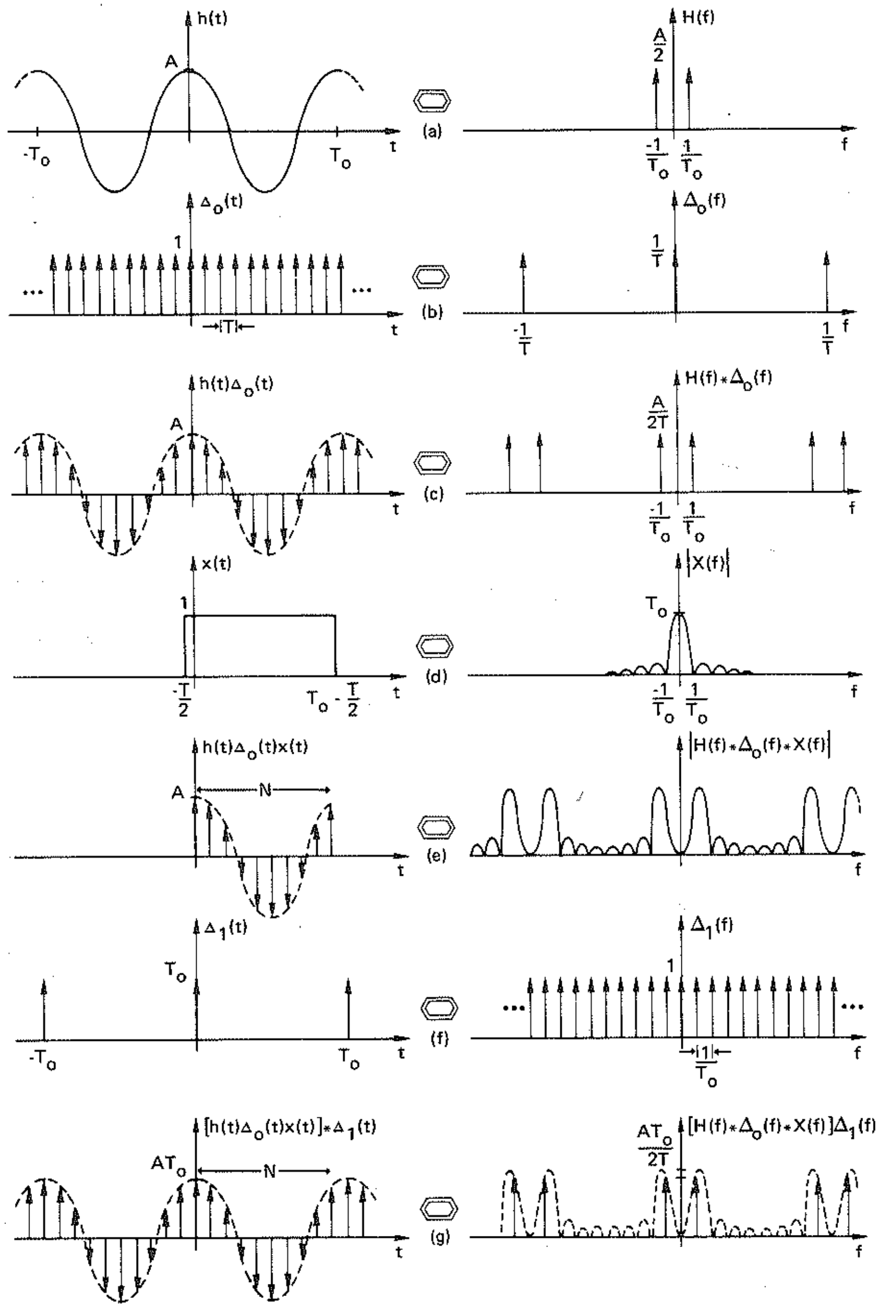

Figure 6-3. Discrete Fourier transform of a band-limited periodic waveform: truncation interval equal to period.

observe that as a result of time domain sampling, the frequency domain has been scaled by the factor $1/T$; the Fourier transform impulse now has an area of $A/2T$ rather than the original area of $A/2$. The sampled waveform is truncated by multiplication with the rectangular function illustrated in Fig. 6-3(d); Fig. 6-3(e) illustrates the sampled and truncated waveform. As shown, we chose the rectangular function so that the N sample values remaining after truncation equate to one period of the original waveform $h(t)$.

The Fourier transform of the finite length sampled waveform [Fig. 6-3(e)] is obtained by convolving the frequency domain impulse functions of Fig. 6-3(c) and the $\sin f/f$ frequency function of Fig. 6-3(d). Figure 6-3(e) illustrates the convolution results; an expanded view of this convolution is shown in Fig. 6-4(b). A $\sin f/f$ function (dashed line) is centered on each impulse of Fig. 6-4(a) and the resultant waveforms are additively combined (solid line) to form the convolution result.

With respect to the original transform $H(f)$, the convolved frequency function [Fig. 6-4(b)] is significantly distorted. However, when this function is sampled by the frequency sampling function illustrated in Fig. 6-3(f) the distortion is eliminated. This follows because the equidistant impulses of the frequency sampling function are separated by $1/T_0$; at these frequencies

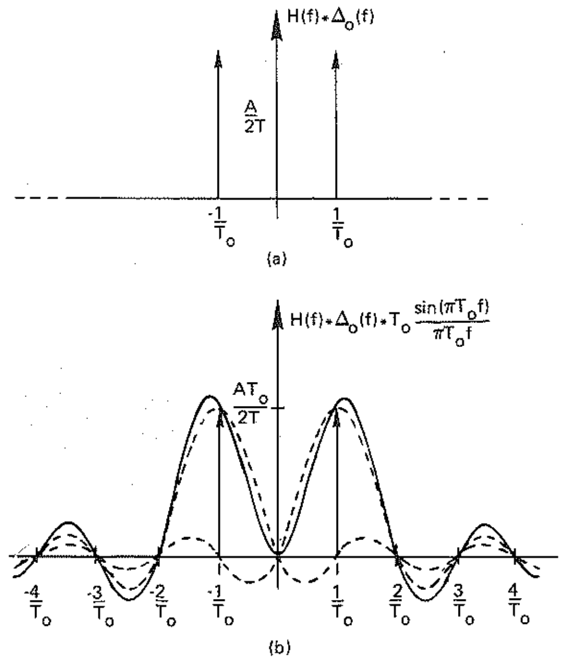

Figure 6-4. Expanded illustration of the convolution of Fig. 6-3(e).

the solid line of Fig. 6-4(b) is zero except at the frequency $\pm 1/T_0$. Frequency $\pm 1/T_0$ corresponds to the frequency domain impulses of the original frequency function $H(f)$. Because of time domain truncation, these impulses now have an area of $AT_0/2T$ rather than the original area of $A/2$. (Fig. 6-4(b) does not take into account that the Fourier transform of the truncation function $x(t)$ illustrated in Fig. 6-4(d) is actually a complex frequency function; however, had we considered a complex function, similar results would have been obtained.)

different area of samples

Multiplication of the frequency function of Fig. 6-3(e) and the frequency sampling function $\Delta_1(f)$ implies the convolution of the time functions shown in Figs. 6-3(e) and (f). Because the sampled truncated waveform [Fig. 6-3(e)] is exactly one period of the original waveform $h(t)$ and since the time domain impulse functions of Fig. 6-3(f) are separated by T_0, then their convolution yields a periodic function as illustrated in Fig. 6-3(g). This is simply the time domain equivalent to the previously discussed frequency sampling which yielded only a single impulse or frequency component. The time function of Fig. 6-3(g) has a maximum amplitude of AT_0, compared to the original maximum value of A as a result of frequency domain sampling.

samples time T_0

Examination of Fig. 6-3(g) indicates that we have taken our original time function, sampled it, and then multiplied each sample by T_0. The Fourier transform of this function is related to the original frequency function by the factor $AT_0/2T$. Factor T_0 is common and can be eliminated. If we desire to compute the Fourier transform by means of the discrete Fourier transform, it is necessary to multiply the discrete time function by the factor T which yields the desired $A/2$ area for the frequency function; Eq. (6-16) thus becomes

$$H\left(\frac{n}{NT}\right) = T\sum_{k=0}^{N-1} h(kT)e^{-j2\pi nk/N} \tag{6-24}$$

equal if: periodic band limit sample > 2T Trunc = T_0

We expect this result since the relationship (6-24) is simply the rectangular rule for integration of the continuous Fourier transform.

This example represents the only class of waveforms for which the discrete and continuous Fourier transforms are exactly the same within a scaling constant. Equivalence of the two transforms requires: (1) the time function $h(t)$ must be periodic, (2) $h(t)$ must be band-limited, (3) the sampling rate must be at least two times the largest frequency component of $h(t)$, and (4) the truncation function $x(t)$ must be non-zero over exactly one period (or integer multiple period) of $h(t)$.

Band-Limited Periodic Waveforms: Truncation Interval Not Equal to Period

If a periodic, band-limited function is sampled and truncated to consist of other than an integer multiple of the period, the resulting discrete and continuous Fourier transform will differ considerably. To examine this

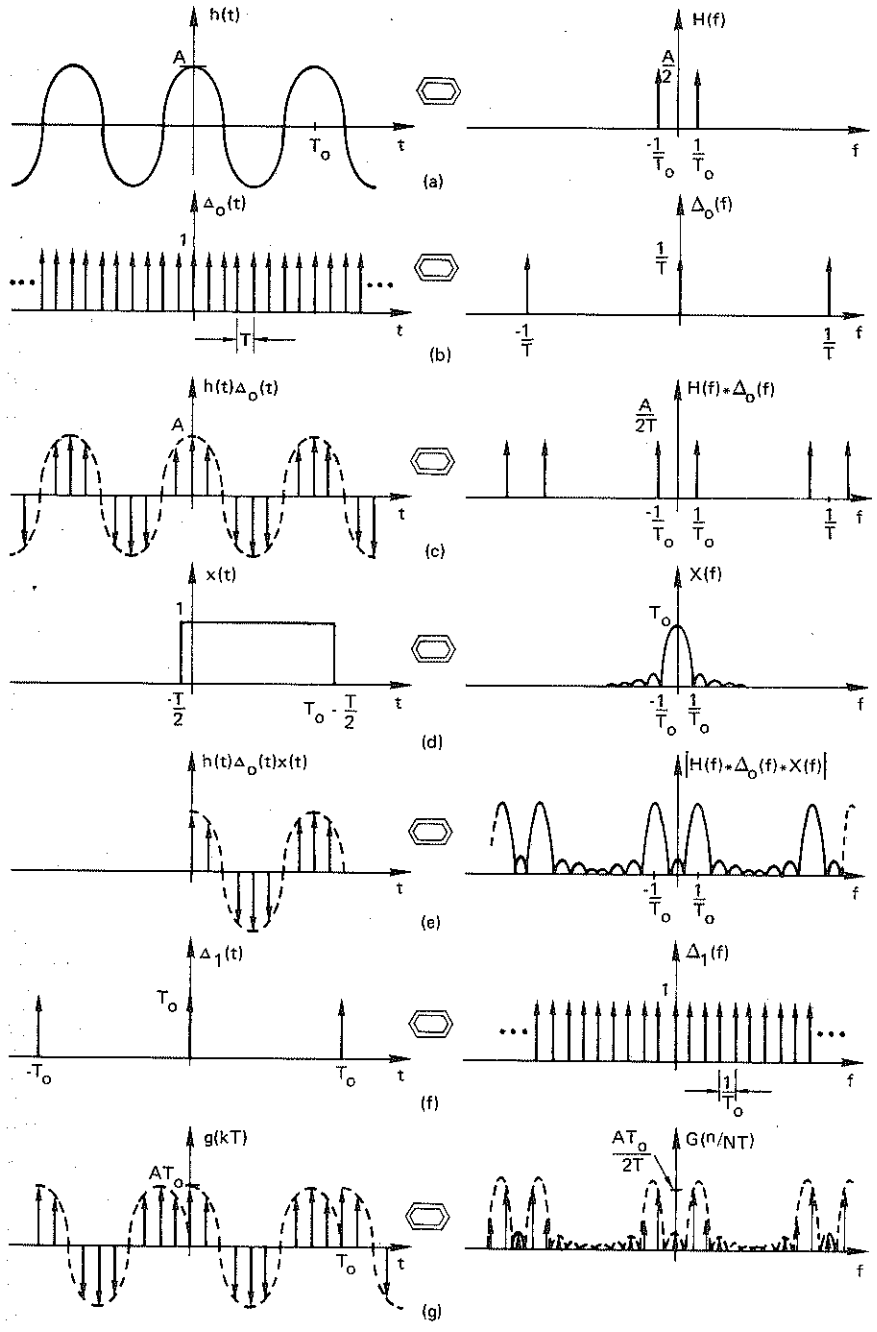

Figure 6-5. Discrete Fourier transform of a band-limited periodic waveform: truncation interval not equal to period.

effect, consider the illustrations of Fig. 6-5. This example differs from the preceding only in the frequency of the sinusoidal waveform $h(t)$. As before, function $h(t)$ is sampled [Fig. 6-5(c)] and truncated [Fig. 6-5(e)]. Note that the sampled, truncated function is not an integer multiple of the period of $h(t)$; therefore, when the time functions of Figs. 6-5(e) and (*f*) are convolved, the periodic waveform of Fig. 6-5(g) results. Although this function is periodic, it is not a replica of the original periodic function $h(t)$. We would not expect the Fourier transform of the time waveforms of Figs. 6-5(a) and (g) to be equivalent. It is of value to examine these same relationships in the frequency domain.

Fourier transform of the sampled truncated waveform of Fig. 6-5(e) is obtained by convolving the frequency domain impulse functions of Fig. 6-5(c) and the $\sin f/f$ function illustrated in Fig. 6-5(d). This convolution is graphically illustrated in an expanded view in Fig. 6-6. Sampling of the

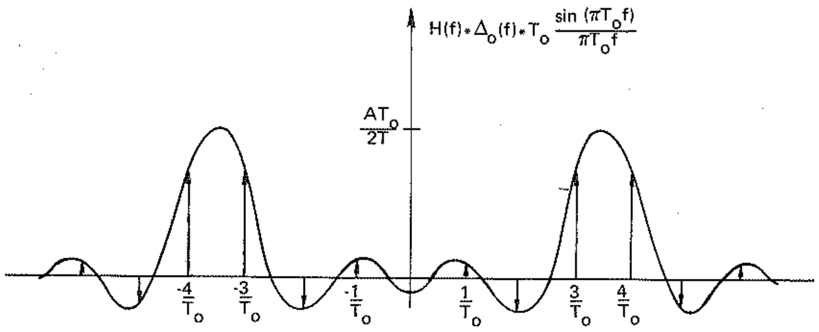

Figure 6-6. Expanded illustration of the convolution of Fig. 6-5(e).

resulting convolution at frequency intervals of $1/T_0$ yields the impulses as illustrated in Fig. 6-6 and, equivalently, Fig. 6-5(g). These sample values represent the Fourier transform of the periodic time waveform of Fig. 6-5(g). Note that there is an impulse at zero frequency. This component represents the average value of the truncated waveform; since the truncated waveform is not an even number of cycles, the average value is not expected to be zero. The remaining frequency domain impulses occur because the zeros of the $\sin f/f$ function are not coincident with each sample value as was the case in the previous example.

$\int \omega$ 0 = average value of trunc.

This discrepancy between the continuous and discrete Fourier transforms is probably the one most often encountered and least understood by users of the discrete Fourier transform. The effect of truncation at other than a multiple of the period is to create a periodic function with sharp *discontinuities* as illustrated in Fig. 6-5(g). Intuitively, we expect the introduction of

these sharp changes in the time domain to result in additional frequency components in the frequency domain. Viewed in the frequency domain, time domain truncation is equivalent to the convolution of a $\sin f/f$ function with the single impulse representing the original frequency function $H(f)$. Consequently, the frequency function is no longer a single impulse but rather a continuous function of frequency with a local maximum centered at the original impulse and a series of other peaks which are termed sidelobes. These sidelobes are responsible for the additional frequency components which occur after frequency domain sampling. This effect is termed *leakage* and is inherent in the discrete Fourier transform because of the required time domain truncation. Techniques for reducing leakage will be explored in Sec. 9-5.

Finite Duration Waveforms

The preceding two examples have explored the relationship between the discrete and continuous Fourier transforms for band-limited periodic functions. Another class of functions of interest is that which is of finite duration such as the function $h(t)$ illustrated in Fig. 6-7. If $h(t)$ is time-limited, its Fourier transform cannot be band-limited; sampling must result in aliasing. It is necessary to choose the sample interval T such that aliasing is reduced to an acceptable range. As illustrated in Fig. 6-7(c), the sample interval T was chosen too large and as a result there is significant aliasing.

If the finite-length waveform is sampled and if N is chosen equal to the number of samples of the time-limited waveform, then it is not necessary to truncate in the time domain. Truncation is omitted and the Fourier transform of the time sampled function [Fig. 6-7(c)] is multiplied by $\Delta_1(f)$, the frequency domain sampling function. The time domain equivalent to this product is the convolution of the time functions shown in Figs. 6-7(c) and (d). The resulting waveform is periodic where a period is defined by the N samples of the original function, and thus is a replica of the original function. The Fourier transform of this periodic function is the sampled function illustrated in Fig. 6-7(e).

No truncation

For this class of functions, if N is chosen equal to the number of samples of the finite-length function, then the only error is that introduced by aliasing. Errors introduced by aliasing are reduced by choosing the sample interval T sufficiently small. For this case the discrete Fourier transform sample values will agree (within a constant) reasonably well with samples of the continuous Fourier transform. Unfortunately, there exist few applications of discrete Fourier transform for this class of functions.

General Periodic Waveforms

Figure 6-7 can also be used to illustrate the relationship between the discrete and continuous Fourier transform for periodic functions which are

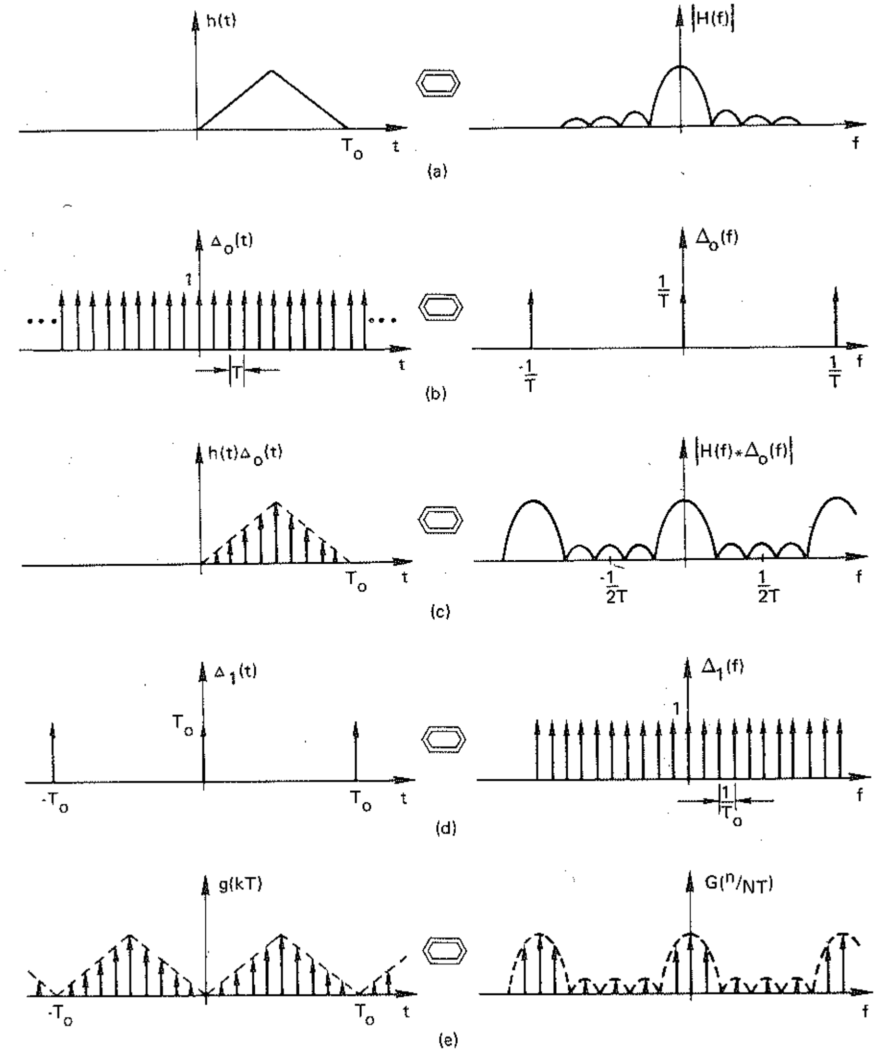

Figure 6-7. Discrete Fourier transform of a time-limited waveform.

not band-limited. Assume that $h(t)$, as illustrated in Fig. 6-7(a), is only one period of a periodic waveform. If this periodic waveform is sampled and truncated at exactly the period, then the resulting waveform will be identical to the time waveform of Fig. 6-7(c). Instead of the continuous frequency function as illustrated in Fig. 6-7(c), the frequency transform will be an

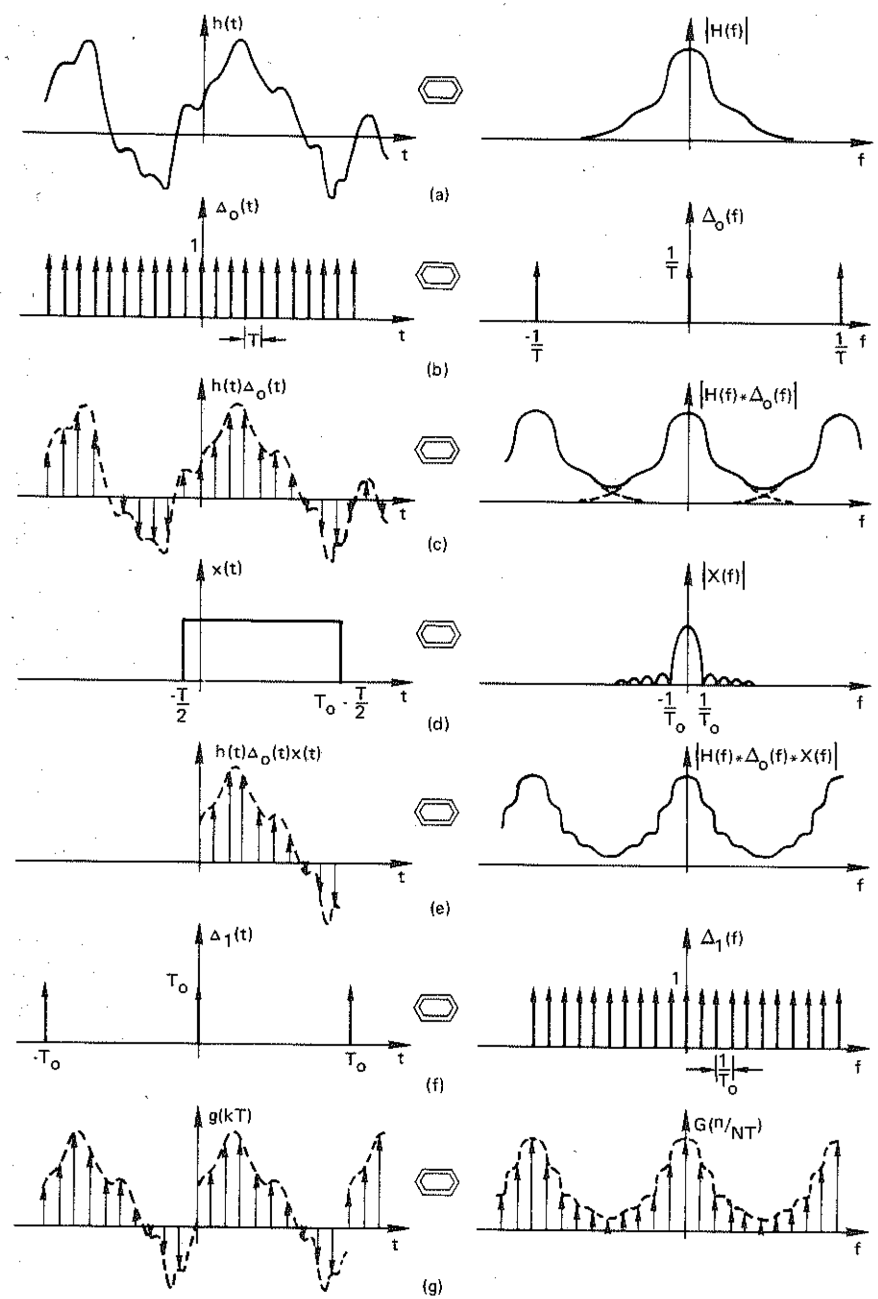

Figure 6-8. Discrete Fourier transform of a general waveform.

infinite series of equidistant impulses separated by $1/T_0$ whose areas are given exactly by the continuous frequency function. Since the frequency sampling function $\Delta_1(f)$, as illustrated in Fig. 6-7(d), is an infinite series of equidistant impulses separated by $1/T_0$ then the result is identical to those of Fig. 6-7(e). As before, the only error source is that of aliasing if the truncation function is chosen exactly equal to an integer multiple of the period. If the time domain truncation is not equal to a period, then results as described previously are to be expected.

General Waveforms

The most important class of functions are those which are neither time-limited nor band-limited. An example of this class of functions is illustrated in Fig. 6-8(a). Sampling results in the aliased frequency function illustrated in Fig. 6-8(c). Time domain truncation introduces rippling in the frequency domain of Fig. 6-8(e). Frequency sampling results in the Fourier transform pair illustrated in Fig. 6-8(g). The time domain function of this pair is a periodic function where the period is defined by the N points of the original function after sampling and truncation. The frequency domain function of the pair is also a periodic function where a period is defined by N points whose values differ from the original frequency function by the errors introduced in aliasing and time domain truncation. The aliasing error can be reduced to an acceptable level by decreasing the sample interval T. Procedures for reducing time domain truncation errors will be addressed in Sec. 9-5.

Summary

We have shown that if care is exercised, then there exist many applications where the discrete Fourier transform can be employed to derive results essentially equivalent to the continuous Fourier transform. The most important concept to keep in mind is that the discrete Fourier transform implies periodicity in both the time and frequency domain. If one will always remember that the N sample values of the time domain function represent one sample of a periodic function, then application of the discrete Fourier transform should result in few surprises.

PROBLEMS

6-1. Repeat the graphical development of Fig. 6-1 for the following functions:

a. $h(t) = |t|e^{-a|t|}$

b. $h(t) = 1 - |t| \quad |t| \leq 1$

$\quad = 0 \quad |t| > 1$

c. $h(t) = \cos(t)$

6-2. Retrace the development of the discrete Fourier transform [Eqs. (6-1) through (6-16)]. Write out in detail all steps of the derivation.

6-3. Repeat the graphical derivation of Fig. 6-3 for $h(t) = \sin(2\pi f_0 t)$. Show the effect of setting the truncation interval unequal to the period. What is the result of setting the truncation interval equal to two periods?

6-4. Consider Fig. 6-7. Assume that $h(t)\ \Delta_0(t)$ is represented by N non-zero samples. What is the effect of truncating $h(t)\ \Delta_0(t)$ so that only $3N/4$ non-zero samples are considered? What is the effect of truncating $h(t)\ \Delta_0(t)$ so that the N non-zero samples and $N/4$ zero samples are considered?

6-5. Repeat the graphical derivation of Fig. 6-7 for $h(t) = \sum_{n=-\infty}^{\infty} e^{-a|t-nT_0|}$. What are the error sources?

6-6. To establish the concept of rippling, perform the following graphical convolutions:

a. An impulse with $\frac{\sin t}{t}$

b. A narrow pulse with $\frac{\sin t}{t}$

c. A wide pulse with $\frac{\sin t}{t}$

d. A single triangle waveform with $\frac{\sin t}{t}$

6-7. Write out several terms of Eq. (6-19) to establish the orthogonality relationship.

6-8. The truncation interval is often termed the "record length." In terms of the record length, write an equation defining the "resolution" or frequency spacing of the frequency domain samples of the discrete Fourier transform.

6-9. Comment on the following: The discrete Fourier transform is analogous to a bank of band-pass filters.

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