TOWARDS A LOGIC FOR PRAGMATICS

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Abstract. The logic for pragmatics extends classical logic in order to characterize the logical properties of the operators of *illocutionary force* such as that of *assertion* and *obligation* [7, 8, 2]. Here we consider the cases of *assertions* and *conjectures*: the assertion that a mathematical proposition α is true is justified by the capacity to present an actual proof of α, while the conjecture is justified the absence of a refutation of α. We give unitary sequent calculi of type **G3i** and **G3i**m [29] with subsystems characterizing intuitionistic logic and its dual [16, 25, 6] and also a fragment of classical reasoning with such operators. Extending Gödel's and McKinsey and A. Tarski's translations of intuitionistic logic into **S4**, we show that our sequent calculi are sound and complete with respect to Kripke's semantics for **S4**. Although the logic for pragmatics does not impose a philosophical view, the ontological committments implicit in the formalism are at least as strong as those of *potential intuitionism* [19, 20].

§1. Preface. The logic for pragmatics, introduced by Dalla Pozza and Garola in [7, 8] and developed in [2, 24], aims at a formal characterization of the logical properties of *illocutionary operators*: it is concerned, i.e., with the operations by which we performs the act of *asserting* a proposition as true, either on the basis of a mathematical proof or by empirical evidence or by the recognition of physical necessity, or the act of taking a proposition as an *obligation*, either on the basis of a moral principle or by inference within a normative system. [1] The discipline of *pragmatics* (as presented, e.g., in [15])

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makes reference to the classical texts of 20th century philosophy and philosophical logic, e.g., by Austin [1], Grice and Searle, and includes a large body of linguistic research in a complex relationship with semantics and other areas of linguistics, which at present lies beyond the scope of our methods. For instance, the focus of our current work is on *impersonal* acts of judgment, leaving the consideration of *speech acts* to future developments. The present task of a *logic for pragmatics* is to characterize the abstract behaviour of a few pragmatic operators, as it is manifested in highly regimented forms of reasoning such as mathematical discourse, or the foundations of laws.

The consideration of the impersonal operator of assertion in Dalla Pozza and Garola's pragmatic interpretation of intuitionistic logic [7] has given an interesting insight in the interpretation of intuitionistic and classical connectives. Their viewpoint can be sketched as follows. There is a logic of *propositions* and a logic of *judgements*. Propositions are entities which can be *true* or *false*, judgements are acts which can be *justified* or *unjustified*. The logic of propositions is about *truth* according to classical semantics, the logic of judgements gives conditions for the *justification* of acts of judgement. An elementary act of judgement is the *assertion* of a proposition α, which is justified by the capacity to exhibit a *proof* of it, if α is a mathematical proposition, or some kind of *empirical evidence*, if α is about states of affairs. It is then claimed that the justification of complex acts of judgement must be in terms of *Heyting's interpretation of intuitionistic connectives*: for instance, a *conditional judgement* where the assertion of β depends of the assertibility of α is justified by a method that transforms any justification for the assertion of α into a justification for the assertion of β.

In modern logic the distinction between propositions and judgements was established by Frege: a proposition expresses the thought which is the content of a judgement and a judgement is the act of recognizing the truth of its content. In Frege's formalism the expression $\vdash \alpha$ expresses the judgement asserting the proposition α; only truth-functional connectives and quantifiers are considered and judgements appear only at the level of the deductive system. It follows that there cannot be nested occurrences of the symbol

"$\vdash$" and that truth-functional connectives cannot be applied to expressions of judgement. For instance the assertion

(1) *Fermat's last theorem holds but I don't believe it.*

cannot be formalized by the ill-formed formula $\vdash (\vdash F \wedge \neg B)$, where F expresses the statement of Fermat's last theorem and B my belief in it.

The distinction between propositions and judgements has recently been taken up by Martin-Löf: in his formalism "α *prop*" expresses the assertion that α is a well-formed proposition, and "α *true*" expresses the judgement that it is known how to verify α. Here *propositions* are given a *verificationist* semantics: to give meaning to a proposition we must know what counts as a verification of it; indeed, by replacing Frege's "$\vdash \alpha$" with "α *true*", Martin-Löf reveals that in his view it is impossible to separate the truth of a proposition from the conditions of its verification.

Unlike Martin-Löf and in agreement with Frege, Dalla Pozza and Garola distinguish between the truth of a proposition and the justification of a judgement, and extend Frege's framework by introducing *pragmatic connectives* with Heyting's semantics while retaining Tarski's semantics for the logic of propositions. In their compatibilist approach classical semantics is *extended* rather than challenged by intuitionistic pragmatics, the latter having a *different subject matter* than the former. The task and the challenge for Dalla Pozza and Garola's approach is to characterize and explain the relations between these two levels, which seem to take the form of a *reflection* of pragmatics on semantics and of the *interactions* between classical and pragmatic connectives.

Concerning the *interactions* between classical and pragmatic connectives, some relevant facts are pointed out in [7, 8], such as

$$\vdash (\alpha \wedge \beta) \;\Leftrightarrow (\vdash \alpha) \cap (\vdash \beta) \quad \text{and} \quad \vdash (\alpha \rightarrow \beta) \;\Rightarrow\; (\vdash \alpha) \supset (\vdash \beta)$$

The pragmatic level is *reflected* into an *extension* of the classical semantic level through modal operators. Such a reflection is explained by the distinction between *expressive* and *descriptive* uses of the pragmatic operators: for instance, a correct formalization of

(1) would be $\vdash (\Box F \wedge \neg B)$, where "$\Box$" *describes* justified assertibility. In the case of the operator of *assertion* the reflection is given by Gödel, McKinsey and Tarski's translation of intuitionistic logic into the classical modal system **S4**, namely

$$\begin{array}{rclcrcl} (\vdash \alpha)^M & = & \Box\alpha & \qquad & (\vartheta_1 \supset \vartheta_2)^M & = & \Box(\vartheta_1^M \to \vartheta_2^M) \\ (\vartheta_1 \cap \vartheta_2)^M & = & \vartheta_1^M \wedge \vartheta_2^M & \qquad & (\vartheta_1 \cup \vartheta_2)^M & = & \vartheta_1^M \vee \vartheta_2^M \end{array}$$

(and $\bigwedge^M = \bot$ where $\bigwedge$ is the unjustifiable act and $\bot$ is falsity; here we define $\sim \vartheta =_{df} \vartheta \supset \bigwedge$). In [8] the same distinction is made with reference to the operator of *obligation*, whose descriptive use is given by the necessity operator of the deontic system **KD**. Through modal reflections, the logics of the illocutionary operators of assertion and obligation are given a classical Kripke semantics on preordered frames and on frames without terminal worlds, respectively. An important question is the *adequacy* of reflection: does Kripke's semantics actually represent all the mathematical structure of the logic of illocutionary operators and does it characterize its most significant properties from a philosophical viewpoint? Should a mathematical treatment of pragmatics be based on the typed λ-calculus, categorical logic or game-theory rather than Kripke semantics? These are well-known questions to the philosophical interpretations of intuitionism throughout the 20th century (cfr. [9]).

Another philosophical question concerning the justification of judgements should be mentioned here, which has recently been raised by Martino and Usberti ([20], pag. 83) in a discussions of the intuitionistic philosophy of mathematics. Can we say that proofs have a *potential existence*, where "*possibility* is not understood in the traditional intuitionistic sense as knowledge of a method" to produce such a proof, but as "knowledge-independent and tenseless" possibility? Professor Prawitz accepts this of possibility:

> "That we can prove A is not to be understood as meaning that it is within our practical reach to prove A, but only that it is possible in principle to prove A. ... Similarly, that there exists a proof of A does not mean that a proof of A will be constructed but only that the possibility is there for constructing a proof of A. ... I see no objection to conceiving the possibility that there is a specific

> method for curing cancer, which we may discover one day, but which may also remain undiscovered." ([23], pag. 153-154)

Martino and Usberti use the expression *"potential intuitionism"* to indicate the point of view of an intuitionist who believes that proofs have a potential existence independently of our present knowledge, and *"orthodox intuitionism"* for the view that there are no potential proofs. Presumably, for an orthodox intuitionist intuitive proofs are nothing but acts of knowing, whose aim is to make a judgement evident and which have no ontological status, not unlike *free choice sequences*, which have no tense-less identity independently of the acts of choice constituting them.

Martino and Usberti claim that the point of view of potential intuitionism inevitably entails a compatibilist philosophy with respect to classical logic:

> "once a tense-less notion of provability has been espoused, the commitment to an objective realm of *propositions* is unavoidable. For, if the possibility to prove a proposition A is conceived as atemporal, then A itself becomes an atemporal entity." ([20], pag.84).

where "proofs and propositions have atemporal existence" means

> "the existence of a proof and of a propositions is independent of the contingent fact that in human history the proof has been found and the truth or falsity of the proposition has been recognized."

It follows that the potential intuitionist can *understand* the law of *potential excluded middle*

> "A is potentially true or A is not potentially true"

in its own framework and therefore reconstruct Tarski's truth definitions in it.

We cannot discuss Martino and Usberti argument here. However, their characterization of *potential intuitionism* seems to fit Martin-Löf's point of view: what makes a judgement *"α true"* evident (and thus justified) is a proof t of α, where the proof is *reified*, so that it can be explicitly represented by the primitive expressions $t : \alpha$ of the formalism. It is remarkable feature of his type theory that it axiomatizes an intuitionistic and predicative notion of what an informal proof is. It should also be mentioned that Martin-Löf does not include in his system the notion of a free choice sequence, which alone makes it possible to derive a contradiction from the law of excluded middle.

1.1. Conjectures and assertions. The contribution of this paper to the project of logic for pragmatics is the treatment of the illocutionary operator of *congecture* "$\mathcal{H}$" regarded as dual of that of *assertion* "$\vdash$"; this opens the way to an extension of intuitionistic logic and Heyting algebras to dual structures such as *co-Heyting algebras*, following Lawvere, Makkai, Reyes, Zolfaghari and others [16, 25]. Here we motivate our work by showing how the standard and dual intuitionistic logic fit in the extended system and also how richer interactions between semantic and pragmatic connectives yield a translation of a fragment of classical logic into intuitionistic pragmatics. Also we must briefly indicate an intended interpretation of the extended language in common sense reasoning, how the extended system fits in the philosophical discussion of intuitionism and how it could be used to formalize some areas of informal reasoning.

What is the justification of an *impersonal* act of conjecture $\mathcal{H}\,\alpha$, where α is a mathematical statement? We claim it is *the absence of a refutation of* α, i.e., *the absence of a proof of the falsity of* α. However, we must explain what "absence" means in this context. We would like to give a characterization of impersonal illocutionary acts in a logical theory, which should hopefully be the basis of a theory of speech acts by *relativization*. Now, speaking at the very beginning of the 21st century, one is justified in conjecturing the falsity of famous statements such as (i) Goldbach's conjecture, (ii) $\mathbf{P} \neq \mathbf{NP}$ and also (iii) the truth or the falsity of the continuum hypothesis: *as a matter of fact*, as long as we know, nobody has produced a proof of (i) and (ii) and also, thanks to Gödel and Cohen, we know that *there can be no proof* of the continuum hypothesis not of its negation, unless we modify our current understanding of what a set is. Perhaps we can say that we have a *conclusive* justification for the conjectures in (iii) and *inconclusive* justification against (i) and (ii): in any case, at present all such acts are *felicitously* made. Nevertheless, (i) and (ii) may very well be true and a proof of them may be around the corner: in a few decades conjecturing their falsity could become infelicitous. It seems to us that if an *impersonal* act of conjecture $\mathcal{H}\,\alpha$ is justified, then it should remain justified when instantiated in any period of history: after all, the circumstances of the present time are relative to the persons now living. Therefore to say that $\mathcal{H}\,\alpha$ is justified by the "absence of a proof" must mean that a proof of $\neg\alpha$ is nowhere to be found, either now, in the past or in the future.

How do we produce a *conclusive* justification of $\mathcal{H}\,\alpha$? Clearly by *proving that there can be no proof of* $\neg\alpha$, where the proof of this impossibility must also be of a mathematical nature. Notice that this proof is already a justification of the *assertion* $\sim \vdash \neg\alpha$, and therefore the consideration of conjectures does not extend the existing pragmatic theory. Can there be an *inconclusive* justification of $\mathcal{H}\,\alpha$? It seems that we are now in an interesting dilemma.

(a) On one hand, we could say that $\mathcal{H}\,\alpha$ is justified inconclusively if there is no proof of the truth of $\neg\alpha$ but also *no proof that there is no proof of* $\neg\alpha$, therefore no proof of α; but then it would never be possible to improve our inconclusive conjecture $\mathcal{H}\,\alpha$ by giving a proof of the truth of α.

(b) Alternatively, we could claim that $\mathcal{H}\,\alpha$ can only be justified conclusively; but makes impersonal conjectures very far removed from the conjectures felicitously made made by us.

Further explanations depend on the ontological status of potential proofs. If there are no potential proofs, then there is no *logical* alternative to (b). If we admit potential proofs, then we can still give a logical status to conjecturing $\mathcal{H}\,\alpha$ with inconclusive justification, but we still need to avoid the definition in (a). The solution comes from an improved explanation of what it means to *assert* that α is true. We claim that the assertion of the truth of α is justified not merely by the *existence* of a proof of the truth of α, but *by the capacity to exhibit an actual proof* t *of* α: an act of assertion that α is true is felicitous if we can explicitly produce the pair $t : \alpha$. At present we cannot definitely characterize what constitutes inconclusive evidence for a justified impersonal act of conjecture; however by contrasting *conjectures* with *assertions* in the refined definition, we conclude that *conjecturing* is similar to *betting* and that asserting provability without having a proof is getting close to bad manners.

$$\begin{tikzcd}
 & & \Diamond p & & \\
 & & & \Diamond\Box\Diamond p \arrow[ul] & \\
p \arrow[uurr] & & \Diamond\Box p \arrow[ur] & & \Box\Diamond p \arrow[ul] \\
 & & & \Box\Diamond\Box p \arrow[ul] \arrow[ur] & \\
 & & \Box p \arrow[ur] \arrow[uull] & &
\end{tikzcd}$$

TABLE 1. The modalities of **S4**

In this perspective, we may distinguish between $\mathcal{H}\,\alpha$ and $\sim\vdash\neg\alpha$ and between a *weak negation* $\frown\delta$ (*it is doubtful that* δ) and the usual intuitionistic *strong negation* $\sim\delta$, which is the assertion of the negation of δ; their modal translations are $(\frown\delta)^M = \Diamond\neg\delta^M)$ and $(\sim\delta)^M = \Box\neg\delta^M)$. It is well-known that there are only seven "modalities" in **S4** (including no modality), the ones in Table 1, and that applying negation to this Table yields a symmetry along the horizontal axis together with a substitution of $\neg p$ for p. More precisely, the fragment of the Lindenbaum algebra on one generator without binary operations is a lattice given by the figure in 1 and by its dual. Notice that of these seven modalities of **S4** only three are expressible in usual intuitionistic logic, namely

$$(\vdash p)^{\Box} \;=\; \Box p \qquad (\sim\sim\vdash p)^{\Box} \;=\; \Box\Diamond\Box p \qquad (\sim\vdash\neg p)^{\Box} \;=\; \Box\Diamond p$$

and that in the language extended with the operator of conjecture and with weak negation we give a pragmatic counterpart to three other modalities of **S4**:

$$(\mathcal{H}\,p)^M \;=\; \Diamond p \qquad (\frown\frown\mathcal{H}\,p)^M \;=\; \Diamond\Box\Diamond p \qquad (\frown\mathcal{H}\neg p)^M \;=\; \Diamond\Box p$$

Next we can define other *conjectural connectives* such as a weak implication "$\delta\succ\delta'$ (*δ may imply δ'*), a weak conjunction $\delta\curlywedge\delta'$ (*possibly δ and possibly δ'*) and a weak disjunction $\delta\curlyvee\delta'$ (*possibly δ or possibly δ'*); thus we can study the proof-theory of co-Heyting algebras (see [6]).

$$\begin{array}{ccccccccc}
 & & \mathcal{H}p & & & & & & \\
 & \nearrow & & \nwarrow & & & & & \\
 & & & & & & \frown\frown\mathcal{H}\,p & & \\
 & & & & & \nearrow & & \nwarrow & \\
p & & & & \frown\mathcal{H}\neg p & & & & \sim\vdash\neg p \\
 & & & & & \nwarrow & & \nearrow & \\
 & \nwarrow & & & & & \sim\sim\vdash p & & \\
 & & & \nearrow & & & & & \\
 & & \vdash p & & & & & &
\end{array}$$

TABLE 2. Asserting and conjecturing

A Heyting algebra is a (distributive) lattice A in which the operation of Heyting implication is defined, which satisfies the adjunction [2]

$$\frac{p \wedge q \leq r}{p \leq q \rightarrow r.}$$

A co-Heyting Algebra C is a (distributive) lattice such that the opposite C^{op} is a Heyting algebra. In a co-Heyting algebra the operation of *co-implication* or *subtraction* is defined, satisfying

$$\frac{r \leq q \vee p}{r \smallsetminus q \leq p}.$$

Conjectural connectives allow us to develop the proof-theory of co-Heyting algebras in our framework. We write υ for formulas that are conjectures or result from conjectural connectives, and ϑ for formulas resulting from assertion or assertive connectives; we let δ $= \vartheta$ or υ. The modal translation of the conjectural connectives is

$$\begin{array}{rclrcl}
(\mathcal{H}\,\alpha)^M & = & \Diamond\alpha & & & \Diamond(\upsilon_1^M \wedge \neg\upsilon_2^M) \\
(\upsilon_1 \curlywedge \upsilon_2)^M & = & \upsilon_1^M \wedge \upsilon_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & = & \upsilon_1^M \vee \upsilon_2^M
\end{array}$$

Since we distinguish between assertive and conjectural expressions, we are *not* working in *bi-Heyting algebra*, i.e., a structure that is both a Heyting algebra and a co-Heyting algebra. But interesting result appear if we extend the framework by allowing a free interaction of assertions and conjectures through *mixed-type* connectives,

[2]The overloading of symbols shall not create confusion between meet, join and Heyting implication and the classical connectives.

for instance

$$\frown\!\sim \upsilon \;\equiv\; \upsilon \quad \text{and} \quad \sim\!\frown \vartheta \;\equiv\; \vartheta.$$

The modal translation of mixed-type implications and subtraction is unchanged; the translation of mixed-type conjunction and disjunction becomes

$$\begin{array}{rclrcl} (\delta_1 \cap \delta_2)^M & = & \Box\delta_1^M \wedge \Box\delta_2^M & (\delta_1 \cup \delta_2)^M & = & \Box\delta_1^M \vee \Box\delta_2^M \\ (\delta_1 \curlywedge \delta_2)^M & = & \Diamond\delta_1^M \wedge \Diamond\delta_2^M & (\delta_1 \curlyvee \delta_2)^M & = & \Diamond\delta_1^M \vee \Diamond\delta_2^M \end{array}$$

For the resulting generalized system **ILP** we prove soundness and completeness with respect Kripke's semantics. Corrado Biasi [5] has also proved the cut-elimination theorem for **ILP**.

In the generalized system **ILP** much richer interactions are found between classical and pragmatic connectives. Considering *weak implication* $\delta_1 \succ \delta_2$ and its dual, *strong subtraction* $\delta_1 \setminus\!\!\setminus \delta_2$ and their modal translation

$$(\delta_1 \succ \delta_2)^M = \Diamond(\delta_1^M \to \delta_2^M) \quad \text{and} \quad (\delta_1 \setminus\!\!\setminus \delta_2)^M = \Box(\delta_1^M \wedge \neg\delta_2^M)$$

we can easily verify that the following rules are valid and semantically invertible with respect to the modal translation.

$$\frac{\vdash \neg\alpha}{\sim \mathcal{H}\,\alpha} \qquad \frac{\mathcal{H}\,\neg\alpha}{\frown\!\vdash \alpha} \qquad \frac{\mathcal{H}\,(\alpha \to \beta)}{\vdash \alpha \succ \mathcal{H}\,\beta} \qquad \frac{\vdash (\alpha \wedge \neg\beta)}{\vdash \alpha \setminus\!\!\setminus \mathcal{H}\,\beta} \qquad \frac{\vdash (\alpha \wedge \beta)}{\vdash \alpha \cap \vdash \beta} \qquad \frac{\mathcal{H}\,(\alpha \vee \beta)}{\mathcal{H}\,\alpha \curlyvee \mathcal{H}\,\beta}$$

As a consequence, a fragment of the classical propositional language in the lower level sematical part can be represented in the intuitionistic pragmatic part.

What applications are expected for our rich formalism? Which forms of scientific or common sense reasoning involving conjectures and assertions could we wish to formalize in it? As Imre Lakatos showed, conjectures play a fundamental role in mathematics, especially from a heuristic and dynamical point of view, but when a mathematical theory is mature for formalization, then usually it can be represented by a formal system whose axioms are assertions and whose rules of inference transform assertions into assertions. The naif view that a concept (such as that of physical space) could be captured once and for all by a unique mathematical formalization may be seen with skepticism today, but the formalization of mathematical theories as conjectures could play a significant role only in a metamathematical consideration of successions of theories as approximations of the diverse uses of a scientific concept. Conjectures play a fundamental role also in natural sciences: theoretical constructs are intrinsically conjectural, as Popper pointed out. However, it has also been argued that the distinction between theoretical constructs and empirical evidence may not be obvious as Popper made it, a discussion which is clearly beyond the scope of this work.

Weak and strong subtraction may be regarded as paradigms of an *investigative* form of common sense reasoning. If the known that the facts ϑ entail a disjunction of conjectures $\upsilon_1, \ldots, \upsilon_{n+1}$, then from the meaning of "subtraction" we know that $\vartheta \smallsetminus \upsilon_{n+1}$ entails the disjunction $\Upsilon = \upsilon_1, \ldots, \upsilon_n$. The $\smallsetminus$-R rule gives us an *operational* interpretation of the meaning of "$\vartheta \smallsetminus \upsilon_{n+1}$": this conjecture is justified as an alternative to Υ on condition that ϑ *is proved* and moreover that *the conjecture* υ_{n+1} *entails* Υ.

A form of reasoning where the distinction between conjectures and assertions plays an essential role in a highly regimented setting is legal reasoning. Consider the sentence

> *"On Sunday, April 26 1998, Monsignor Juan Gerardi Conedera, Auxiliary Bishop of Guatemala City, was killed by a member of a paramilitary death squad"*

Consider also the (fictional) scenarios in which such a statement might have been made, in English or in Spanish, by different subjects with different intentions: (a) as an assertion by the murdered, reporting to his boss, (b) as a suggestion by gangsters to intimidate political opponents, (c) as a statement by the prosecutor during the trial, (d) as a confession by the murdered during the trial, (e) as a part of the sentence of guilt read by a judge at the end of the trial, (f) as a political statement in the US Senate, aimed at closing down the Army School of the Americas, where the murderers had been trained.[3] Notice that although the statements (a) - (f) would have different illocutionary forces and diverse intentions and effects, in a legal procedure the force of an *impersonal assertion* could be

[3]Our fictional scenarions are based on real events, see, e.g., `http://www.peacehost.net/soaw-w/gerardi.html` and `http://leahy.senate.gov/press/199804/980428.html`.

recognized only to the statement (e), while under the presumption of innocence of the defendant proper of a fair trial statement (c) can only be an *accusatory conjecture*. Since further evidence can cause the trial to be reopened, it seems that the evidence for such an impersonal assertion should be regarded as inconclusive.

§2. The pragmatic language $\mathcal{L}^P$.

DEFINITION 1. (*Syntax*) (i) The language $\mathcal{L}^P$ is built from an infinite set of *propositional letters* p, p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; these expressions are called *radical formulas*. The *elementary formulas* of the pragmatic language are obtained by prefixing a radical formula with a sign of *illocutionary force* "$\vdash$" and "$\mathcal{H}$". There are *elementary constants*, $\bigwedge$ for absurdity, and $\bigvee$ for validity. Finally, the *sentential formulas* of $\mathcal{L}^P$ are built from the elementary formulas and the constant $\bigwedge$, using the *pragmatic connectives* $\sim$, $\cap$, $\cup$, $\supset$, $\frown$, $\succ$, $\curlywedge$ and $\curlyvee$.

(ii) (*Formation Rules*) The *pragmatic* language $\mathcal{L}^P$ is the union of the sets **Rad** of *radical formulas* and **Sent** of *sentential formulas*. These sets are defined inductively by the following grammar:

$$\alpha \;:=\; p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid$$

$$\delta \;:=\; \vartheta \mid \upsilon \mid$$

$$\vartheta \;:= \vdash \alpha \mid \bigwedge \mid \bigvee \mid \sim\delta \mid \delta \supset \delta \mid \delta \cap \delta \mid \delta \cup \delta \mid$$

$$\upsilon \;:=\; \mathcal{H}\alpha \mid \bigwedge \mid \bigvee \mid \frown\delta \mid \delta \succ \delta \mid \delta \curlyvee \delta \mid \delta \curlywedge \delta \mid$$

We use the letters α, β, α_1, ... to denote *radical* formulas, η, η_1, ... for *elementary sentential* formulas, ϑ, ϑ_1, ... for *assertive expressions* and υ, υ_1, ... for *conjectural expressions*.

The *intuitionistic fragment* of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary sentences to those with *atomic radical* only:

$$\bigwedge, \quad \bigvee, \quad \vdash p \quad \text{and} \quad \mathcal{H}p.$$

DEFINITION 2. (*Informal Interpretation*) (i) *Radical* formulas are interpreted as propositions, with the Tarskian classical semantics, as usual.

(ii) *Sentential expressions* ϑ and υ are interpreted as interpreted as impersonal illocutionary acts of *assertion* and *conjecture*, respectively. Illocutionary acts (and the sentential expressions expressing them) can be *"justified"* or *"unjustified"*:

1. $\bigwedge$ is never *justified* and $\bigvee$ is always *justified.*
2. $\vdash \alpha$ is *justified* if and only if there is a proof that α is true; it is *unjustified* otherwise.
3. $\mathcal{H}\, \alpha$ is *justified* if there is no refutation of α, i.e., no proof that α is false; it is *unjustified* otherwise.
4. $\sim \delta$ is *justified* if and only if there is a proof that δ is unjustified; it is *unjustified* otherwise.
5. $\frown \delta$ is *justified* if and only if there is no proof that δ is justified; it is *unjustified* otherwise.
6. $\delta_1 \supset \delta_2$ is *justified* if and only if there is a proof that a justification of δ_1 can be transformed into a justification of δ_2; it is *unjustified*, otherwise.
7. $\delta_1 \succ \delta_2$ is *justified* if and only if there is no proof that δ_1 is justified and δ_2 is unjustified; it is *unjustified*, otherwise.
8. $\delta_1 \setminus\!\!\setminus \delta_2$ is *justified* if and only if there is a proof that a there is justification of δ_1 and no justification of δ_2; it is *unjustified*, otherwise.
9. $\delta_1 \smallsetminus \delta_2$ is *justified* if and only if there is no proof that δ_1 is unjustified or δ_2 is justified; it is *unjustified*, otherwise.
10. $\vartheta_1 \cap \vartheta_2$ is *justified* if and only if both ϑ_1 and ϑ_2 are justified; it is *unjustified* otherwise. Similarly, $\vartheta_1 \cup \vartheta_2$ is justified if and only if either ϑ_1 or ϑ_2 is justified.
11. $\upsilon \cap \delta$ and $\delta \cap \upsilon$ are justified if and only if there are proofs that both υ and δ are justified; they are *unjustified* otherwise. Similarly, $\upsilon \cup \delta$ and $\delta \cup \upsilon$ are justified if and only if there is a proof that either υ or δ is justified.
12. $\upsilon_1 \curlywedge \upsilon_2$ is justified if and only if both υ_1 and υ_2 are justified; it is *unjustified* otherwise. Similarly, $\upsilon_1 \curlyvee \upsilon_2$ is justified if and only if either υ_1 or υ_2 is justified.
13. $\vartheta \curlywedge \delta$ and $\delta \curlywedge \vartheta$ are justified if and only if there is no proof that either ϑ or δ is unjustified; they are unjustified otherwise. Similarly, $\vartheta \curlyvee \delta$ and $\delta \curlyvee \vartheta$ are justified if and only if there is no proof that both ϑ and δ are unjustified.

2.1. Topological interpretation. A mathematical model for the system $\mathcal{L}^P$ is obtained through a topological interpretation.

Definition 3. (*topological interpretation*). Let S be a set, let $\cap$, $\cup$ and $\setminus$ be the usual operations of intersection, union and (binary) complementation defined on the powerset $\wp(S)$ of S, let $(X)^C$ be $S \setminus X$ and let $\mathbf{I} : \wp(S) \to \wp(S)$ and $\mathbf{C} : \wp(S) \to \wp(S)$ be the *interior* and *closure* operators, satisfying

$$\begin{array}{rl}
\mathbf{I}(X) \subseteq X & X \subseteq \mathbf{C}(X) \\
\mathbf{I}(X) \subseteq \mathbf{I}(\mathbf{I}(X)) & \mathbf{C}(\mathbf{C}(X)) \subseteq \mathbf{C}(X) \\
X \subseteq Y \Rightarrow \mathbf{I}(X) \subseteq \mathbf{I}(Y) & X \subseteq Y \Rightarrow \mathbf{C}(X) \subseteq \mathbf{C}(Y) \\
\mathbf{C}(X) = (\mathbf{I}(X^C))^C & \mathbf{I}(X) = (\mathbf{C}(X^C))^C
\end{array}$$

A topological interpretation δ^* of the full language $\mathcal{L}^P$ is given by assigning to each atomic formula P a subset P^* of S and then by proceeding as follows:

$$\begin{array}{rcl@{\qquad}rcl}
(\bigwedge)^* & =_{df} & \emptyset & (\bigvee)^* & =_{df} & S \\
(\vdash \alpha)^* & =_{df} & \mathbf{I}(\alpha^*) & (\mathcal{H}\, \alpha)^* & =_{df} & \mathbf{C}(\alpha^*) \\
(\sim \delta)^* & =_{df} & \mathbf{I}((\delta^*)^C) & (\frown \delta)^* & =_{df} & \mathbf{C}((\delta^*)^C) \\
(\delta_1 \supset \delta_2)^* & =_{df} & \mathbf{I}((\delta_1^*)^C \cup \delta_2^*)) & (\delta_1 \smallsetminus \delta_2)^* & =_{df} & \mathbf{C}((\delta_1^*) \setminus \delta_2^*) \\
(\delta_1 \smallsetminus\!\!\smallsetminus \delta_2)^* & =_{df} & \mathbf{I}((\delta_1^*) \setminus \delta_2^*)) & (\delta_1 \succ \delta_2)^* & =_{df} & \mathbf{C}((\delta_1^*)^C \cup \delta_2^*) \\
(\delta_1 \cap \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cap \mathbf{I}(\delta_2^*) & (\delta_1 \curlyvee \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cup \mathbf{C}(\delta_2^*) \\
(\delta_1 \cup \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cup \mathbf{I}(\delta_2^*) & (\delta_1 \curlywedge \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cap \mathbf{C}(\delta_2^*)
\end{array}$$

2.2. Modal interpretation. Another mathematical interpretation is obtained through an extension of Gödel, McKinsey and Tarski's modal translation $(\ )^\Box$ into the logic **S4**. The language of **S4**, Kripke's semantics and sequent calculus for it are in the Appendix. The language $\mathcal{L}^P$ is translate in **S4** as follows:

Definition 4. (**S4** *translation*)

$$\begin{array}{rcl@{\qquad}rcl}
(\bigwedge)^M & =_{df} & \bot & (\bigvee)^M & =_{df} & \top \\
(\vdash \alpha)^M & =_{df} & \Box\alpha & (\mathcal{H}\, \alpha)^M & =_{df} & \Diamond\alpha \\
(\sim \delta)^M & =_{df} & \Box\neg\delta^M & (\frown \delta)^M & =_{df} & \Diamond\neg\delta^M \\
(\delta_1 \supset \delta_2)^M & =_{df} & \Box(\delta_1^M \to \delta_2^M) & (\delta_1 \smallsetminus \delta_2)^M & =_{df} & \Diamond(\delta_1^M \wedge \neg\delta_2^M) \\
(\delta_1 \smallsetminus\!\!\smallsetminus \delta_2)^M & =_{df} & \Box(\delta_1^M \wedge \neg\delta_2^M) & (\delta_1 \succ \delta_2)^M & =_{df} & \Diamond(\delta_1^M \to \delta_2^M) \\
(\vartheta_1 \cap \vartheta_2)^M & =_{df} & \vartheta_1^M \wedge \vartheta_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & =_{df} & \upsilon_1^M \vee \upsilon_2^M \\
(\vartheta_1 \cup \vartheta_2)^M & =_{df} & \vartheta_1^M \vee \vartheta_2^M & (\upsilon_1 \curlywedge \upsilon_2)^M & =_{df} & \upsilon_1^M \wedge \upsilon_2^M \\
\text{If } \delta_i \text{ is an } \upsilon, & \text{for} & i = 1 \text{ or } 2, \text{ then} & \text{If } \epsilon_i \text{ is a } \vartheta & \text{for} & i = 1 \text{ or } 2, \text{ then} \\
(\delta_1 \cap \delta_2)^M & =_{df} & \Box\delta_1^M \wedge \Box\delta_2^M & (\epsilon_1 \curlyvee \epsilon_2)^M & =_{df} & \Diamond\epsilon_1^M \vee \Diamond\epsilon_2^M \\
(\delta_1 \cup \delta_2)^M & =_{df} & \Box\delta_1^M \vee \Box\delta_2^M & (\epsilon_1 \curlywedge \epsilon_2)^M & =_{df} & \Diamond\epsilon_1^M \wedge \Diamond\epsilon_2^M
\end{array}$$

§3. Sequent calculus for the logic of pragmatics. The sequent calculus for the logic of pragmatics is gigantic. It is a *unitary* system [11], in the sense that it must contain fragments which formalize classical and intuitionistic reasoning, respectively: the classical fragment contains rules for the *radical part* of the pragmatic language with classical semantics; the intuitionistic fragment contains rules for the pragmatic connectives only, thus in this fragment the radical parts are regarded as atomic and remain unchanged throughout a derivation. Moreover, to represent the reflection of the pragmatic part into the radical part within the calculus, the sequent calculus for **S4** should also be added to the classical part.

Here it is convenient to keep the fragments separate. We shall deal mostly with calculi for the *intuitionistic fragment* **ILP**, of which we prove soundness, completeness and finite model property for Kripke's semantics through the **S4** translation; the cut-elimination theorem for **ILP** has been proved by [5]. For a fragment of the classical language we shall also consider a *basic classical sequent calculus* whose rules act on the *radical part* of the formulas in the sequents; we shall also show that there is a translation of this fragment in **ILP** such that a sequent is provable in the basic sequent calculus if and only if its translation is provable in **ILP**.

The official calculus **ILP** for the Intuitionistic Logic for Pragmatics is a system of type **G3i** in the classification of Gentzen systems by Troestra and Schwichtenberg [29], where the rules of weakening and contraction are *implicit*. Gentzen's familiar restriction for intuitionistic sequents is generalized, by using sequents with privileged areas in the antecedent and in the succedent and by requiring that each sequent must contain at most one privileged formula.

Definition 5. All the sequents S are of the form

$$\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon$$

where

- Θ is a sequence of assertive formulas $\vartheta_1, \ldots, \vartheta_m$;
- Υ is a sequence of conjectural formulas $\upsilon_1, \ldots, \upsilon_n$;
- ϵ is conjectural and ϵ' is assertive and at most one of ϵ, ϵ' occurs in S.

The rules of **ILP** are given in the Appendix II. The main result of this paper is the following theorem:

THEOREM 1. *The intuitionistic sequent calculus* **ILP** *without the rules of cut is sound and complete with respect to the modal interpretation in* **S4**. *The finite model property holds for* **ILP**.

In order to prove the completeness theorem for **ILP**, we reduce the problem to the completeness of **S4** and use the "semantic tableaux" procedure for **S4** given in Appendix I. More precisely, given an **ILP** sequent S of the form $\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon$ we consider its modal translation S^M, namely $\Theta^M, \epsilon^M\ \Rightarrow\ \epsilon'^M, \Upsilon^M$, and apply the "semantic tableaux procedure" to S^M. If S^M is falsifiable, in a finite number of steps the procedure yields a Kripke model $\mathcal{M}$ on a preordered frame which falsifies S^M, and it is regarded as a countermodel for S. Otherwise, S^M is derivable in the sequent calculus for **S4** and we must show that S is derivable in **ILP**. We find it convenient to introduce an auxiliary system **FILP** equivalent to **ILP** and to prove that if S^M is derivable in the sequent calculus for **S4** then S is derivable in **FILP**.

§4. FILP. The auxiliary system **FILP** of Full Intuitionistic Logic of Pragmatics generalizes intuitionistic sequent calculi with multiple succedent, such as the systems **G3i**m in [29] or the logic **FILL** (*Full Intuitionistic Linear Logic*) by De Paiva and others (from which we take the acronym). As **FILL** relaxes the intuitionistic restriction on the succedent, so in **FILP** the distinction between two areas in the antecedent and sucedent of sequents is removed and the restriction on the pair ϵ, ϵ' is relaxed whenever this is possible from a logical point of view. In this way, **FILP** retains exactly those restrictions on the sequent-premises S of its rules which are needed for S^M to preserve the restrictions on the modal inferences $\Box$-R and $\Diamond$-L of **S4**. The rules of **ILP** and **FILP** for which it is not possible to relax the restriction on the sequent premises are marked with an asterisk **(*)**. Because of its closeness to sequent calculus for **S4**, the system **FILP** may have an independent interest in the logic for pragmatics.

identity and pragmatic axioms

logical axiom:
$$\delta, \Theta, \Upsilon' \;\Rightarrow\; \delta, \Theta', \Upsilon$$

absurdity axiom:
$$\Theta, \bigwedge, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon$$

assertion-conjecture:
$$\vdash \alpha, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \; \mathcal{H}\, \alpha$$

validity axiom:
$$\Theta, \Upsilon' \;\Rightarrow\; \Theta', \bigvee, \Upsilon$$

structural rules

left exchange:
$$\frac{\Theta_0, \vartheta_0, \vartheta_1, \Theta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta_0, \vartheta_1, \vartheta_0, \Theta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

right exchange:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon_0, \upsilon_0, \upsilon_1, \Upsilon_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon_0, \upsilon_1, \upsilon_0, \Upsilon_1}$$

TABLE 3. **FILP**, identity and structural rules

LEMMA 1. *A sequent* $\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon$ *is derivable in* **FILP** *(without cut) if and only if* $\vdash_{S4} \Theta^M, \Upsilon'^M \;\Rightarrow\; \Theta'^M, \Upsilon^M$ *is derivable in* **S4** *(without cut).*

The *"only if"* part is left to the reader. To prove the *"if"* part, let d be a derivation in **S4** of a sequent S^M, where S is a **FILP** sequent. Given a sequent derivation d and a formula-occurrence α in a sequent S in d we can define the notion of *ancestor* [*descendant*] of α in d as usual and so it is clear what it means to say that a formula β in a sequent S is traceable to a a formula α in a sequent S', when S' occurs above S. To simplify the proof we make some assumptions on the structure of d which are summarized in the following proposition.

PROPOSITION 1. *Let* S *be a* **FILP** *sequent. If* S^M *is derivable in the sequent calculus for* **S4***, then there exists a derivation* d *of* S^M *with the following properties:*

(a) *Let* $\mathcal{I}$ *be an application of* $\vee$*-L* [$\wedge$*-R*]. *If the principal formula of* $\mathcal{I}$ *is* $\Box\gamma_1 \vee \Box\gamma_2$ [$\Diamond\gamma_1 \wedge \Diamond\gamma_2$], *then the inference immediately above* $\mathcal{I}$ *on both branches is* $\Box$*-L* [$\Diamond$*-R*] *with principal formula the active formula of* $\mathcal{I}$*.*

ASSERTIVE LOGICAL RULES

connective of type $\vartheta \to \vartheta$

(*) $\sim$-R:
$$\frac{\Theta, \vartheta \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \Rightarrow\, \sim \vartheta, \Theta', \Upsilon}$$

$\sim$-L:
$$\frac{\sim \vartheta, \Theta, \Upsilon' \;\Rightarrow\; \vartheta, \Theta', \Upsilon}{\sim \vartheta, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\vartheta \times \vartheta \to \vartheta$

(*) $\supset$-R:
$$\frac{\Theta, \vartheta_1 \;\Rightarrow\; \vartheta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \supset \vartheta_2, \Theta', \Upsilon}$$

$\supset$-L:
$$\frac{\vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon \qquad \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\cap$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon \qquad \Theta, \Upsilon' \;\Rightarrow\; \vartheta_2, \Theta', \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \cap \vartheta_2, \Theta', \Upsilon}$$

$\cap$-R:
$$\frac{\vartheta_0, \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_0 \cap \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\cup$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_0, \vartheta_1, \Theta', \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_0 \cup \vartheta_1, \Theta', \Upsilon}$$

$\cup$-L:
$$\frac{\vartheta_0, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_0 \cup \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\smallsetminus$-R:
$$\frac{\Theta \;\Rightarrow\; \vartheta_1, \Upsilon \qquad \vartheta_2, \Theta, \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \smallsetminus \vartheta_2, \Theta', \Upsilon}$$

$\smallsetminus$-L:
$$\frac{\vartheta_0 \smallsetminus \vartheta_1, \vartheta_0, \Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon}{\vartheta_0 \smallsetminus \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

Table 4. Sequent calculus for **FILP**, the standard fragment

Similarly, let $\mathcal{I}$ be an application of $\wedge$-L $[\vee$-R$]$. If the principal formula of $\mathcal{I}$ is $\Box\gamma_1 \wedge \Box\gamma_2$ $[\Diamond\gamma_1 \vee \Diamond\gamma_2]$, then the two inferences immediately above $\mathcal{I}$ are applications of $\Box$-L $[\Diamond$-R$]$ and descendants of their principal formulas are active in $\mathcal{I}$.

(b) *Let $\mathcal{I}$ be an application of $\Box$-L $[\Diamond$-R$]$ and let $\beta = \neg\gamma$ or $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$. If the principal formula of $\mathcal{I}$ is $\Box\beta$ $[\Diamond\beta]$, then the inference $\mathcal{I}'$ immediately above $\mathcal{I}$ is an application of $\neg$-L or $\to$-L or $\wedge$-L immediately below an inference $\neg$-L $[\neg$-R or $\to$-R or $\wedge$-R immediately below a $\neg$-L$]$ respectively, and the principal formula of $\mathcal{I}'$ is the active formula β of $\mathcal{I}$.*

(c) *Let $\mathcal{I}$ be an application of $\Box$-R $[\Diamond$-L$]$ and let $\beta = \neg\gamma$ or $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$. If the principal formula of $\mathcal{I}$ is $\Box\beta$ $[\Diamond\beta]$, then*

CONJECTURAL RULES

connective of type $\upsilon \to \upsilon$

$\frown$-R:
$$\frac{\Theta, \Upsilon', \upsilon \;\Rightarrow\; \Theta', \Upsilon, \frown \upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \frown \upsilon}$$

(*) $\frown$-L:
$$\frac{\Theta \;\Rightarrow\; \Upsilon, \upsilon}{\Theta, \frown \upsilon, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\upsilon \times \upsilon \to \upsilon$

$\succ$-R:
$$\frac{\Theta, \Upsilon', \upsilon_1 \;\Rightarrow\; \Theta', \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \succ \upsilon_2}$$

(*) $\succ$-L:
$$\frac{\Theta, \;\Rightarrow\; \Upsilon, \upsilon_1 \qquad \Theta, \upsilon_2 \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon', \upsilon_1 \succ \upsilon_2 \;\Rightarrow\; \Theta' \Upsilon}$$

$\curlywedge$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_0 \qquad \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_0 \curlywedge \upsilon_1}$$

$\curlywedge$-L:
$$\frac{\Theta, \upsilon_0, \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta, \upsilon_0 \curlywedge \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta' \Upsilon}$$

$\curlyvee$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1, \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \curlyvee \upsilon_2}$$

$\curlyvee$-L:
$$\frac{\Theta, \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \Theta, \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta, \upsilon_1 \curlyvee \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\smallsetminus$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1, \upsilon_0 \smallsetminus \upsilon_2 \qquad \Theta, \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_0 \smallsetminus \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

(*) $\smallsetminus$-L:
$$\frac{\Theta, \upsilon_1 \;\Rightarrow\; \Upsilon, \upsilon_2}{\Theta, \upsilon_1 \smallsetminus \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

TABLE 5. Sequent calculus for **FILP**, the dual fragment

the inference $\mathcal{I}'$ immediately above $\mathcal{I}$ is an application of $\neg - R$ or $\to$-R or $\wedge$-R immediately below an inference $\neg - L$ [$\neg$-L or $\to$-L or $\wedge$-L immediately below a $\neg$-L] respectively, and the principal formula of $\mathcal{I}'$ is the active formula β of $\mathcal{I}$.

(d) *Let $\mathcal{I}$ be an application of $\vee$-R, $\wedge$-R, $\wedge$-L, $\vee$-L with principal formula β of the form*
(I) $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ *in the antecedent or* $\Box\gamma_0 \vee \Box\gamma_1$ *in the succedent;*
(II) $\Diamond\gamma_0 \vee \Diamond\gamma_1$ *in the antecedent or* $\Box\gamma_0 \wedge \Box\gamma_1$ *in the succedent.*
Then the sequent-conclusion of $\mathcal{I}$ has the form

$$\Pi, \underline{\Box\Gamma}, \Diamond\Delta', \Lambda \;\Rightarrow\; \Lambda', \Box\Gamma', \underline{\Diamond\Delta}, \Pi'$$

where Π, Π' are pairwise disjoint sequences of atoms and where Λ and Λ' are sequences of formulas of the form (I) or (II).

MIXED ASSERTIVE RULES

connective of type $\upsilon \to \vartheta$

(*) $\sim$-R:
$$\frac{\Theta, \upsilon \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \Rightarrow \sim \upsilon, \Theta', \Upsilon}$$

$\sim$-L:
$$\frac{\sim \upsilon, \Theta, \Upsilon' \;\Rightarrow\; \upsilon, \Theta', \Upsilon}{\sim \upsilon, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\vartheta \times \upsilon \to \vartheta, \upsilon \times \vartheta \to \vartheta, \upsilon \times \upsilon \to \vartheta$

(*) $\supset$-R:
$$\frac{\Theta, \delta_1 \;\Rightarrow\; \delta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \supset \delta_2, \Theta', \Upsilon}$$

$\supset$-L:
$$\frac{\delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \delta_1, \Upsilon \qquad \delta_2, \delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\cap$-R:
$$\frac{\Theta \;\Rightarrow\; \delta_1, \Upsilon \qquad \Theta \;\Rightarrow\; \delta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \cap \delta_2, \Theta', \Upsilon}$$

$\cap$-L:
$$\frac{\delta_0, \delta_1, \delta_0 \cap \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_0 \cap \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\cup^i$-R:
$$\frac{\Theta \;\Rightarrow\; \delta_i, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_0 \cup \delta_1, \Theta, \Upsilon}$$

$\cup$-L:
$$\frac{\delta_0, \delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \delta_1, \delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\setminus\!\!\setminus$-R:
$$\frac{\Theta \;\Rightarrow\; \delta_1, \Upsilon \qquad \delta_2, \Theta \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \setminus\!\!\setminus \delta_2, \Theta', \Upsilon}$$

$\setminus\!\!\setminus$-L:
$$\frac{\delta_0 \setminus\!\!\setminus \delta_1, \Theta, \delta_0, \Upsilon' \;\Rightarrow\; \delta_1, \Theta', \Upsilon}{\delta_0 \setminus\!\!\setminus \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

TABLE 6. **FILP**, mixed assertive rules

The proof of the proposition can be obtained by implementing conditions (a), (b), (c) and (d) as a search-strategy in the "semantic tableaux" procedure.

If d is a sequent derivation, the *size* $s(d)$ of d is 1 plus the number of inferences in d (not counting exchange and weakening rules). The proof of the lemma is by induction on the size of the given derivation d of S^M in **S4**, assumed to satisfy conditions (a), (b) (c) and (d) of the Proposition; in the proof we construct a **FILP** derivation d^- of S. We consider the last inference of d, having classified the inferences in four cases, we indicate how to prove the inductive step in in each case and give all details only for some example.

MIXED CONJECTURAL RULES

connective of type $\vartheta \to \upsilon$

$\frown$-R:
$$\frac{\Theta, \Upsilon', \vartheta \;\Rightarrow\; \Theta', \Upsilon, \frown \vartheta}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \frown \vartheta}$$

(*) $\frown$-L
$$\frac{\Theta \;\Rightarrow\; \Upsilon, \vartheta}{\Theta, \frown \vartheta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\vartheta \times \upsilon \to \upsilon, \upsilon \times \vartheta \to \upsilon, \vartheta \times \vartheta \to \upsilon,$

$\succ$-R:
$$\frac{\Theta, \delta_1, \Upsilon' \;\Rightarrow\; \Theta', \delta_2, \Upsilon, \delta_1 \succ \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \succ \delta_2}$$

(*) $\succ$-L:
$$\frac{\Theta, \;\Rightarrow\; \delta_1, \Upsilon \qquad \Theta, \delta_2 \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon', \delta_1 \succ \delta_2 \;\Rightarrow\; \Theta', \Upsilon}$$

$\curlywedge$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1, \delta_0 \qquad \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1, \delta_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1}$$

(*) $\curlywedge^i$-L:
$$\frac{\Theta, \delta_i \;\Rightarrow\; \Upsilon}{\Theta, \delta_0 \curlywedge \delta_1, \Upsilon' \;\Rightarrow\; \Theta' \Upsilon}$$

$\curlyvee$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \curlyvee \delta_2, \delta_1, \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \curlyvee \delta_2}$$

(*) $\curlyvee$-L:
$$\frac{\Theta, \delta_1 \;\Rightarrow\; \Upsilon \qquad \Theta, \delta_2 \;\Rightarrow\; \Upsilon}{\Theta, \delta_1 \curlyvee \delta_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\smallsetminus$-R:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1, \delta_1 \smallsetminus \delta_2 \qquad \Theta, \delta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \smallsetminus \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \smallsetminus \delta_2}$$

(*) $\smallsetminus$-L:
$$\frac{\Theta, \delta_1 \;\Rightarrow\; \Upsilon, \delta_2}{\Theta, \delta_1 \smallsetminus \delta_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

TABLE 7. **FILP**, mixed conjectural rules

Case 0. If a sequent S^M is an axiom of one of the forms

$$\Gamma, \Box\alpha \;\Rightarrow\; \Box\alpha, \Delta \quad \text{or} \quad \Gamma, \Diamond\alpha \;\Rightarrow\; \Diamond\alpha, \Delta \quad \text{or} \quad \Gamma, \bot \;\Rightarrow\; \Delta \quad \text{or} \quad \Gamma \;\Rightarrow\; \Delta, \top$$

where Γ and Δ are translations of $\mathcal{L}^P$ formulas, then S is a logical axiom or an absurdity or validity axiom, respectively, of **FILP**. If S^M has the form $\Gamma, \Box\alpha \;\Rightarrow\; \Diamond\alpha, \Delta$ then S is an assumption-conjecture axiom of **FILP**.

Otherwise the derivation d has size greater than 1 and we consider the last inference $\mathcal{I}$ of d. There are four cases:

Case 1. *Propositional* **S4** *rules corresponding to invertible pragmatic rules.* This case excludes inferences with principal formula $\Box\gamma_0 \vee \Box\gamma_1$ or $\Box\gamma_0 \wedge \Box\gamma_1$ *in the succedent* or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ or $\Diamond\gamma_0 \vee \Diamond\gamma_1$ *in* *the antecedent*: for instance, the rule corresponding to an inference $\vee$-R with principal formula $\Box\gamma_0 \vee \Box\gamma_1$ is a right mixed assertive disjunction $\cup$-R which is non-invertible.

Subcase 1.1. If the last inference $\mathcal{I}$ has principal formula $\vartheta_0^M \wedge \vartheta_1$, $\vartheta_0 \vee \vartheta_1$ $\upsilon_0 \wedge \upsilon_1$ or $\upsilon_0 \vee \upsilon_1$, then the sequent-premises are also translations of a **FILP** sequent and we build the derivation d^- by applying

- either an assertive rule $\cap$-R, $\cap$-L, $\cup$-R, $\cup$-L;
- or a conjectural rule $\curlywedge$-R, $\curlywedge$-L, $\curlyvee$-R, $\curlyvee$-L.

Subcase 1.2. Suppose the last inference $\mathcal{I}$ has principal formula $\Box\gamma_0 \wedge \Box\gamma_1$ or $\Box\gamma_0 \vee \Box\gamma_1$ in the antecedent $\Diamond\gamma_0 \vee \Diamond\gamma_1$ or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ in the succedent.

If the last inference $\mathcal{I}$ is $\vee$-L, then by clause (a) of the Proposition d has the form

$$\vee\text{-L}\,\dfrac{\Box\text{-L}\,\dfrac{\overset{d_{1,1}}{\Theta^M, \Upsilon'^M, \gamma_0, \Box\gamma_0 \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_0 \Rightarrow \Upsilon^M} \qquad \Box\text{-L}\,\dfrac{\overset{d_{2,1}}{\Theta^M, \Upsilon'^M, \gamma_1, \Box\gamma_1, \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_1, \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_0 \vee \Box\gamma_1, \Rightarrow \Upsilon^M}$$

Let d_1 and d_2 the immediate subderivations of d. By applying $\vee$-L to the sequent-conclusions of $d_{1,1}$ and d_2 we derive a sequent which is translation of

$$S_1 : \ \Theta, \Upsilon', \delta_0, \delta_0 \cup \delta_1 \Rightarrow \Upsilon$$

letting $\gamma_i^M = \delta_i$. Moreover $s(d_{1,1}) + s(d_2) + 1 < s(d_1) + s(d_2) + 1 = s(d)$ thus we may apply the induction hypothesis and obtain a derivation of S_1. In a similar way we obtain a derivation of

$$S_2 : \ \Theta, \Upsilon', \delta_1, \delta_0 \cup \delta_1 \Rightarrow \Upsilon$$

We build the derivation d^- by applying

- a mixed assertive rule $\cup$-L.

The cases when $\mathcal{I}$ is a $\wedge$-L with principal formula $\Box\gamma_0 \wedge \Box\gamma_1$ or a $\vee$-R [or $\wedge$-R] with principal formula $\Diamond\gamma_0 \vee \Diamond\gamma_1$ [or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$] are similar and dealt with by an application of

- a mixed assertive rule $\cap$-L,
- a mixed conjectural rule $\curlywedge$-R [or $\curlyvee$-R].

Case 2. *Modal rules $\Box L$ or $\Diamond R$ corresponding to invertible pragmatic rules.* The principal formula of such an inference $\mathcal{I}$ is either $\Box\beta$ in the antecedent or $\Diamond\beta$ in the succedent, where β is $\neg\gamma$, $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$ and where γ, γ_1 and γ_2 are translations of $\mathcal{L}^P$ formulas.

Suppose $\Diamond\beta = \Diamond(\delta_1^M \wedge \neg\delta_2^M)$. By clause (b) in the Proposition, d has the form

$$\Diamond\text{-R}\ \dfrac{\wedge\text{-R}\ \dfrac{\begin{array}{c} d_{1,1} \\ \Gamma \Rightarrow \delta_1^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta \end{array} \qquad \neg\text{-R}\ \dfrac{\begin{array}{c} d_{1,2} \\ \delta_2^M, \Gamma, \Rightarrow \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta \end{array}}{\Gamma, \Rightarrow \neg\delta_2^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}}{\Gamma \Rightarrow \delta_1^M \wedge \neg\delta_2^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}}{\Gamma \Rightarrow \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}$$

where $\Gamma = \Theta^M$, Υ'^M and $\Delta = \Theta'^M$, Υ^M. The endsequents of $d_{1,1}$ and of $d_{1,2}$ are translations of **FILP** sequents and $s(d_{1,1}) < s(d)$, $s(d_{1,2} < s(d)$ hence we can apply the inductive hypothesis and obtain the desired derivation d^- by applying $\smallsetminus$-R.

If the principal formula of $\mathcal{I}$ has another form $\Box\beta$ to the left or $\Diamond\beta$ to the right, we proceed in a similar way, using

- either the assertive rules $\sim$-L, $\supset$-L, $\smallsetminus\!\!\smallsetminus$-L;
- or the conjectural rules $\frown$-R, $\succ$-R, $\smallsetminus$-R;
- or the mixed assertive rules $\sim$-L, $\supset$-L, $\smallsetminus\!\!\smallsetminus$-L;
- or the mixed conjectural rules $\frown$-R, $\sim$-R, $\smallsetminus$-R.

Case 3. *Modal rules* $\Box$-R *or* $\Diamond$-L *corresponding to non-invertible pragmatic rules.* The principal formula of such an inference $\mathcal{I}$ is either $\Box\beta$ in the succedent or $\Diamond\beta$ in the antecedent, where β is $\neg\gamma$, $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$ and where γ, γ_1 and γ_2 are translations of $\mathcal{L}^P$ formulas.

Let $\diamond\beta = \Diamond(\delta_1^M \wedge \neg\delta_2^M)$. By clause (c) in the Proposition , the derivation d has the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} d_{1,1} \\ \Box\Gamma, \delta_1^M \Rightarrow \delta_2^M, \Diamond\Delta \end{array}}{\Box\Gamma, \delta_1^M, \neg\delta_2^M \Rightarrow \Diamond\Delta}\ \neg\text{-L}}{\Box\Gamma, \delta_1^M \wedge \neg\upsilon_2^M \Rightarrow \Diamond\Delta^M}\ \wedge\text{-L}}{\Box\Gamma, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Rightarrow \Upsilon^M}\ \Diamond\text{-L}$$

where $\Box\Gamma = \Theta^M$ and $\Diamond\Delta = \Upsilon^M$ and the desired derivation d^- is

$$\dfrac{\begin{array}{c} d^-_{1,1} \\ \Theta, \delta_1 \Rightarrow \delta_2, \Upsilon \end{array}}{\Theta, \delta_1 \smallsetminus \delta_2, \Rightarrow \Upsilon} \smallsetminus\text{-L}$$

If the principal formula of $\mathcal{I}$ has another form $\Box\beta$ in the succedent or $\Diamond\beta$ in the antecedent we proceed in a similar way, by applying one of the following rules:

- the assertive rule $\sim$-R or $\supset$-R or $\setminus\!\!\setminus$-R;
- the conjectural rule $\frown$-L or $\succ$-R or $\smallsetminus$-L;
- the mixed assertive rule $\sim$-R or $\supset$-R or $\setminus\!\!\setminus$-R;
- the mixed conjectural rule $\frown$-L or $\succ$-L or $\smallsetminus$-L.

Case 4. *Propositional rules corresponding to non-invertible pragmatic rules.* The remaining cases are those of inferences whose principal formula β has one of the following forms:

(I) $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ in the antecedent or $\Box\gamma_0 \vee \Box\gamma_1$ in the succedent;
(II) $\Diamond\gamma_0 \vee \Diamond\gamma_1$ in the antecedent or $\Box\gamma_0 \wedge \Box\gamma_1$ in the succedent.

where γ_0 and γ_1 are translations of $\mathcal{L}^P$ formulas. By clause (d) of the Proposition, we may assume that the endsequent S of d has the form

$$\Pi, \underline{\Box\Gamma}, \Diamond\Delta', \Lambda \;\Rightarrow\; \Lambda', \Box\Gamma', \underline{\Diamond\Delta}, \Pi'$$

where Π, Π' are pairwise disjoint sequences of atoms and where Λ, Λ' are sequences of formulas of the form (I) or (II). We consider the part $\overline{d}$ of d which is below all applications of $\Box$-R or $\Diamond$-L; thus $\overline{d}$ is a tree whose leaves are either axioms, or sequents of the form

$$\overline{S_\ell}: \qquad \Box\Gamma, \Diamond\alpha_\ell \;\Rightarrow\; \Diamond\Delta \qquad \text{or} \qquad \Box\Gamma \Rightarrow \Box\alpha_\ell, \Diamond\Delta$$

In each branch $\mathcal{B}$ of $\overline{d}$ below an $\overline{S_\ell}$ we find an application of weakening with conclusion S^+_ℓ and then a sequence $\mathcal{I}_1, \ldots, \mathcal{I}_k$ of applications of $\vee$-R, $\wedge$-R, $\vee$-L, or $\wedge$-L, whose principal formula β is (an ancestor of a formula) in Λ or in Λ'. Among these **S4** inferences we are searching for one which may be *relevant* for our desired **FILP** derivation. We consider the inferences $\mathcal{I}_j$, of a branch $\mathcal{B}$ starting with $j = 1$. Let $S_{j,0}$ [and $S_{j,1}$] be the sequent-premises of $\mathcal{I}_j$. We have the following cases:

(a) $\beta = \Box\gamma_0 \vee \Box\gamma_1$ and β is not traceable to α_ℓ, i.e., α_ℓ is an ancestor neither of $\Box\gamma_0$ nor of $\Box\gamma_1$. In this case we remove

$S_{j,0}$ and the inference $\mathcal{I}_j$ and replace β for the pair $\Box\gamma_0$, $\Box\gamma_1$ in S^+_ℓ. Similarly, if β is $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ and is not traceable to α_ℓ.

(b) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is not traceable to α_ℓ. We remove the inference $\mathcal{I}_j$ and replace β for the $\Box\gamma_i$ which occurs in S^+_ℓ, for $i = 0$ or 1. Similarly, if β is $\Diamond\gamma_0 \vee \Diamond\gamma_1$ and is not traceable to α_ℓ.

(c) $\beta = \Box\gamma_0 \vee \Box\gamma_1$ and β is traceable to α_ℓ. In this case we say that the search has found a *relevant* inference.

(d) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is traceable to α_ℓ through the active formula $\Box\gamma_i$ and also the active formula $\Box\gamma_{1-i}$ is traceable to some $\alpha_{\ell'}$ in some other leaf $S_{\ell'}$ of $\overline{d}$. In this case also we have found a *relevant* inference and we consider (nondeterministically) a branch $\mathcal{B}'$ which starts from such $S_{\ell'}$. Similarly, in the case of $\beta = \Diamond\gamma_0 \vee \Diamond\gamma_1$ and $\Diamond\gamma_0$ and $\Diamond\gamma_1$ are traceable to α_ℓ, $\alpha_{\ell'}$.

(e) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is traceable to α_ℓ through the active formula $\Box\gamma_i$ but the active formula $\Box\gamma_{1-i}$ is not traceable to the $\alpha_{\ell'}$ in any other leaf $S_{\ell'}$ of $\overline{d}$. In this case we consider (nondeterministically) a branch $\mathcal{B}'$ which starts from such a $S_{\ell'}$.

Notice that in each branch $\mathcal{B}$ the search may find a relevant inference only once, and also that steps (a), (b), (e) reduce the size of $\overline{d}$: thus in the end any branch contains at most one inference $\mathcal{I}_j$ and the resulting derivation d' has size not greater than d.

We apply the induction hypothesis to the premises of the $\Box$-R or $\Diamond$-L occurring in the remaining branches of $\overline{d}$. We have three cases:

(i) Case (c) succeeds: the desired derivation d^- is obtained by an application of $\cup$-R or of $\curlywedge$-L;

(ii) Case (d) succeeds: the desired derivation d^- is obtained by an application of $\cap$-R or $\curlyvee$-L.

(iii) otherwise: since α is an ancestor of a formula in $\Diamond\Delta'$ or $\Box\Gamma'$ we are back to **Case 3**.

This concludes the proof of the Lemma.

4.1. Equivalence of ILP and FILP. If $\Theta = \vartheta_1, \ldots, \vartheta_m$, we write $\cup\Theta$ for $\vartheta_1 \cup \ldots \cup \vartheta_m$; similarly, we write $\curlywedge\Upsilon$ for $\upsilon_1 \curlywedge \ldots \curlywedge \upsilon_n$; notice that generalized associativity holds for both $\cup$ and $\curlywedge$.

LEMMA 2. *If*

$$\Theta, \Upsilon' \Rightarrow \Theta', \Upsilon$$

is derivable in the sequent calculus for **FILP**, *then*

$$\Theta\ ; \Rightarrow \curlywedge\Upsilon' \supset \cup\Theta'; \Upsilon$$

is derivable in the sequent calculus for **ILP**.

The proof is by induction on the length of the given **FILP** derivation d. It is a lengthy exercise, whose details can be found in [5]. We consider only one case.

Let d end with an application of the $\cap$-L rule of type $\vartheta \times \upsilon \to \vartheta$, corresponding to the **ACA.4** rule

$$\frac{\Theta, \vartheta \cap \upsilon, \vartheta, \upsilon, \Upsilon' \Rightarrow \Theta', \Upsilon}{\Theta, \vartheta \cap \upsilon\ , \Upsilon' \Rightarrow \Theta', \Upsilon}$$

By inductive hypothesis we have an **ILP** derivation of d_1^*

$$\Theta, \vartheta \cap \upsilon, \vartheta;\ \Rightarrow\ (\curlywedge\Upsilon' \curlywedge \upsilon) \supset \cup\Theta; \Upsilon$$

In **ILP** we have the following derivation d^*

$$\dfrac{\dfrac{\dfrac{\dfrac{; \curlywedge\Upsilon' \Rightarrow ; \curlywedge\Upsilon'}{; \Rightarrow \curlywedge\Upsilon' \supset \cup\Theta'; \curlywedge\Upsilon'}}{; \Rightarrow \upsilon \supset (\curlywedge\Upsilon' \supset \cup\Theta'); \curlywedge\Upsilon'} \qquad \dfrac{; \upsilon \Rightarrow ; \upsilon}{; \Rightarrow \upsilon \supset (\curlywedge\Upsilon' \supset \cup\Theta'); \upsilon}}{; \Rightarrow \upsilon \supset (\curlywedge\Upsilon' \supset \cup\Theta'); \curlywedge\Upsilon' \curlywedge \upsilon} \qquad \dfrac{\dfrac{\cup\Theta'; \Rightarrow \cup\Theta';}{\cup\Theta'; \Rightarrow \curlywedge\Upsilon' \supset \cup\Theta';}}{\cup\Theta'; \Rightarrow \upsilon \supset (\curlywedge\Upsilon' \supset \cup\Theta');}}{(\curlywedge\Upsilon' \curlywedge \upsilon) \supset \cup\Theta'; \Rightarrow \upsilon \supset (\curlywedge\Upsilon' \supset \cup\Theta');}$$

Writing $\delta = \curlywedge\Upsilon' \supset \cup\Theta$, and applying cut_1 to d_1^* and d^* we obtain a derivation d_1^{**} of $\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow\ \upsilon \supset \delta; \Upsilon$. Hence we obtain the following **ILP** derivation:

$$\dfrac{\dfrac{\begin{matrix} d_1^{**} \\ \Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \upsilon \supset \delta; \Upsilon \end{matrix} \qquad \dfrac{\text{ACA.4}\ \dfrac{\Theta, \vartheta \cap \upsilon, \vartheta, \vartheta; \upsilon \Rightarrow ; \Upsilon, \upsilon}{\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \delta; \Upsilon, \upsilon} \qquad \delta, \Theta, \vartheta \cap \upsilon\, \vartheta; \Rightarrow \delta; \Upsilon}{\delta, \Theta, \vartheta \cap \upsilon\, \vartheta; \Rightarrow \delta; \Upsilon}}{\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \delta; \Upsilon}\ cut_1}{\Theta, \vartheta \cap \upsilon; \Rightarrow \delta; \Upsilon}\ \text{ACA.5}$$

§5. Sequent calculus for classical $\mathcal{L}^P$. We are looking for a set of inference rules that modify the *radical part* of pragmatic sentential expressions in a compositional way, inferring formulas with a more complex radical part from simpler ones. Once again, the guideline is given by the **S4** translation. As suggested in the preface, we look for illocutionary operators $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ and a pair of connectives $\circ$ and $\bullet$, where $\circ$ is classical and $\bullet$ is pragmatic, such that

$$(\mathcal{O}(\alpha_1 \circ \alpha_2))^M \equiv (\mathcal{O}'\alpha_1)^M \bullet (\mathcal{O}''\alpha_2)^M$$

When such a relation holds, then we are on a good path to find a sequent calculus where both the *left* and *right* rules *preserve validity* and are *semantically invertible* in the **S4** translation. A set of rules satisfying our requirements is given in the following table 8: this is a fragment of classical reasoning for which the soundness and completeness theorem with respect to the semantic interpretation in **S4** can be easily proved.

right assert-negation:
$$\frac{\Theta \,;\ \mathcal{H}\alpha \ \Rightarrow \,;\ \Upsilon}{\Theta \,;\ \Rightarrow\ \vdash\neg\alpha \,;\ \Upsilon}$$

left assert-negation
$$\frac{\Theta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon,\ \mathcal{H}\alpha}{\vdash\neg\alpha, \Theta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\Upsilon}$$

right hyp-negation:
$$\frac{\Theta,\ \vdash\alpha \,;\ \Rightarrow \,;\ \Upsilon}{\Theta \,;\ \Rightarrow \,;\ \Upsilon,\ \mathcal{H}\neg\alpha}$$

left hyp-negation
$$\frac{\Theta \,;\ \Rightarrow\ \vdash\alpha \,;\ \Upsilon}{\Theta \,;\ \mathcal{H}\neg\alpha \ \Rightarrow \,;\ \Upsilon}$$

right hyp-impl:
$$\frac{\Theta,\ \vdash\alpha \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon,\ \mathcal{H}(\alpha\to\beta),\ \mathcal{H}\beta}{\Theta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon,\ \mathcal{H}(\alpha\to\beta)}$$

left hyp-impl:
$$\frac{\Theta \,;\ \Rightarrow \vdash\alpha \,;\ \Upsilon \qquad \Theta \,;\ \mathcal{H}\beta \ \Rightarrow \,;\ \Upsilon}{\Theta \,;\ \mathcal{H}(\alpha\to\beta) \ \Rightarrow \,;\ \Upsilon}$$

right assert-subtract:
$$\frac{\Theta \,;\ \Rightarrow\ \vdash\alpha \,;\ \Upsilon \qquad \Theta \,;\ \mathcal{H}\beta \ \Rightarrow \,;\ \Upsilon}{\Theta \,;\ \Rightarrow\ \vdash(\alpha\wedge\neg\beta) \,;\ \Upsilon}$$

left assert-subtract:
$$\frac{\Theta,\ \vdash(\alpha\wedge\neg\beta),\ \vdash\alpha \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon,\ \mathcal{H}\beta}{\Theta,\ \vdash(\alpha\wedge\neg\beta) \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon}$$

right assert-and:
$$\frac{\Theta \,;\ \Rightarrow\ \vdash\alpha \,;\ \Upsilon \qquad \Theta \,;\ \Rightarrow\ \vdash\beta \,;\ \Upsilon}{\Theta \,;\ \Rightarrow\ \vdash(\alpha\wedge\beta) \,;\ \Upsilon}$$

left assert-and:
$$\frac{\Theta,\ \vdash\alpha,\ \vdash\beta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon}{\Theta,\ \vdash(\alpha\wedge\beta) \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \Upsilon}$$

right hyp-or:
$$\frac{\Theta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \mathcal{H}\alpha,\ \mathcal{H}\beta, \Upsilon}{\Theta \,;\ \epsilon \ \Rightarrow\ \epsilon' \,;\ \mathcal{H}(\alpha\vee\beta), \Upsilon}$$

left hyp-or
$$\frac{\Theta \,;\ \mathcal{H}\alpha \ \Rightarrow \,;\ \Upsilon \quad \Theta \,;\ \mathcal{H}\beta \ \Rightarrow \,;\ \Upsilon}{\Theta,\ \mathcal{H}(\alpha\vee\beta) \,;\ \Rightarrow \,;\ \Upsilon}$$

TABLE 8. Classical sequent calculus

DEFINITION 6. (i) Consider the following grammar for *radical formulas*:

$$\begin{array}{lcl} \mathbf{P} & := & p \mid \neg\mathbf{N} \mid \mathbf{P}\wedge\mathbf{P} \mid \mathbf{P}\wedge\neg\mathbf{N} \mid \\ \mathbf{N} & := & p \mid \neg\mathbf{P} \mid \mathbf{N}\vee\mathbf{N} \mid \mathbf{P}\rightarrow\mathbf{N} \mid \end{array}$$

(ii) Consider the sublanguage of $\mathcal{L}^P$ where elementary pragmatic expressions are generated by the following rules:

$$\vartheta \ := \ \vdash \mathbf{P} \qquad\qquad \upsilon \ := \ \mathcal{H}\,\mathbf{N}.$$

Let us call such a language *the basic classical language.*

(iii) The *basic classical sequent calculus* is system of sequent calculus for classical logic where sequents are restricted to elementary formulas in the basic classical language, i.e., sequents have one of the forms

$$\vdash \alpha_1, \ldots, \vdash \alpha_m \ ; \ \Rightarrow \ \vdash \alpha \ ; \ \ \mathcal{H}\,\beta_1, \ldots, \mathcal{H}\,\beta_n$$

$$\vdash \alpha_1, \ldots, \vdash \alpha_m \ ; \ \ \mathcal{H}\,\beta \ \Rightarrow \ ; \ \ \mathcal{H}\,\beta_1, \ldots, \mathcal{H}\,\beta_n$$

where the α_i, α are of the form $\mathbf{P}$ and the β_j, β are of the form $\mathbf{N}$.

THEOREM 2. *The basic classical sequent calculus is sound and complete with respect to the modal interpretation in* **S4**.

To prove the theorem, notice that in the semantics of **S4** there is a countermodel to the translation of the sequent-conclusion if and only if there is a countermodel to the translation of at least one sequent-premise. For sequents consisting of elementary formulas whose radical is in the basic classical language, there is always a rule in the basic sequent calculus which can be applied, until we reach a sequent where all elementary formulas have atomic radicals. Therefore we can apply the semantical procedure of section 6.2.2 to the translations of the sequents.

Consider the following translation $(\ )^P$:

$$\begin{array}{rcl@{\qquad}rcl} (\mathbf{P})^P & =_{df} & \vdash p & \text{if } \mathbf{P} & := & p \\ (\mathbf{N})^P & =_{df} & \mathcal{H}\,p & \text{if } \mathbf{N} & := & p \\ (\neg\mathbf{N})^P & =_{df} & \sim(\mathbf{N}^P) & (\mathbf{P}\wedge\mathbf{P})^P & =_{df} & \mathbf{P}^P \cap \mathbf{P}^P \\ (\neg\mathbf{P})^P & =_{df} & \frown(\mathbf{P}^P) & (\mathbf{N}\vee\mathbf{N})^P & =_{df} & \mathbf{N}^P \curlyvee \mathbf{N}^P \\ (\mathbf{P}\rightarrow\mathbf{N})^P & =_{df} & \mathbf{P}^P \succ \mathbf{N}^P) & (\mathbf{P}\wedge\neg\mathbf{N})^P & =_{df} & \mathbf{P}^P \setminus\!\!\setminus \mathbf{N}^P) \end{array}$$

where the conditions $\mathbf{P} \ := \ p$ and $\mathbf{N} \ := \ p$ in the first two rules refer to the productions of the grammar generating the radical formulas.

THEOREM 3. *Let S be a sequent consisting of elementary formulas in the basic classical language. Then S is derivable in the classical sequent calculus if and only if S^P is derivable in the intuitionistic sequent calculus.*

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§6. APPENDIX I.The modal language and the semantics for K and S4.

Definition 7. (*Syntax*) (i) The language $\mathcal{L}^m$ is built from an infinite set **Atoms** of *propositional letters* p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; and the modal operators $\Box$ and $\Diamond$.

(ii) (*Formation Rules*) The expressions of the language $\mathcal{L}^m$ are given by the following grammar, where p ranges over **Atoms**:

$$\alpha \ := \ p \mid \bot \mid \top \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \Box\alpha \mid \Diamond\alpha \mid$$

6.1. Frames and Kripke models.

DEFINITION 8. (*Frames and Kripke models*) (i) A *frame* is a pair $\mathcal{F} = (W, \sqsubseteq)$ where

- W is a set (of *"possible worlds"*);
- $\sqsubseteq \quad \subset \quad W \times W$ is a relation (the *"accessibility relation"* between possible worlds).

(ii) A *Kripke model* is a triple $\mathcal{M} = (W, \sqsubseteq, \Vdash)$ where $\mathcal{F} = (W, \sqsubseteq)$ is a frame and $\Vdash \subseteq W \times \mathbf{Atoms}$ is the *forcing relation*, usually written in infix notation: $w \Vdash p$ means *"p is true in the possible world w"* and $w \nVdash p$ means *"p is false in the possible world w"*.

(iii) The relation $\Vdash$ is extended to a relation $\Vdash \quad \subset \quad W \times \mathcal{L}^m$ according to the following rules:

1. $w \nVdash \bot$ and $w \Vdash \top$, for all $w \in W$;
2. $w \Vdash \neg\alpha$ iff $w \nVdash \alpha$;
3. $w \Vdash (\alpha \wedge \beta) = V$ iff $w \Vdash \alpha$ and $w \Vdash \beta$;
4. $w \Vdash (\alpha \vee \beta)$ iff $w \Vdash \alpha$ or $w \Vdash \beta$;
5. $w \Vdash (\alpha \to \beta)$ iff either $w \nVdash \alpha$ or $w \Vdash B$;
6. $w \Vdash \Box\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that $w' \sqsubseteq w$;
7. $w \Vdash \Diamond\alpha$ iff $w' \Vdash \alpha$ for some $w' \in W$ such that $w' \sqsubseteq w$.

If Γ and Δ are sequences of formulas in $\mathcal{L}^m$, then the *sequent* $\Gamma \Rightarrow \Delta$ is *true in* $w \in W$ iff $w \Vdash (\bigwedge \Gamma \to \bigvee \Delta)$.

(iv) We say that a formula α is *valid in a model* $\mathcal{M} = (W, \sqsubseteq, \Vdash)$, in symbols $\models_{\mathcal{M}} \alpha$, iff for every $w \in W$ we have $w \Vdash \alpha$. Similarly, given a *sequent* $S = \Gamma \Rightarrow \Delta$ we say that S is *valid in* $\mathcal{M}$ iff for every $w \in W$, S is true in w.

(v) We say that a formula α is *valid in a frame* $\mathcal{F}$ iff for every $\mathcal{M}$ over $\mathcal{F}$ we have $\models_{\mathcal{M}} \alpha$. Similarly, a sequent S is *valid in a frame* $\mathcal{F}$ iff it is valid in every Kripke model over $\mathcal{F}$.

(vi) A formula α [a sequent S] is *valid* in the system **K** iff α [S] it is valid in all Kripke models $\mathcal{M}$.

(vii) A formula α [a sequent S] is *valid* in the system **S4** iff α [S] is valid in all preordered frames, i.e., all frames where the accessibility relation $\sqsubseteq$ is reflexive and transitive.

6.2. Sequent calculi G3c, K and S4. Gentzen-Kleene's sequent calculus **G3c** for classical propositional logic (cfr.[29], p. 77) is given by the following sequent-axioms and rules of inference. Notice that the rules of *weakening* and *contraction* are implicit.

DEFINITION 9. (i) Given a notion of semantic validity, a rule of the sequent calculus $\dfrac{S_1, \dots, S_n}{S}$ *preserves validity* if for every instance of the rule, the sequent conclusion S is valid whenever the sequent-premises S_1, ..., S_n are all valid; a rule is *semantically invertible* if for every instance of the rule the sequent-premises are all valid whenever the sequent-conclusion is valid.

PROPOSITION 2. *(i) The rules of the system* **G3c** *preserve validity and are semantically invertible for any modal semantics;*
(ii) the modal rules for the systems **K** *and* **S4** *preserve validity and are semantically invertible in the semantics of the system* **S4***;*
(iii) the rules of weakening *preserve validity but are not semantically invertible.*

6.2.1. *Semantic Tableaux procedure for* **K**. The "semantic tableaux" procedure decides whether a sequent S is valid in the semantics for **K** by building a *refutation tree* labelled with sequents and with S at the root; if S is valid, then it return a derivation of S in the sequent calculus for **K**; if S not valid, it returns a counterexample $\mathcal{M}$ which refutes S.

DEFINITION 10. *(semantic tableaux procedure)* Start with tree τ_0 consisting of the root S; at stage $n+1$, for every leaf S' of the tree τ_n check whether the sequent S' matches the conclusion of a rule of inference (in some given order, e.g., checking the one-premise rules first). If yes, invert that rule; otherwise, the leaf in question is a sequent of the form

$$p_1, \dots, p_k, \Box\Gamma, \Diamond\alpha_1, \dots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \dots, \Box\beta_n, \Diamond\Delta, q_1, \dots, q_\ell \qquad (\dagger)$$

Rewrite the sequent (†) as a *hypersequent* as follows:

$$\Rightarrow [p_1, \dots, p_k \Rightarrow q_1, \dots, q_\ell] \dots [\Box\Gamma, \Diamond\alpha_i \Rightarrow \Diamond\Delta] \dots [\Box\Gamma, \Rightarrow \Box\beta_j, \Diamond\Delta] \dots \qquad (\ddagger)$$

We call this step a *disjunctive ramification.* Now there are three cases:

(a) the sequent $p_1, \dots, p_k \Rightarrow q_1, \dots, q_\ell$ is valid, because $p_i = q_j$ for some $i \leq k$, $j \leq \ell$ or because $p_i = \bot$ for some $i \leq k$: in this case the sequent (†) is a *logical axiom* or a *falsity axiom* or a *truth axiom* and the procedure halts on this branch, which is *closed*.

(b) otherwise, if (†) is not an axiom and $m = 0 = n$, then the procedure halts on this branch leaving it *open*;

(c) otherwise, (†) is not an axiom and $m + n > 0$: in this case the procedures *branches* by inverting the $\Diamond$-L or $\Box$-R rules in the remaining $m + n$ sequents of the hypersequent.

Definition 11. We define inductively what it means for a refutation tree τ to be *closed* (starting from the *leaves*):

- a *logical axiom*, a *falsity axiom* oy a *truth axiom* is closed;
- if τ results from τ_0 by a *one-premise* inference rule, then τ is closed iff τ_0 is closed;
- if τ results from τ_0 and τ_1 by a *two-premises* inference rule, then τ is closed iff τ_0 and τ_1 are both closed;
- if τ ends with a hypersequent and results from $\tau_1, \ldots, \tau_{m+n}$ by a *disjunctive ramification*, then τ is closed iff at least one τ_i is closed, for $i \leq m + n$.

Fact 1: *The semantic tableax procedure for* **K** *terminates.*

Fact 2: *If a refutation tree τ with conclusion S is closed, then we can obtain a derivation of S in the sequent calculus for* **K** *as follows:*

- for each disjunctive ramification branching from a sequent of the form (‡) with subtrees $\tau_1, \ldots, \tau_{m+n}$, first we *prune* τ by selecting a *closed* subtree τ_k, by removing the others and the hypersequent notation; the endsequent of τ_k has the form $\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta$ or $\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta$ and now we apply *weakening* to obtain the sequent (†).

Fact 3: *If a refutation tree τ with conclusion S is open, the we can construct a Kripke model $\mathcal{M}$ which refutes S*:

- for every two-premises logical rule, if the sequent-conclusion is open, then we select one of the sequent-premises which is open. In this way we eventually obtain a tree τ' where all branches are open.

- Consider all fragments of branches $\beta_1, \ldots, \beta_z$ obtained from τ' by removing every hypersequent and every conclusion of a modal inference;
  (i) identify β_i with a possible world w_i;
  (ii) put $w_i \sqsubseteq w_j$ if and only if the lowermost sequent of β_i is the premise of a **KR** occurring immediately above a sequent S^* of the form (†) and S^* is the uppermost sequent of β_j;
  (iii) let $w_i \Vdash p_i$ if and only if p_i occurs in the antecedent of a sequent S^* of the form (†) and S^* is the uppermost sequent of β_i.

From facts 1-3 we obtain the following theorem:

THEOREM 4. *The semantic tableaux procedure for* **K** *is sound and complete with respect to the semantics of* **K**. *The system* **K** *has the finite model property.*

6.2.2. *Semantic Tableaux procedure for* **S4**. In the case of **S4** the procedure is modified by inverting the $\Box$-left and $\Diamond$-right in the same way as the propositional rules, but we must deal with the fact that in this way the procedure may enter infinite loops. The first problem is that the $\Box$-left and $\Diamond$-right rules could be iterated forever with the same principal formula. It is enough to mark the modal formula which is principal formula of such an inference and remove the mark later when some $\Box$-right or $\Diamond$-left rule is inverted; in other words we take modal rules of the forms

$$\Box\ \textit{left}: \quad \frac{\alpha, \Gamma, \underline{\Box\alpha}, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}}{\Box\alpha, \Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}} \qquad \Box\ \textit{right}: \quad \frac{\Box\Gamma \Rightarrow \alpha, \Diamond\Delta}{\underline{\Box\Gamma} \Rightarrow \Box\alpha, \underline{\Diamond\Delta}}$$

$$\Diamond\ \textit{left}: \quad \frac{\Box\Gamma, \alpha \Rightarrow \Diamond\Delta}{\underline{\Box\Gamma}, \Diamond\alpha \Rightarrow \underline{\Diamond\Delta}} \qquad \Diamond\ \textit{right}: \quad \frac{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \alpha, \underline{\Diamond\alpha}, \underline{\Diamond\Lambda}}{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \Diamond\alpha, \underline{\Diamond\Lambda}}$$

A *disjunctive branching* in **S4** has the form

$$\frac{\Rightarrow [\Pi \Rightarrow \Pi'], \ldots, \dfrac{\Box\Gamma, \alpha_i \Rightarrow \Diamond\Delta}{[\underline{\Box\Gamma}, \Diamond\alpha_i \Rightarrow \underline{\Diamond\Delta}]}, \ldots, \dfrac{\Box\Gamma \Rightarrow \beta_j, \Diamond\Delta}{[\underline{\Box\Gamma}, \Rightarrow \Box\beta_j, \underline{\Diamond\Delta}]}, \forall i \leq m, \forall j \leq n}{\Pi, \underline{\Box\Gamma}, \Diamond\alpha_1, \ldots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \ldots, \Box\beta_n, \underline{\Diamond\Delta}, \Pi'}$$

where $\Pi = p_1, \ldots, p_k$ and $\Pi' = q_1, \ldots, q_\ell$.

The second source of non-termination is the fact that in general an inversion of the $\Box$-left and of the $\Diamond$-right rules increases the logical complexity of the sequent instead of reducing it. However, since the procedure satisfies the *subformula property* and there is only a finite number of modal subformulas in any given sequent, eventually on any branch the procedure must invert a $\Box$-right or $\Diamond$-left rule with a sequent-conclusion S such that the same rule with the same sequent-conclusion S had already inverted at some point below in the refutation tree (here we consider sequents S *modulo* exchange and contraction). Let $< \mathcal{I}, \mathcal{I}' >$ be such a pair of inferences, where $\mathcal{I}'$ occurs above $\mathcal{I}$. In this case we *identify* the sequent-premise of $\mathcal{I}'$ with the sequent premise of $\mathcal{I}$ and the procedure stops on that branch. Notice that as a consequence of such a *gluing* there will be a loop in the transitive closure of the accessibility relation $\sqsubseteq$ of the countermodel constructed in **Fact 3**. Other details are left to the reader. It follows that

THEOREM 5. *The semantic tableaux procedure for* **S4** *is sound and complete with respect to the semantics of* **S4**. *The system* **S4** *has the finite model property.*

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SEQUENT CALCULUS G3c FOR CLASSICAL LOGIC

axioms: $p, \Gamma \Rightarrow \Delta, p$ *falsity axioms:* $\bot, \Gamma \Rightarrow \Delta$ *truth axioms:* $\Gamma \Rightarrow \Delta, \top$

right exchange:
$$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta, \Delta'}{\Gamma \Rightarrow \Delta, \beta, \alpha, \Delta'}$$

left exchange:
$$\frac{\Gamma, \alpha, \beta, \Gamma' \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Gamma' \Rightarrow \Delta}$$

right $\neg$:
$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

left $\neg$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}$$

right $\wedge$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

left $\wedge$:
$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$$

right $\to$:
$$\frac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \to \beta, \Delta}$$

left $\to$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \to \beta, \Gamma \Rightarrow \Delta}$$

right $\vee$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

left $\vee$:
$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}$$

EXTENSION TO MODAL SYSTEMS

weakenings

$$\frac{\Box\Gamma, \Diamond\alpha \ \Rightarrow \ \Diamond\Delta}{\Pi, \Box\Gamma, \Diamond\alpha, \Diamond\Delta' \ \Rightarrow \ \Box\Gamma', \Diamond\Delta, \Pi'} \qquad \frac{\Box\Gamma \ \Rightarrow \ \Box\alpha, \Diamond\Delta}{\Pi, \Box\Gamma, \Diamond\Delta' \ \Rightarrow \ \Box\alpha, \Box\Gamma', \Diamond\Delta, \Pi'}$$

where Π, Π' are sequences of atoms.

modal rules for K

K-$\Box$-*rule*:
$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta}$$

K-$\Diamond$-*rule*:
$$\frac{\Gamma, \alpha \Rightarrow \Delta}{\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta}$$

modal rules for S4

$\Box$ *left*:
$$\frac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}$$

$\Box$ *right*:
$$\frac{\Box\Gamma \ \Rightarrow \ \alpha, \Diamond\Delta}{\Box\Gamma \ \Rightarrow \ \Box\alpha, \Diamond\Delta}$$

$\Diamond$ *left*:
$$\frac{\Box\Gamma, \alpha \ \Rightarrow \ \Diamond\Delta}{\Box\Gamma, \Diamond\alpha \ \Rightarrow \ \Diamond\Delta}$$

$\Diamond$ *right*:
$$\frac{\Gamma \Rightarrow \Delta, \Diamond\alpha, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}$$

TABLE 9. Sequent calculi for **K** and **S4**

APPENDIX II: The rules of ILP

identity rules

S.1: logical axiom:
$$\vartheta, \Theta \,;\, \Rightarrow\, \vartheta \,;\, \Upsilon$$

S.2: logical axiom:
$$\Theta \,;\, \upsilon \,\Rightarrow\, ;\, \Upsilon, \upsilon$$

S.3: absurdity axiom:
$$\Theta, \bigwedge \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon$$

S.4: absurdity axiom:
$$\Theta \,;\, \bigwedge \,\Rightarrow\, ;\, \Upsilon$$

S.5: assertion-conjecture:
$${}^{\vdash}\alpha, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon,\, {}^{\mathcal{H}}\alpha$$

S.6: validity axiom:
$$\Theta \,;\, \Rightarrow\, \bigvee \,;\, \Upsilon$$

S.7: validity axiom:
$$\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \bigvee$$

S.8: cut$_1$:
$$\frac{\Theta \,;\, \Rightarrow\, \vartheta \,;\, \Upsilon \qquad \vartheta, \Theta' \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon'}{\Theta, \Theta' \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \Upsilon'}$$

S.9: cut$_2$:
$$\frac{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \upsilon \qquad \Theta' \,;\, \upsilon \,\Rightarrow\, \Upsilon'}{\Theta, \Theta' \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \Upsilon'}$$

structural rules

S.10: exchange:
$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}$$

S.11: exchange:
$$\frac{\Theta, \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

TABLE 10. **ILP**, identity and structural rules

ASSERTIVE LOGICAL RULES

connective of type $\vartheta \to \vartheta$

(*) *A.1: right negation:*

$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \sim \vartheta \; ; \; \Upsilon}$$

A.2: left negation:

$$\frac{\sim \vartheta, \Theta ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\sim \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

connectives of type $\vartheta \times \vartheta \to \vartheta$

(*) *A.3: right* $\supset$*:*

$$\frac{\Theta, \vartheta_1 \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \supset \vartheta_2 \; ; \; \Upsilon}$$

A.4: left $\supset$*:*

$$\frac{\vartheta_1 \supset \vartheta_2, \Theta ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \vartheta_2, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.5: right $\cap$*:*

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \Theta \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \cap \vartheta_2 \; ; \; \Upsilon}$$

A.6: left $\cap$*:*

$$\frac{\vartheta_0, \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_0 \cap \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.7,8: right $\cup^i$*:*

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_i \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_0 \cup \vartheta_1 \; ; \; \Upsilon}$$

for $i = 0, 1$.

A.9: left $\cup$*:*

$$\frac{\vartheta_0, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon \quad \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_0 \cup \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

(*) *A.10: right* $\setminus\!\!\setminus$*:*

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \vartheta_2, \Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \setminus\!\!\setminus \vartheta_2 \; ; \; \Upsilon}$$

A.11: left $\setminus\!\!\setminus$*:*

$$\frac{\vartheta_1 \setminus\!\!\setminus \vartheta_2, \Theta, \vartheta_1 \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\vartheta_1 \setminus\!\!\setminus \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.12: left $\setminus\!\!\setminus$*:*

$$\frac{\vartheta_1 \setminus\!\!\setminus \vartheta_2, \Theta, \vartheta_1 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_1 \setminus\!\!\setminus \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

TABLE 11. Sequent calculus for **ILP**, the standard fragment

CONJECTURAL RULES

connective of type $\upsilon \to \upsilon$

C.1: right $\frown$:

$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\; ;\, \Upsilon, \frown \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \frown \upsilon}$$

(*) *C.2: left* $\frown$:

$$\frac{\Theta ; \;\Rightarrow\; ;\, \Upsilon, \upsilon}{\Theta \,;\, \frown \upsilon \;\Rightarrow\; ;\, \Upsilon}$$

connectives of type $\upsilon \times \upsilon \to \upsilon$

C.3: right $\succ$:

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \upsilon_2, \Upsilon, \upsilon_1 \succ \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.4: right $\succ$:

$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \succ \upsilon_2}$$

(*) *C.5: left* $\succ$:

$$\frac{\Theta \,; \;\Rightarrow\; ;\, \Upsilon, \upsilon_1 \qquad \Theta \,; \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_1 \succ \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

C.6: right $\curlywedge$:

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_0 \qquad \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_0 \curlywedge \upsilon_1}$$

C.7,8: left $\curlywedge$:

$$\frac{\Theta \,;\, \upsilon_i \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_0 \curlywedge \upsilon_1 \;\Rightarrow\; ;\, \Upsilon}$$

for $i = 0, 1$.

C.9: right $\curlyvee$:

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1, \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \curlyvee \upsilon_2}$$

C.10: left $\curlyvee$:

$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon \qquad \Theta \,;\, \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_1 \curlyvee \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

C.11: right $\smallsetminus$:

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \qquad \Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon, \upsilon_0 \smallsetminus \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

(*) *C.12: left* $\smallsetminus$:

$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon, \upsilon_2}{\Theta \,;\, \upsilon_1 \smallsetminus \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

TABLE 12. Sequent calculus for **ILP**, the dual fragment

MIXED-TYPE NEGATIONS

connective of type $\upsilon \to \vartheta$

(*) *CA.1: right* $\sim$:
$$\frac{\Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \sim \upsilon \ ; \ \Upsilon}$$

CA.2: left negation$_1$
$$\frac{\Theta ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon}{\sim \upsilon, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

connective of type $\vartheta \to \upsilon$

AC.1: right $\frown$:
$$\frac{\Theta, \vartheta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \frown \vartheta}$$

(*)*AC.2: left* $\frown$:
$$\frac{\Theta ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\Theta \ ; \ \frown \vartheta \ \Rightarrow \ ; \ \Upsilon}$$

MIXED-TYPE SUBTRACTIONS

connective of type $\vartheta \times \upsilon \to \vartheta$

(*) *ACA.9: right* $\smallsetminus\!\!\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \smallsetminus\!\!\smallsetminus \upsilon \ ; \ \Upsilon}$$

ACA.10: left $\smallsetminus\!\!\smallsetminus$:
$$\frac{\vartheta, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon}{\vartheta \smallsetminus\!\!\smallsetminus \upsilon, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \vartheta$

(*) *CAA.10: right* $\smallsetminus\!\!\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ ; \ \Upsilon, \upsilon \qquad \vartheta, \Theta \ ; \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \upsilon \smallsetminus\!\!\smallsetminus \vartheta \ ; \ \Upsilon}$$

CAA.11: left $\smallsetminus\!\!\smallsetminus$:
$$\frac{\upsilon \smallsetminus\!\!\smallsetminus \vartheta, \Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon}{\upsilon \smallsetminus\!\!\smallsetminus \vartheta, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

CAA.12: left $\smallsetminus\!\!\smallsetminus$:
$$\frac{\upsilon \smallsetminus\!\!\smallsetminus \vartheta, \Theta \ ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\upsilon \smallsetminus\!\!\smallsetminus \vartheta, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

connective of type $\upsilon \times \upsilon \to \vartheta$

(*) *CCA.9: right* $\smallsetminus\!\!\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ ; \ \Upsilon, \upsilon_1 \qquad \Theta \ ; \ \upsilon_2 \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \upsilon_1 \smallsetminus\!\!\smallsetminus \upsilon_2 \ ; \ \Upsilon}$$

CCA.10: left $\smallsetminus\!\!\smallsetminus$:
$$\frac{\upsilon_1 \smallsetminus\!\!\smallsetminus \upsilon_2, \Theta \ ; \ \upsilon_1 \ \Rightarrow \ ; \ \Upsilon, \upsilon_1}{\upsilon_1 \smallsetminus\!\!\smallsetminus \upsilon_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

CCA.11: left $\smallsetminus\!\!\smallsetminus$:
$$\frac{\upsilon_1 \smallsetminus\!\!\smallsetminus \upsilon_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_1}{\upsilon_1 \smallsetminus\!\!\smallsetminus \upsilon_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

connective of type $\vartheta \times \upsilon \to \upsilon$

ACC.9: right $\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon, \vartheta \smallsetminus \upsilon \qquad \Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon, \vartheta \smallsetminus \upsilon}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \vartheta \smallsetminus \upsilon}$$

(*) *ACC.10: left* $\smallsetminus$:
$$\frac{\Theta, \vartheta \ ; \ \Rightarrow \ ; \ \Upsilon, \upsilon}{\Theta \ ; \ \vartheta \smallsetminus \upsilon \ \Rightarrow \ ; \ \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \upsilon$

CAC.10: right $\smallsetminus$:
$$\frac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon \qquad \vartheta, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon \smallsetminus \vartheta}$$

(*) *CAC.11: left* $\smallsetminus$:
$$\frac{\Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \upsilon \smallsetminus \vartheta \ \Rightarrow \ ; \ \Upsilon}$$

(*) *CAC.12: left* $\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\Theta \ ; \ \upsilon \smallsetminus \vartheta \ \Rightarrow \ ; \ \Upsilon}$$

connective of type $\vartheta \times \vartheta \to \upsilon$

AAC.10: right $\smallsetminus$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon, \vartheta_1 \smallsetminus \vartheta_2 \qquad \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon,}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \vartheta_1 \smallsetminus \vartheta_2}$$

(*) *AAC.11: left* $\smallsetminus$:
$$\frac{\Theta, \vartheta_1 \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \vartheta_1 \smallsetminus \vartheta_2 \ \Rightarrow \ ; \ \Upsilon}$$

TABLE 13. Mixed-type negations and subtractions

MIXED-TYPE ASSERTIVE LOGICAL RULES

connectives of type $\vartheta \times \upsilon \to \vartheta$

(*) *ACA.1: right* $\supset$:
$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon, \upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta \supset \upsilon \; ; \; \Upsilon}$$

ACA.2: left $\supset$:
$$\frac{\vartheta \supset \upsilon, \Theta ; \; \Rightarrow \vartheta \; ; \; \Upsilon \qquad \vartheta \supset \upsilon, \Theta \; ; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\vartheta \supset \upsilon, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon \; ; \; \Upsilon}$$

(*) *ACA.3: right* $\cap$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon \qquad \Theta \; ; \; \Rightarrow \; ; \; \upsilon, \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta \cap \upsilon \; ; \; \Upsilon}$$

ACA.4: left $\cap$:
$$\frac{\vartheta \cap \upsilon, \Theta, \vartheta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\vartheta \cap \upsilon, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

ACA.5: left $\cap$:
$$\frac{\vartheta \cap \upsilon, \Theta, \vartheta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta \cap \upsilon, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

(*) *ACA.6: right* $\cup$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta \cup \upsilon \; ; \; \Upsilon}$$

(*) *ACA.7: right* $\cup$:
$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \upsilon, \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta \cup \upsilon \; ; \; \Upsilon}$$

ACA.8: left $\cup$:
$$\frac{\vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon \qquad \vartheta \cup \upsilon, \Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\vartheta \cup \upsilon, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

connectives of type $\upsilon \times \vartheta \to \vartheta$

(*) *CAA.1: right* $\supset$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon \supset \vartheta \; ; \; \Upsilon}$$

(*) *CAA.2: right* $\supset$:
$$\frac{\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon \supset \vartheta \; ; \; \Upsilon}$$

CAA.3: left $\supset$:
$$\frac{\Theta ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \qquad \vartheta, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\upsilon \supset \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

(*) *CAA.4: right* $\cap$:
$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \upsilon, \Upsilon \qquad \Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon \cap \vartheta \; ; \; \Upsilon}$$

CAA.5: left $\cap$:
$$\frac{\upsilon \cap \vartheta, \Theta, \vartheta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\upsilon \cap \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

CAA.6: left $\cap$:
$$\frac{\upsilon \cap \vartheta, \Theta, \vartheta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\upsilon \cap \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

(*) *CAA.7: right* $\cup$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon \cup \vartheta \; ; \; \Upsilon}$$

(*) *CAA.8: right* $\cup$:
$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \upsilon, \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon \cup \vartheta \; ; \; \Upsilon}$$

CAA.9: left $\cup$:
$$\frac{\upsilon \cup \vartheta, \Theta \; ; \; \upsilon \; \Rightarrow ; \; \Upsilon \qquad \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\upsilon \cup \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

connectives of type $\upsilon \times \upsilon \to \vartheta$

(*) *CCA.1: right* $\supset$:
$$\frac{\Theta \; ; \; \upsilon_1 \; \Rightarrow \; ; \; \Upsilon, \upsilon_2}{\Theta \; ; \; \Rightarrow \; \upsilon_1 \supset \upsilon_2 \; ; \; \Upsilon}$$

CCA.2: left $\supset$:
$$\frac{\Theta ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_1 \qquad \upsilon_1 \supset \upsilon_2, \Theta \; ; \upsilon_2 \; \Rightarrow \; ; \; \Upsilon}{\upsilon_1 \supset \upsilon_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon ; \; ; \; \Upsilon}$$

(*) *CCA.3: right* $\cap$:
$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \upsilon_0, \Upsilon \qquad \Theta \; ; \; \Rightarrow \; ; \; \upsilon_1, \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon_0 \cap \upsilon_1 \; ; \; \Upsilon}$$

CCA.4,5: left $\cap$:
$$\frac{\Theta, \upsilon_0 \cap \upsilon_1 \; ; \; \upsilon_i \; \Rightarrow \; ; \; \Upsilon}{\Theta, \upsilon_0 \cap \upsilon_1 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$
for $i = 0, 1$.

(*) *CCA.6, 7: right* $\cup$:
$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \upsilon_i, \Upsilon}{\Theta \; ; \; \Rightarrow \; \upsilon_1 \cup \upsilon_2 \; ; \; \Upsilon}$$
for $i = 0, 1$.

CCA.8: left $\cup$:
$$\frac{\upsilon_0 \cup \upsilon_1, \Theta \; ; \; \upsilon_0 \; \Rightarrow \; ; \; \Upsilon \quad \upsilon_0 \cup \upsilon_1, \Theta \; ; \; \upsilon_1 \; \Rightarrow \; ; \; \Upsilon}{\upsilon_0 \cup \upsilon_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

TABLE 14. Mixed-type assertive logical rules

MIXED-TYPE CONJECTURAL LOGICAL RULES

connective of type $\vartheta \times \upsilon \to \upsilon$

ACC.1: right $\succ$:
$$\frac{\Theta, \vartheta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta \succ \upsilon}$$

(*) *ACC.2: left $\succ$:*
$$\frac{\Theta ; \; \Rightarrow \vartheta \; ; \; \Upsilon \qquad \Theta ; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta \succ \upsilon \; \Rightarrow \; ; \; \Upsilon}$$

ACC.3: right $\curlywedge$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon, \vartheta \curlywedge \upsilon \qquad \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta \curlywedge \upsilon}$$

(*) *ACC.4: left $\curlywedge$:*
$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta \curlywedge \upsilon \; \Rightarrow \; ; \; \Upsilon}$$

(*) *ACC.5: left $\curlywedge$:*
$$\frac{\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta \curlywedge \upsilon \; \Rightarrow \; ; \; \Upsilon}$$

ACC.6: right $\curlyvee$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \upsilon, \Upsilon, \vartheta \curlyvee \upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta \curlyvee \upsilon}$$

ACC.7: right $\curlyvee$:
$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \upsilon, \Upsilon, \vartheta \curlyvee \upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta \curlyvee \upsilon}$$

(*)*ACC.8: left $\curlyvee$:*
$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon \qquad \Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta \curlyvee \upsilon \; \Rightarrow \; ; \; \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \upsilon$

CAC.1: right $\succ$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon, \upsilon \succ \vartheta}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \succ \vartheta}$$

CAC.2: right $\succ$:
$$\frac{\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon, \upsilon \succ \vartheta}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \succ \vartheta}$$

(*) *CAC.3: left $\succ$:*
$$\frac{\Theta ; \; \Rightarrow \; ; \; \Upsilon, \upsilon \qquad \vartheta, \Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon \succ \vartheta \; \Rightarrow \; ; \; \Upsilon}$$

CAC.4: right $\curlywedge$:
$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon, \qquad \Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon, \upsilon \curlywedge \vartheta}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \curlywedge \vartheta}$$

(*) *CAC.5: left $\curlywedge$:*
$$\frac{\Theta ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon \curlywedge \vartheta \; \Rightarrow \; ; \; \Upsilon}$$

(*)*CAC.6: left $\curlywedge$:*
$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon \curlywedge \vartheta \; \Rightarrow \; ; \; \Upsilon}$$

CAC.7: right $\curlyvee$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \curlyvee \vartheta}$$

CAC.8: right $\curlyvee$:
$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \curlyvee \vartheta}$$

(*) *CAC.9: left $\curlyvee$:*
$$\frac{\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon \qquad \Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon \curlyvee \vartheta \; \Rightarrow \; ; \; \Upsilon}$$

connective of type $\vartheta \times \vartheta \to \upsilon$

AAC.1: right $\succ$:
$$\frac{\Theta, \vartheta_1 \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta_1 \succ \vartheta_2}$$

AAC.2: right $\succ$:
$$\frac{\Theta, \vartheta_1 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta_1 \succ \vartheta_2}$$

(*) *AAC.3: left $\succ$:*
$$\frac{\Theta ; \; \Rightarrow \vartheta_1 \; ; \; \Upsilon \qquad \vartheta_2, \Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta_1 \succ \vartheta_2 \; \Rightarrow \; ; \; \Upsilon}$$

AAC.4: right $\curlywedge$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon, \vartheta_0 \curlywedge \vartheta_1 \qquad \Theta \; ; \; \Rightarrow \; \vartheta_1 ; \; \Upsilon, \vartheta_0 \curlywedge \vartheta_1}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta_0 \curlywedge \vartheta_1}$$

(*) *AAC.5, 6: left $\curlywedge$:*
$$\frac{\Theta, \vartheta_i \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta_0 \curlywedge \vartheta_i \; \Rightarrow \; ; \; \Upsilon}$$
for $i = 0, 1$.

AAC.7,8: right $\curlyvee$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_i \; ; \; \Upsilon, \vartheta_0 \curlyvee \vartheta_1,}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \vartheta_0 \curlyvee \vartheta_1}$$
for $i = 0, 1$.

(*) *AAC.9: left $\curlyvee$:*
$$\frac{\Theta, \vartheta_0 \; ; \; \Rightarrow \; ; \; \Upsilon \qquad \Theta, \vartheta_1 \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \vartheta_1 \curlyvee \vartheta_2 \; \Rightarrow \; ; \; \Upsilon}$$

TABLE 15. Mixed-type conjectural logical rules