Quadrature and assembly for FEM in two dimensions

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June 9, 2014

1 Quadrature formulas

1.1 By interpolation (mass matrix)

1.2 By trapezoidal rule

$$\int_{\ell_j} g(x,y)\mathrm{d}x\mathrm{d}y \approx |\Delta_j| \frac{g(x_{\ell_{j,1}}, y_{\ell_{j,1}}) + g(x_{\ell_{j,2}}, y_{\ell_{j,2}}) + g(x_{\ell_{j,3}}, y_{\ell_{j,3}})}{3}$$

1.2.1 Mass matrix by trapezoidal rule

Let us start computing

$$M_{ii} = \sum_{\ell_{m,k}=i} \int_{\ell_m} \varphi_{\ell_{m,k}}(x)\varphi_{\ell_{m.k}}(x)\mathrm{d}x = \sum_{\ell_{m,k}=i} \frac{|\Delta_m|}{6}$$

$$M_{ij} = \sum_{\substack{\ell_{m,k}=i \\ \ell_{m,h}=j}} \int_{\ell_m} \varphi_{\ell_{m,h}}(x)\varphi_{\ell_{m,k}}(x)\mathrm{d}x = \sum_{\substack{\ell_{m,k}=i \\ \ell_{m,h}=j}} \frac{|\Delta_m|}{12}, \quad i \neq j$$

If we take the sum over j, we get

$$\sum_j M_{ij} = \sum_{\ell_{m,k}=i} \frac{|\Delta_m|}{6} + 2\sum_{\ell_{m,k}=i} \frac{|\Delta_m|}{12} = \sum_{\ell_{m,k}=i} \frac{|\Delta_m|}{3}$$

If we try to approximate by trapezoidal rule the computation of the mass matrix, we get

$$M_{ii} \approx \sum_{\ell_{j,k}=i} \frac{|\Delta_m|}{3}$$

$$M_{ij} \approx 0, \quad i \neq j$$

It is equivalent to the operation of *lumping*, that is to sum up all the elements of each row of the exact mass matrix.

1.3 By barycentric formulas

$$\int_\Omega f(x,y)\varphi_i(x,y)\mathrm{d}x\mathrm{d}y \approx \sum_{\ell_{j,k}=i} \bar{f}_j \frac{|\Delta_j|}{3}$$

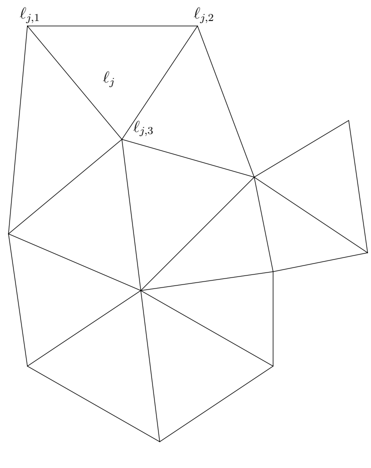
where

$$\bar{f}_j = \frac{f(x_{\ell_{j,1}}, y_{\ell_{j,1}}) + f(x_{\ell_{j,2}}, y_{\ell_{j,2}}) + f(x_{\ell_{j,3}}, y_{\ell_{j,3}})}{3}$$

1.4 By Gauss–Legendre quadrature

There exist high order Gauss–Legendre quadrature formulas for triangles.

2 Assembly



The assembly in the two-dimensional case is not much different from the one-dimensional case.

First of all, the number of points is m and the number of triangles is n. Then, we consider the basis function $\varphi_{\ell_{j,k}}$ which has value 1 on node $\ell_{j,k}$ and 0 on nodes $\ell_{j,h}$, $h \in \{1,2,3\}$, $h \neq k$ of the triangle ℓ_j. It has the form

$$\varphi_{\ell_{j,k}}(x,y) = \frac{a_{j,k} + b_{j,k}x + c_{j,k}y}{2\Delta_j} = \begin{vmatrix} 1 & \overset{k}{1} & 1 \\ x_{\ell_{j,1}} & x & x_{\ell_{j,3}} \\ y_{\ell_{j,1}} & y & y_{\ell_{j,3}} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 \\ x_{\ell_{j,1}} & x_{\ell_{j,2}} & x_{\ell_{j,3}} \\ y_{\ell_{j,1}} & y_{\ell_{j,2}} & y_{\ell_{j,3}} \end{vmatrix}$$

where Δ_j is the area (with sign) of triangle ℓ_j. We need to compute

$$\int_{\ell_j}\left(\frac{\partial\varphi_{\ell_{j,k}}(x,y)}{\partial x}\frac{\partial\varphi_{\ell_{j,h}}(x,y)}{\partial x}+\frac{\partial\varphi_{\ell_{j,k}}(x,y)}{\partial y}\frac{\partial\varphi_{\ell_{j,h}}(x,y)}{\partial y}\right)\mathrm{d}x\mathrm{d}y,\quad h,k=1,2,3$$

for the stiffness matrix (and also derivatives with respect to y) and

$$\int_{\ell_j} f(x,y)\varphi_{\ell_{j,k}}(x,y)\mathrm{d}x\mathrm{d}y$$

for the right hand side. We have

$$\int_{\ell_j}\frac{\partial\varphi_{\ell_{j,k}}(x,y)}{\partial x}\frac{\partial\varphi_{\ell_{j,h}}(x,y)}{\partial x}\mathrm{d}x\mathrm{d}y=\int_{\ell_j}\frac{b_{j,k}}{2\Delta_j}\frac{b_{j,h}}{2\Delta_j}\mathrm{d}x\mathrm{d}y=\frac{b_{j,k}b_{j,h}}{4|\Delta_j|}$$
$$\int_{\ell_j}\frac{\partial\varphi_{\ell_{j,k}}(x,y)}{\partial y}\frac{\partial\varphi_{\ell_{j,h}}(x,y)}{\partial y}\mathrm{d}x\mathrm{d}y=\int_{\ell_j}\frac{c_{j,k}}{2\Delta_j}\frac{c_{j,h}}{2\Delta_j}\mathrm{d}x\mathrm{d}y=\frac{c_{j,k}c_{j,h}}{4|\Delta_j|}$$

and

$$\int_{\ell_j} f(x,y)\varphi_{\ell_{j,k}}(x,y)\mathrm{d}x\mathrm{d}y\approx\tilde{f}_{\ell_{j,k}}$$

The algorithm for the assembly is

- $A_{ij}=0,\ 1\le i,j\le m,\ f_i=0,\ 1\le i\le m$
- FOR $j=1,\ldots,n$

    FOR $k=1,\ldots,3$

    $A_{\ell_{j,k}\ell_{j,k}}=A_{\ell_{j,k}\ell_{j,k}}+\frac{b_{j,k}b_{j,k}}{4|\Delta_j|}+\frac{c_{j,k}c_{j,k}}{4|\Delta_j|}$, $f_{\ell_{j,k}}=f_{\ell_{j,k}}+\tilde{f}_{\ell_{j,k}}$

    FOR $h=k+1,\ldots,3$ ($h=1,\ldots,3$, $h\neq k$ non-symm. case)

    $A_{\ell_{j,k}\ell_{j,h}}=A_{\ell_{j,k}\ell_{j,h}}+\frac{b_{j,k}b_{j,h}}{4|\Delta_j|}+\frac{c_{j,k}c_{j,h}}{4|\Delta_j|}$

    $A_{\ell_{j,h}\ell_{j,k}}=A_{\ell_{j,k}\ell_{j,h}}$ (only symm. case)

    END

    END

  END