Molenkamp–Crowley test

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June 9, 2014

1 Pure advection

Let us consider the pure advection equation

$$\begin{cases} \partial_t u(t,x,y) + \boldsymbol{b}(x,y)\cdot\nabla u(t,x,y) = 0, & (t,x,y)\in(0,T)\times\Omega, \quad \Omega\subset\mathbb{R}^2 \\ u(0,x,y) = u_0(t,x,y) \end{cases}$$

where $\boldsymbol{b}(x,y) = [b_1(x,y), b_2(x,y)] = [y,-x]$. The exact solution at time t and point (x_t, y_t) is given by

$$u(t,x_t,y_t) = u_0(\bar{x},\bar{y})$$

where $x_t = x_t(t)$ and $y_t = y_t(t)$ with

$$\begin{cases} x_t'(s) = b_1(x_t(s), y_t(s)) \\ y_t'(s) = b_2(x_t(s), y_t(s)) \\ x_t(0) = \bar{x} \\ y_t(0) = \bar{y} \end{cases} \tag{1}$$

In fact, $u(0, x_t(0), y_t(0)) = u_0(\bar{x},\bar{y})$ and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}u_0(\bar{x},\bar{y}) = \frac{\mathrm{d}}{\mathrm{d}t}u(t,x_t,y_t) = \partial_t u(t,x_t,y_t) + \boldsymbol{b}(x_t,y_t)\cdot\nabla u(t,x_t,y_t)$$

This method to solve the pure advection equation is called *forward characteristics*. The drawback is that if we select a mesh point $(\bar{x},\bar{y})$ and solve equation (1) up to time t, the point (x_t, y_t) may not be a mesh point. We can consider then a new variable r and a new couple $(x_{-t}(r), y_{-t}(r))$ defined by

$$r = t - s, \quad x_{-t}(r) = x_t(s), \quad y_{-t}(r) = y_t(s)$$

It is clear that $(x_{-t}(0), y_{-t}(0)) = (x_t(t), y_t(t))$, $(x_{-t}(t), y_{-t}(t)) = (x_t(0), y_t(0))$ and moreover

$$\begin{cases} x'_{-t}(r) = -b_1(x_{-t}(r), y_{-t}(r)) \\ y'_{-t}(r) = -b_2(x_{-t}(r), y_{-t}(r)) \end{cases}$$

and therefore if we consider

$$\begin{cases} x'_{-t}(r) = -b_1(x_{-t}(r), y_{-t}(r)) \\ y'_{-t}(r) = -b_2(x_{-t}(r), y_{-t}(r)) \\ x_{-t}(0) = x \\ y_{-t}(0) = y \end{cases} \tag{2}$$

we have

$$u(t, x, y) = u(t, x_t, y_t) = u_0(x_t(0), y_t(0)) = u_0(x_{-t}(t), y_{-t}(t))$$

If we denote by X_t the operator

$$X_t \colon \Omega \to \Omega, \quad (x, y) \mapsto (x_{-t}(t), y_{-t}(t))$$

we can write

$$u(t, x, y) = u_0 \circ X_t(x, y)$$

Therefore, given a mesh point (x, y), if we solve equation (2) up to time t we have the solution u at time t at a mesh point. This method is called *backward characteristics*, since solving forward (from 0 to t) equation (2) is equivalent to solve backward (from t to 0) equation (1).

For our particular choice of $\boldsymbol{b}(x, y)$, equation (2) is trivial and its solution is

$$\begin{bmatrix} x_{-t}(t) \\ y_{-t}(t) \end{bmatrix} = \exp\left(t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, the solution $u(t, x, y)$ is the clockwise rotation of angle t of the initial solution $u_0(x, y)$. This particular pure advection equation is called *Molenkamp–Crowley* test.

1.1 A Discontinuous Galerkin formulation

It is possible to consider the following DG formulation

$$\int_\Omega \left(\frac{u^{n+1} - u^n}{k} + \boldsymbol{b} \cdot u^{n+1}\right) v \mathrm{d}\Omega - 2 \int_{\mathcal{E}} \left(\frac{1}{2}|\boldsymbol{n} \cdot \boldsymbol{b}| - \frac{1}{2}\boldsymbol{n} \cdot \boldsymbol{b}\right) [u^n] v \mathrm{d}\Omega = 0$$