Sobolev spaces

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I collect in these notes some facts about Sobolev spaces (see [2]). Some very nice examples are in [1].

1 $\Omega \subseteq \mathbb{R}$

1.1 $H^1(\Omega)$

We can define $H^1(\Omega)$ in the following way: it is the subspace of $L^2(\Omega)$ of functions u for which there exists $g \in L^2(\Omega)$ such that

$$\int_\Omega u\varphi' = -\int_\Omega g\varphi \quad \forall \varphi \in C_{\mathrm{c}}^\infty(\Omega)$$

We will denote g by u'. This definition is equivalent to the definition with distributional derivatives.

Theorem 1. *If $u \in H^1(\Omega)$, there exists (unique) $\tilde{u} \in C(\bar{\Omega})$ such that*

$$u = \tilde{u} \text{ almost everywhere}$$

and

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\mathrm{d}t$$

We will call $\tilde{u}$ the continuous representative of the class of equivalence of u. We will often indicate it simply by u when necessary. For instance, if we want to write $u(x_0)$, $x_0 \in \Omega$. The scalar product in $H^1(\Omega)$ is

$$(u, v) = \int_\Omega uv + \int_\Omega u'v'$$

with the inducted norm.

1.2 $H^m(\Omega)$

We can define $H^m(\Omega)$ in the following way: it is the subspace of $L^2(\Omega)$ of functions u for which there exists $g1, g_2, \dots, g_m \in L^2(\Omega)$ such that

$$\int_\Omega u\varphi^{(j)} = (-1)^j \int_\Omega g_j \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

We will denote g_j by $u^{(j)}$ $(u', u'', \dots, u^{(m)})$.

1.3 $H_0^1(\Omega)$

$H_0^1(\Omega)$ is the closure of $C_c^1(\Omega)$ in $H^1(\Omega)$. If $\Omega = \mathbb{R}$, then $H_0^1(\mathbb{R}) = H^1(\mathbb{R})$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, its closure is $H_0^1(\Omega)$ itself.

Theorem 2. *If $u \in H^1(\Omega)$, then $u \in H_0^1(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

If $\Omega = (a, b)$, this is precisely a case in which the function u such that $u(a) = u(b) = 0$ is the continuous representative (of the class of equivalence) of u. Another way to characterize $H_0^1(\Omega)$ is the following: $u \in H_0^1(\Omega)$ if and only if $\bar{u} \in H^1(\mathbb{R})$, where $\bar{u}(x) = u(x)$ if $x \in \Omega$ and $\bar{u}(x) = 0$ if $x \in \mathbb{R} \setminus \Omega$.

2 $\Omega \subseteq \mathbb{R}^N$, $N > 1$

2.1 $H^1(\Omega)$

The definition is analogous, we have to replace the derivative with all the partial derivatives. One main difference with the one-dimensional case is that there are functions, like the following

$$u(x, y) = \left(\log \frac{1}{\sqrt{x^2 + y^2}} \right)^k, \quad 0 < k < 1/2$$

that belongs to $H^1(\Omega)$, $\Omega = B(0, 1) \subset \mathbb{R}^2$, but it is noway possible to find a continuous representative $\tilde{u}$ for it. So, Theorem 1 does not hold.

2.2 $H_0^1(\Omega)$

$H_0^1(\Omega)$ is the closure of $C_c^1(\Omega)$ in $H^1(\Omega)$. If $\Omega = \mathbb{R}^N$, then $H_0^1(\mathbb{R}^N) = H^1(\mathbb{R}^N)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, its closure is $H_0^1(\Omega)$ itself.

Now, Theorem 2 cannot be stated in the same way: in general, there is no continuous representative which is zero at $\partial\Omega$. Still, it is correct to think to functions in $H_0^1(\Omega)$ as to the functions in $H^1(\Omega)$ which "are zero at $\partial\Omega$". Let us see in which sense.

Theorem 3. *If Ω is sufficiently regular and $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $u \in H^1_0(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

Moreover, it is possible to characterize $H^1_0(\Omega)$ as above: $u \in H^1_0(\Omega)$ if and only if $\bar{u} \in H^1(\mathbb{R}^N)$, where $\bar{u}(x) = u(x)$ if $x \in \Omega$ and $\bar{u}(x) = 0$ if $x \in \mathbb{R} \setminus \Omega$.

Theorem 4. *If Ω is a bounded open subset of $\mathbb{R}^N$ with $\partial\Omega$ sufficiently regular, then there exists a unique linear continuous operator $T\colon H^1(\Omega) \to L^2(\partial\Omega)$ such that*

$$\begin{aligned} &Tu = u|_{\partial\Omega} && \text{if } u \in H^1(\Omega) \cap C(\bar{\Omega}) \\ &\|Tu\|_{L(\partial\Omega)} \leq c\|u\|_{H^1(\Omega)} \end{aligned}$$

The operator T is called trace operator *and the function $Tu \in L^2(\partial\Omega)$ is called* trace *of u on $\partial\Omega$.*

The operator γ_0 is not surjective on $L^2(\partial\Omega)$. The set of functions in $L^2(\partial\Omega)$ which are traces of functions in $H^1(\Omega)$ is a subspace of $L^2(\partial\Omega)$ denoted by $H^{1/2}(\partial\Omega)$. We have $H^1(\partial\Omega) \subseteq H^{1/2}(\partial\Omega) \subseteq H^0(\partial\Omega) = L^2(\partial\Omega)$. If u is more regular, so is $u|_{\partial\Omega}$ in the sense that

$$T\colon H^k(\Omega) \to H^{k-1/2}(\partial\Omega) \subseteq H^{k-1}(\partial\Omega)$$

Given $u \in H^1(\Omega)$ and $u^{\mathrm{D}} \in H^{1/2}(\partial\Omega)$, if we require that "$u = u^{\mathrm{D}}$ on $\partial\Omega$" or "$u|_{\partial\Omega} = u^{\mathrm{D}}$", we really mean that $Tu = u^{\mathrm{D}}$ (almost everywhere).

Finally, we can define

$$H^1_0(\Omega) = \ker(T) = \{u \in H^1(\Omega)\colon Tu = 0\}$$

References

[1] M. Bramanti, Introduzione alla formulazione debole dei problemi ai limiti per EDP, `http://www1.mate.polimi.it/~bramanti/corsi/pdf_metodi/sobolev2.pdf`.

[2] H. Brezis, Analisi funzionale, Liguori editore, 1986.