Some notes on ADR equations

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The Advection-Diffusion-Reaction equation is

$$\begin{cases} -\mathrm{div}(\mu\nabla u) + b\cdot\nabla u + \sigma u = f, & u \in \Omega \subset \mathbb{R}^n \\ u = 0, & u \in \partial\Omega \end{cases}$$

1 Estimates

1.1 $\mathrm{div}(b), \sigma \in L^2(\Omega)$

In this case we have

$$\int_\Omega v b\cdot\nabla v \mathrm{d}\Omega + \int_\Omega \sigma v^2 \mathrm{d}\Omega = \int_\Omega v^2\left(-\frac{1}{2}\mathrm{div}(b) + \sigma\right)\mathrm{d}\Omega$$

ans using Cauchy–Schwartz inequality

$$\int_\Omega \left|v^2\left(-\frac{1}{2}\mathrm{div}(b)+\sigma\right)\right|\mathrm{d}\Omega \le \left\|v^2\right\|_{L^2(\Omega)}\left\|-\frac{1}{2}\mathrm{div}(b)+\sigma\right\|_{L^2(\Omega)}$$

Therefore we require $v \in L^4(\Omega)$. Moreover

$$\begin{aligned}\left|\int_\Omega \sigma uv \mathrm{d}\Omega\right| &\le \|\sigma\|_{L^2(\Omega)}\|uv\|_{L^2(\Omega)} \le \|\sigma\|_{L^2(\Omega)}\|u\|_{L^4(\Omega)}\|v\|_{L^4(\Omega)} \le \\ &\le C\|\sigma\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\end{aligned}$$

In fact $H^1(\Omega) \subset L^4(\Omega)$ with a continuous immersion.

1.2 $\mathrm{div}(b), \sigma \in L^\infty(\Omega)$

In this case we have

$$\int_\Omega v b\cdot\nabla v \mathrm{d}\Omega + \int_\Omega \sigma v^2 \mathrm{d}\Omega = \int_\Omega v^2\left(-\frac{1}{2}\mathrm{div}(b) + \sigma\right)\mathrm{d}\Omega$$

ans using Hölder's inequality

$$\int_\Omega \left| v^2 \left(-\frac{1}{2}\mathrm{div}(b) + \sigma \right) \right| \mathrm{d}\Omega \leq \left\| v^2 \right\|_{L^1(\Omega)} \left\| -\frac{1}{2}\mathrm{div}(b) + \sigma \right\|_{L^\infty(\Omega)}$$

Therefore $v \in L^2(\Omega)$. Moreover

$$\left| \int_\Omega \sigma uv \mathrm{d}\Omega \right| \leq \|\sigma\|_{L^\infty(\Omega)} \|uv\|_{L^1(\Omega)} \leq \|\sigma\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq$$
$$\leq \|\sigma\|_{L^\infty(\Omega)} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

2 One-dimensional AD problem

For the problem

$$\begin{cases} -\mu u''(x) + bu'(x) = 0 \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

we have that the solutions of the differences equation

$$-(\mathrm{Pe}+1) + 2\rho + (\mathrm{Pe}-1)\rho^2 = 0$$

are

$$\rho_1 = (1+\mathrm{Pe})/(1-\mathrm{Pe}), \quad \rho_2 = 1$$

where $\mathrm{Pe} = |b|\, h/(2\mu)$ is the grid Peclét number. In case it is one, the differences equation is

$$-(\mathrm{Pe}+1) + 2\rho = 0$$

with solution $\rho = (\mathrm{Pe}+1)/2$. Therefore, u_i (on inner nodes) is

$$u_i = A\rho^i$$

and by imposing boundary condition $u(0) = 0$ we get $A = 0$. Therefore, the numerical solution is $u_0 = u_1 = u_{M-1} = 0$ and $u_M = u(1) = 1$.

The analytical solution

$$u(x) = \frac{\exp\left(\frac{b}{\mu}x\right) - 1}{\exp\left(\frac{b}{\mu}\right) - 1}$$

We can try to find ε such that $u(1-\varepsilon) \approx 0$. It is

$$f(\varepsilon) = u(1-\varepsilon) \approx f(0) + f'(0)\varepsilon = 1 + \frac{-\frac{b}{\mu}\exp\left(\frac{b}{\mu}\right)}{\exp\left(\frac{b}{\mu}\right) - 1}\varepsilon = 0$$

from which

$$\varepsilon = \frac{\mu}{b}\frac{\exp\left(\frac{b}{\mu}\right)-1}{\exp\left(\frac{b}{\mu}\right)} = \mathcal{O}\left(\frac{\mu}{b}\right)$$

Therefore the boundary layer width is $\mathcal{O}\left(\frac{\mu}{b}\right)$.

References

[1] A. Quarteroni, Modellistica numerica per problemi differenziali, 3a edizione, Springer, 2006, Chap. 5.

[2] A. Pratelli, Spazi di Sobolev, `http://www-dimat.unipv.it/~pratelli/2007_2008/Approfondimenti_Analisi_Matematica/Dispense.pdf`.