A Kripke–style semantics for the intuitionistic logic of pragmatics **ILP**

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Abstract

We give a Kripke–style semantics for the intuitionistic logic of pragmatics **ILP** and show completeness with respect to this semantics. In order to prove the completeness theorem we give a decision procedure that given an **ILP**–sequent S either returns a cut–free derivation of S or constructs a finite counter–model if S is not provable. Thus we have the finite model property and also a new proof that the cut rule is eliminable in **ILP**.

1 Introduction

1.1 Principles of the logic of pragmatics

1.1.1 The philosophical context

The project of a *formal pragmatics*, introduced by Dalla Pozza and Garola [6], arises in philosophical logic and aims at resolving the conflict between classical and intuitionistic logic by presenting the latter as an *extension* or *integration* of the former, rather than an *alternative* to it. Dalla Pozza and Garola take off from Frege's distinction between *propositions* and *judgements*: the former are entities which can be *true* or *false*; the latter are acts which have a propositional content but are *justified* or *unjustified* (in given contexts) rather than true or false. In Frege's notation [8], a judgement $\vdash \alpha$ expresses the act of asserting that α is true, while a proposition $-\alpha$ expresses the content of the act of assertion. Frege noticed that as a consequence

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1. there can be no nested occurrences of $\vdash$, and

2. truth–functional connectives don't apply to expressions of judgements.

But what counts as a justification of an act of judgement $\vdash\alpha$? If α is a non–mathematical sentence then $\vdash\alpha$ is justified by some kind of evidence that α is true. If α is a mathematical sentence then $\vdash\alpha$ is justified by a *proof* that α is true. Furthermore, a necessary and sufficient condition to infer an assertion $\vdash\alpha_2$ from an assertion $\vdash\alpha_1$ is that there exists a method which transforms a proof [evidence] of α_1 into a proof [evidence] of α_2. The existence of such a method justifies the implication $\vdash\alpha_1 \supset \vdash\alpha_2$, which is therefore *intuitionistic* implication. From these considerations Dalla Pozza and Garola argue that intuitionistic logic, as a *logic of judgements*, has a different subject matter than classical logic, as the *logic of propositions*.

Why is the specific content of intuitionistic logic characterized as *formal pragmatics*? The operation by which a proposition is transformed into an act of assertion may be regarded as an *illocutionary operator* in the sense of Austin [1]; by the above considerations, intuitionistic logic is about the logical properties of such an operator. In [5] the framework has been extended to a logic of *judgements* and *norms*: here the act setting the *obligation* to bring about (the state of affairs described by the proposition) α is expressed by $\circ\!\!-\alpha$; the resulting formal system may be regarded as an axiomatization of the logical properties of the illocutionary operator of obligation.

It could be objected that every act of assertion or of obligation is performed by a subject in a given state of information and situation; indeed one of the main functions of pragmatics in linguistic theory is to fix the context for a semantic interpretation of an expression. In general norms are valid within a *normative system* N: this dependance could be expressed by the notation $\circ\!\!-_N\,\alpha$. Although a fully developed formal pragmatics should characterize the properties of *relativized judgements and obligations*, it is legitimate to develop first a theory of the properties of the operators which hold *abstracting from particular subjects and situations*: in particular $\vdash\alpha$ expresses an *impersonal* act of judgement and $\circ\!\!-\alpha$ expresses a norm which is valid in every rational normative system.

A more subtle issue can be raised about the status of molecular formulas and pragmatic connectives. If $\vdash\alpha_1$ and $\vdash\alpha_2$ *express* an illocutionary act, does the conjunction $\vdash\alpha_1 \cap \vdash\alpha_2$ express a *composite act* or a *relation* between acts? The paper [6] does not answer to this question; however, since a relation between acts may be regarded as a binary function which returns a *truth-value*, the interpretation of molecular expressions as acts seems inescapable. We cannot further discuss the issue here.

Therefore in [6, 5] the language $\mathcal{L}^P$ of formal pragmatics and its semantics are as follows:

1. Radical formulas are of the form
   $\alpha := p \mid \neg\, \alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2$
   where p ranges over an infinite set of propositional letters and $\neg$, $\wedge$, $\vee$ and $\rightarrow$ are the usual classical connectives; radical formulas are interpreted according to Tarski's semantics.

2. Elementary formulas are of the form
   $\eta := \vdash \alpha \mid \multimapinv \alpha \mid \bigwedge$
   where $\vdash$ and $\multimapinv$ are signs of illocutionary force and the symbol $\bigwedge$ stands for an illocutionary act which is never justified.

3. Sentential formulas are of the form
   $\delta := \eta \mid \delta_1 \supset \delta_2 \mid \delta_1 \cap \delta_2 \mid \delta_1 \cup \delta_2$
   where $\supset$, $\cap$ and $\cup$ are the pragmatic connectives interpreted according to Heyting's semantics of proofs.

What are the relations between the different layers of the formal system for pragmatics? How does one prove that an intuitionistic pragmatics is *compatible* with classical semantics? A partial answer is given by Gödel [9], McKinsey and Tarski [14] interpretation of intuitionistic logic into the classical modal system **S4** in [6]. Namely, one *extends* the radical part of the language with the following grammar:

$$\beta := p \mid \neg\beta \mid \beta_1 \wedge \beta_2 \mid \beta_1 \vee \beta_2 \mid \beta_1 \rightarrow \beta_2 \mid \bot \mid \Box\beta$$

which is interpreted by Kripke's semantics for **S4** (on preordered frames). Then one sets the modal translation of the sentential formulas as follows:

$$\begin{aligned} \textstyle\bigwedge^\star &= \bot \\ (\vdash \alpha)^\star &= \Box\ \alpha \\ (\delta_1 \supset \delta_2)^\star &= \Box(\delta_1^\star \rightarrow \delta_2^\star) \\ (\delta_1 \cap \delta_2)^\star &= \delta_1^\star \wedge \delta_2^\star \\ (\delta_1 \cup \delta_2)^\star &= \delta_1^\star \vee \delta_2^\star \end{aligned}$$

When the elementary formulas include an operator of obligation [5], then the radical part is also extended with a modal operator "$\circ$", which is given Kripke's sematics for **KD** (on frames without terminal points). As the deontic expressions $\multimapinv \alpha$ belongs to an *intuitionistic* sentential language, the modal translation must be as follows:

$$(\multimapinv \alpha)^\star = \Box \circ \alpha$$

According to [5], the modal interpretation of the elementary formulas establishes a relation between the *expressive use* and the *descriptive use* of the operators of illocutionary force and thus of the whole pragmatic language: a Tarskian semantics can be assigned to the pragmatic language only in its descriptive use, not in its expressive use. The existence of a correspondence between the expressive use and the descriptive use of the pragmatic language does not justify the claim that the latter *fully represents* the former; in particular it does not rule out the possibility that the expressive use may be more adequately represented by other mathematical constructions (such as computational, categorical, game-theoretical interpretations of intuitionistic logic) and that Kripke's semantics may be a kind of *abstract interpretation* of the whole pragmatic system.

The *intuitionistic fragment* of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary formulas to those with atomic radical only. In addition to the projection of the whole pragmatic system down to the semantic level, given by the modal interpretation, one would also expect an inverse action of the *semantic* level on the *pragmatic* level, which would certainly provide a stronger case for the compatibility claim by Dalla Pozza and Garola. Some indications for this line of research are in [6, 5], but we will not consider them here and limit ourselves to the intuitionistic fragment.

1.1.2 The pragmatic system as a framework for AI

In [2] the system of formal pragmatics has been given a Gentzen-style sequent calculus presentation and also developed to provide a mathematically principled approach to Artificial Intelligence applications.

Consider the problem of axiomatizing the foundations of laws, or more generally, of representing legal reasoning within a normative system. A sequent calculus for intuitionistic logic extended with an intuitionistic deontic modality would not suffice to formalize actual normative systems. For instance, according to Kelsen's theory [11] a typical norm occurring in a normative system could be formalized by an axiom[1]

$$\mathcal{O}\!\!-(\neg\alpha_1), \vdash \alpha_1 \Rightarrow \mathcal{O}\!\!-\alpha_2 \qquad (\dagger)$$

where α_1 represents an illicit act and α_2 an appropriate sanction. The axiom expresses the obligation for a judge to inflict the sanction α_2 in case conclusive evidence has been gathered that the illicit act α_1 has been performed.

[1]The formalization $\vdash \alpha_1 \Rightarrow \mathcal{O}\!\!-\alpha_2$ would perhaps be more faithful to Kelsen's theory, the obligation being adressed only to the judge.

The example (†) shows that, firstly, in order to deal with normative systems we need a proof system for *mixed* assertive and prescriptive illocutionary forces and, secondly, one that could express non–trivial relations between norms and assertions. For instance, we would like to express the principle that if $\vdash \alpha_2$ is a consequence of $\vdash \alpha_1$, then the prohibition of α_2 entails the prohibition of α_1.

But for which notion of consequence does such a principle hold? As it is argued in [2] only a *causal relation* validates it. Indeed, the axioms

$$\circ\!\!-\alpha_1, \vdash(\alpha_1 \to \alpha_2) \Rightarrow \circ\!\!-\alpha_2$$
$$\circ\!\!-\alpha_1, \vdash \alpha_1 \supset \vdash \alpha_2 \Rightarrow \circ\!\!-\alpha_2$$

are counterintuitive if we consider for example $\alpha_1 =$ "Hitler is dead" and $\alpha_2 =$ "The Titanic has sunk". On the other hand, writing $\Supset$ for *causal implication*, consider the axioms

$$\circ\!\!-\alpha_1, \vdash \alpha_1 \Supset \vdash \alpha_2 \Rightarrow \circ\!\!-\alpha_2$$
$$\circ\!\!-(\neg\alpha_2), \vdash \alpha_1 \Supset \vdash \alpha_2 \Rightarrow \circ\!\!-(\neg\alpha_1)$$

where the sentence α_1 is "The gun is shot" and α_2 is "The person is killed". These axioms become plausible if the method trasforming a justification of $\vdash \alpha_1$ into a justification $\vdash \alpha_2$ is a causal relation.

A formalization of the connective of causal implication is itself a difficult and controversial issue, let alone the production of a mixed system involving causality. The only possible choice at this stage was to axiomatize some features of causal implication, as they result from an analysis of very specific uses, namely, *extensionality* and *relevance*. According to the interpretation in [2], $\vdash \alpha_1 \Supset \vdash \alpha_2$ is justified if and only if there is a scientific law $\forall x.A(x) \to B(x)$ such that $\alpha_1 = A(t)$ and $\alpha_2 = B(t)$ for some term t. This interpretation doesn't support the use of weakening. Furthermore this interpretation is extensional, i.e. the correctness of $\vdash \alpha_1 \Supset \vdash \alpha_2$ depends only on the existence of a scientific law $\forall x.A(x) \to B(x)$ and the justification of $\vdash \alpha_1$ where $\alpha_1 = A(t)$ and $\alpha_2 = B(t)$. In this strictly extensional interpretation we don't fully understand what is means to say that $(\vdash \alpha_1 \Supset \vdash \alpha_2) \Supset \vdash \alpha_3$ and thus causal implication has its intended meaning only in positive occurrences. Thus we consider only formulas in which the occurrences of causal implication are of the form

$$\vdash \alpha_1 \Supset (\ldots \Supset (\vdash \alpha_n \Supset \vdash \alpha) \ldots)$$

Futhermore we suppose that the interpretation is symmetric, i.e. if $\cong$ denotes interderivability then

$$\vdash \alpha_1 \Supset (\vdash \alpha_2 \Supset \vdash \alpha_3) \cong \vdash \alpha_2 \Supset (\vdash \alpha_1 \Supset \vdash \alpha_3)$$

The paper [2] presents a mixed relevant and intuitionistic sequent calculus **ILP** for the intuitionistic fragment of the pragmatic system with operators for assertion, obligation and causal implication and gives a syntactic proof of consistency of the system, as a corollary of the cut–elimination theorem.

Adding the pragmatic connective of causal implication to the language $\mathcal{L}^P$ raises several issues. Is there a distinction between *expressive* and *descriptive* use of causal implication? Can we say that $\vdash \alpha_1 \mathbin{\supset\!\!\!\!\circ} \vdash \alpha_2$ expresses a *different act* from $\vdash \alpha_1 \supset \vdash \alpha_2$, depending on whether the method leading from a justification of $\vdash \alpha_1$ to the justification of $\vdash \alpha_2$ is causal or not? What Kripke semantics shall we assign to the descriptive use of causal implication? We cannot discuss these philosophical issues here. To interpret causal implication we shall choose the *monoidal semantics* for relevant implication originally due to Urquhart [17], but we are aware that this choice is not the only one possible. Such a choice has the consequence of bringing the mixed system for formal pragmatics close to mixed systems for intuitionistic and linear implication, such as Girard's **LU** [10] and Pym and O'Hearn's **BI** [15] (perhaps closer to the former than to the latter). The restrictions on the uses of causal implication clearly makes our system a very small fragment of such general mixed systems. For this reason and others, comparing our approach with theirs is clearly beyond the scope of this paper.

1.2 Basic ideas of the present work

The aim of this paper is to give a Kripke–style semantics to the sequent calculus **ILP** which formalizes the principles of the intuitionistic logic of pragmatics explained above.

As explained above, the Kripke-style semantics is not given to the pragmatic operators of $\mathcal{L}^P$ in their expressive use, but to their descriptive counterpart. Thus with respect to the presentation in [2] we extend the radical formulas at the semantic level with the new connectives $\Box$ for the **S4** modality, $\circ$ for the **KD** modality, $\multimap$ for relevant implication and $\mathbf{1}$ for absurdity.

As already suggested in [13] for intuitionistic logic the completeness of the intuitionistic logic of pragmatics is obtained by the completeness of the interpreted formulas with respect to the given Kripke–style semantics and by the existence of the interpretation.

In order to show the completeness theorem we give a proof–search procedure and a procedure that constructs finite counter–models if the sequent is not provable. For the proof–search procedure we consider only proofs in particular form, called canonical proofs: the advantage of canonical proofs is that we can use one of the well known proof–search procedures for intuitionistic propositional logic which must only be extended to deal with the connectives for assertability, obligation and causal implication. Also the procedure that constructs counter–models for unprovable sequents extends procedures for intuitionistic propositional logic. As a consequence of this the finite model property follows from the finiteness of the counter–models for propositional intuitionistic logic. From the fact that the proof–search procedure considers only canonical proofs, which are in particular cut–free proofs, it follows that we can give a new proof of cut elimination.

The paper is structured in the following way. In section 2 we introduce the pragmatic language $\mathcal{L}^P$ and the sequent calculus **ILP** given in [2]. Furthermore, we extend the radical part of the language and define an interpretation of sentential formulas in the extended radical part. Then, in section 3 we define the Kripke–style semantics for the formulas of the extended radical part which is based on the Kripke semantics of modal [12, 3] and relevant logic [17]. Furthermore, we show the soundness theorem. Finally, in section 4 we give a proof–search procedure with respect to proofs in canonical form and a procedure to construct counter–models. By this way we show the completeness theorem, decidability of **ILP**, finite model property and cut elimination.

2 Intuitionistic logic of pragmatics

2.1 Pragmatic language

Definition 2.1 (pragmatic language) *The formulas of the pragmatic language $\mathcal{L}^P$ are defined inductively by the following rules.*

1. *Radical formulas are of the form*
   $\alpha := p \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2$
   where p ranges over an infinite set of propositional letters and $\neg$, $\wedge$, $\vee$ and $\rightarrow$ are the usual classical connectives.

2. *Elementary formulas are of the form*
   $\eta_0 := \vdash \alpha$ *and*
   $\eta := \bigwedge \mid \circ\!\!-\alpha$
   where $\vdash$ and $\circ\!\!-$ are signs of illocutionary force and the symbol $\bigwedge$ stands for an illocutionary act which is never justified.

3. *Sentential formulas are of the form*
   $\zeta := \eta_0 \mid \eta_0 \supset_c \zeta$ *and*
   $\delta := \eta \mid \zeta \mid \delta_1 \supset \delta_2 \mid \delta_1 \cap \delta_2 \mid \delta_1 \cup \delta_2$
   where the symbol $\supset_c$ stands for causal implication and $\supset$, $\cap$ and $\cup$ are the pragmatic connectives.

$$\bigwedge^\star = \bot \qquad (\vdash \alpha)^\star = \Box\alpha \qquad ({\circ\!\!-}\,\alpha)^\star = \Box\circ\alpha$$

$$(\vdash \alpha \mathrel{\supset\!\!\!\!\!{\scriptscriptstyle c}\;} \zeta)^\star = (\vdash \alpha)^\star \multimap \zeta^\star \qquad (\delta_1 \supset \delta_2)^\star = \Box(\delta_1^\star \to \delta_2^\star)$$

$$(\delta_1 \cap \delta_2)^\star = \delta_1^\star \wedge \delta_2^\star \qquad (\delta_1 \cup \delta_2)^\star = \delta_1^\star \vee \delta_2^\star$$

Figure 1: Extended modal interpretation

The intuitionistic fragment of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary formulas to those with atomic radical only.

We consider an extended version $\mathcal{L}_e^P$ of the pragmatic language in which we extend the radical part of the language with the modal operators $\Box$ and $\circ$, the connective $\multimap$ and the logical constant $\bot$. In order to model relevant logic we need also the connective $\otimes$ and the logical constant $\mathbf{1}$ where $\otimes$ and $\mathbf{1}$ stand for tensor product and identity of the tensor product, respectively. Thus the radical formulas of $\mathcal{L}_e^P$ which are denoted with β are of the following form.

$$\beta := p \mid \neg\beta \mid \beta_1 \wedge \beta_2 \mid \beta_1 \vee \beta_2 \mid \beta_1 \to \beta_2 \mid \bot \mid \Box\beta \mid \circ\beta \mid \beta_1 \multimap \beta_2 \mid \beta_1 \otimes \beta_2 \mid \mathbf{1}$$

We project the formulas of the intuitionistic fragment of the pragmatic language $\mathcal{L}^P$ into the extended radical part of the extended pragmatic language $\mathcal{L}_e^P$ as represented in figure 1. This is extends the Gödel [9], McKinsey and Tarski [14] modal interpretation of intuitionistic propositional logic with the **KD** modality $\circ$ for norms and the relevant implication $\multimap$ for causality.

2.2 Sequent calculus

The sequent calculus **ILP** formalizes derivations of sentential formulas of the intuitionistic fragment of the pragmatic language $\mathcal{L}^P$. Let Γ and Δ denote finite multisets of sentential formulas. We write $\mathbf{Z}$ for the non empty multiset $\zeta_1, \ldots, \zeta_n$, $\vdash\mathbf{A}$ for the non empty multiset $\vdash\alpha_1, \ldots, \vdash\alpha_m$ and ${\circ\!\!-}\mathbf{A}$ for the non empty multiset ${\circ\!\!-}\,\alpha_1, \ldots, {\circ\!\!-}\,\alpha_m$. The rules of the sequent calulus **ILP** are given in figure 2 (where $i = 1, 2$). Note that the antecedent of the sequents has two areas, the relevant area on the left and the pragmatic area on the right of the semicolon. As a consequence of this there are only positive occurrences of causal implications and causal implications are not derivable from an absurdity.

Identity rules

$$\frac{}{\delta; \Rightarrow \delta}\,Ax$$

$$\frac{\Gamma; \Delta \Rightarrow \delta \quad \delta, \Gamma'; \Delta \Rightarrow \delta'}{\Gamma, \Gamma'; \Delta \Rightarrow \delta'}\,Cut_r \qquad \frac{; \Delta \Rightarrow \delta \quad \Gamma; \delta, \Delta \Rightarrow \delta'}{\Gamma; \Delta \Rightarrow \delta'}\,Cut_p$$

Structural rules

$$\frac{\delta, \delta, \Gamma; \Delta \Rightarrow \delta'}{\delta, \Gamma; \Delta \Rightarrow \delta'}\,c_r \qquad \frac{\Gamma; \delta, \delta, \Delta \Rightarrow \delta'}{\Gamma; \delta, \Delta \Rightarrow \delta'}\,c_p$$

$$\frac{\Gamma, \delta; \Delta \Rightarrow \delta'}{\Gamma; \delta, \Delta \Rightarrow \delta'}\,p \qquad \frac{\Gamma; \Delta \Rightarrow \delta'}{\Gamma; \delta, \Delta \Rightarrow \delta'}\,w$$

Logical rules

$$\frac{}{\Gamma; \bigwedge, \Delta \Rightarrow \delta}\,\bigwedge Ax$$

$$\frac{\Gamma, \vdash \alpha; \Rightarrow \zeta}{\Gamma; \Rightarrow \vdash \alpha \mathbin{{\scriptstyle c}\!\!\!\supset} \zeta}\,R \mathbin{{\scriptstyle c}\!\!\!\supset} \qquad \frac{\Gamma; \Rightarrow \vdash \alpha \quad \zeta, \Gamma'; \Delta \Rightarrow \delta}{\vdash \alpha \mathbin{{\scriptstyle c}\!\!\!\supset} \zeta, \Gamma, \Gamma'; \Delta \Rightarrow \delta}\,L \mathbin{{\scriptstyle c}\!\!\!\supset}$$

$$\frac{\Gamma; \Delta, \delta_1 \Rightarrow \delta_2}{\Gamma; \Delta \Rightarrow \delta_1 \supset \delta_2}\,R \supset \qquad \frac{; \Delta \Rightarrow \delta_1 \quad \Gamma; \delta_2, \Delta \Rightarrow \delta}{\Gamma; \delta_1 \supset \delta_2, \Delta \Rightarrow \delta}\,L \supset$$

$$\frac{\Gamma; \Delta \Rightarrow \delta_1 \quad \Gamma; \Delta \Rightarrow \delta_2}{\Gamma; \Delta \Rightarrow \delta_1 \cap \delta_2}\,R \cap \qquad \frac{\Gamma; \delta_i, \Delta \Rightarrow \delta}{\Gamma; \delta_1 \cap \delta_2, \Delta \Rightarrow \delta}\,L \cap$$

$$\frac{\Gamma; \Delta \Rightarrow \delta_i}{\Gamma; \Delta \Rightarrow \delta_1 \cup \delta_2}\,R \cup \qquad \frac{\Gamma; \delta_1, \Delta \Rightarrow \delta \quad \Gamma; \delta_2, \Delta \Rightarrow \delta}{\Gamma; \delta_1 \cup \delta_2, \Delta \Rightarrow \delta}\,L \cup$$

Mixed rules

$$\frac{\vdash \mathbf{A}, \mathbf{Z}; \Rightarrow \vdash \alpha}{\circ\!\!- \mathbf{A}, \mathbf{Z}; \Rightarrow \circ\!\!- \alpha}\,\mathbin{{\scriptstyle c}\!\!\!\supset} / \circ\!\!-$$

Figure 2: The sequent calculus **ILP**

We specify now briefly some useful properties of the sequent calculus **ILP** which represents the basic ideas underlying the decision procedure given in section 4.1. Note that we consider the proof of a sequent $\Gamma; \Delta \Rightarrow \delta$ as a tree where the sequents form the nodes and the instances of rules induce the edges between the nodes. Furthermore, we suppose that the final sequent of the proof is the root of the tree.

Lemma 2.2 *A sequent $\Gamma; \Delta \Rightarrow \delta$ is provable without cuts in* **ILP** *if and only if there exists a proof of it such that all occurrences of the $L \Supset$ rule have as left premise an instance of the logical axiom Ax.*

Proof Note that in the left subtree there can occur only instances of $L \Supset$ or c_r rules. Suppose that the left premise of an $L \Supset$ rule ($\dagger$) is the conclusion of an $L \Supset$ rule. We can permute the order of the rules in the following way.

$$\dfrac{\dfrac{\Gamma_1; \Rightarrow \vdash \alpha' \quad \zeta', \Gamma_2; \Rightarrow \vdash \alpha}{\vdash \alpha' \Supset \zeta', \Gamma_1, \Gamma_2; \Rightarrow \vdash \alpha} \qquad \zeta, \Gamma; \Delta \Rightarrow \delta}{\vdash \alpha \Supset \zeta, \vdash \alpha' \Supset \zeta', \Gamma_1, \Gamma_2, \Gamma; \Delta \Rightarrow \delta}\,\dagger \quad \rightsquigarrow \quad \dfrac{\Gamma_1; \Rightarrow \vdash \alpha' \qquad \dfrac{\zeta', \Gamma_2; \Rightarrow \vdash \alpha \quad \zeta, \Gamma; \Delta \Rightarrow \delta}{\zeta', \vdash \alpha \Supset \zeta, \Gamma_2, \Gamma; \Delta \Rightarrow \delta}\,\dagger}{\vdash \alpha \Supset \zeta, \vdash \alpha' \Supset \zeta', \Gamma_1, \Gamma_2, \Gamma; \Delta \Rightarrow \delta}$$

If we measure the weight of an $L \Supset$ rule ($\dagger$) by the number of nodes in the left subtree then the permutation decreases the weight of the rule. The same argument holds if the left premise of an $L \Supset$ rule ($\dagger$) is the conclusion of an c_r rule. Thus we can show the hypothesis by induction on the weight of an $L \Supset$ rule. □

Definition 2.3 (quasi–canonical proof) *We say that a cut–free proof is in quasi–canonical form if every path from the leaves to the root can be split into three subpaths such that the following conditions are satisfied.*

1. *The first subpath is induced only by instances of the c_r, $R \Supset$, $L \Supset$ and $\Supset/\multimapinv$ rules.*

2. *The second subpath is induced only by instances of the p rule.*

3. *The third subpath is induced only by instances of the c_p, w, $R \supset$, $L \supset$, $R \cap$, $L \cap$, $R \cup$ and $L \cup$ rules.*

Note that for quasi–canonical proofs we have that the pragmatic area of occurrences of the c_r and $L \Supset$ rules is empty. Furthermore, for proofs in **ILP** we have that in the same branch there can never be an instance of the $R \Supset$ and an instance of the $\Supset/\multimapinv$ rule.

Definition 2.4 (canonical proof) *We say that a proof in quasi–canonical form is in canonical form if in every path from the leaves to the root all the instances of the c_r and $L \supset$ rules occur before the instances of the $R \supset$ rule or the instance of the $\supset / \multimapinv$ rule.*

It is easy to see that every cut–free proof in **ILP** can be transformed in a proof in quasi–canonical form by permuting the order of the rules. We can also show that every proof in quasi–canonical form can be transformed in a proof in canonical form by permuting the order of the rules. As a consequence we have the following

Lemma 2.5 *A sequent $\Gamma; \Delta \Rightarrow \delta$ is provable without cuts in* **ILP** *if and only if there exists a proof of it in canonical form.*

3 Kripke–style semantics

3.1 Kripke models

We use the modal interpretation of sentential formulas given in section 2.1 and give a Kripke–style semantic to the extended radical part of the extended pragmatic language $\mathcal{L}^P_e$ which combines and extends the Kripke semantics of modal [12, 3] and relevant logic [17].

Definition 3.1 (Kripke model) *A Kripke frame is a quadruple $M = (\mathcal{W}, \mathcal{V}, \cdot, \prec)$ such that the following properties are satisfied.*

1. *$\mathcal{W} = (W, \cdot, 1, \sqsubseteq)$ is a preordered commutative and idempotent monoid.*
2. *$\mathcal{V} = (V, \cdot)$ is a is a commutative and idempotent semigroup.*
3. *The operation $\cdot : W \times V \to V$ and the relation $\prec \subseteq W \times V$ are such that*
   (a) *for all $w \in W$ there exists $v \in V$ such that $w \prec v$;*
   (b) *for all $w, w' \in W$ and $v \in V$ such that $w \sqsubseteq w'$ and $w' \prec v$ we have that $w \prec v$;*
   (c) *for all $w_1, w_2 \in W$ and $v \in V$ such that $w_1 \cdot w_2 \prec v$ there exists $v_1, v_2 \in V$ such that $w_1 \prec v_1$, $w_2 \prec v_2$ and $v = v_1 \cdot v_2$;*
   (d) *for all $w_1, w_2 \in W$ and $v \in V$ such that $w_1 \cdot w_2 \prec v$ there exists $v' \in V$ such that $w_1 \prec v'$ and $w_2 \cdot v' = v$.*

1. $w \not\Vdash \bot$ always
2. $w \Vdash \mathbf{1}$ if and only if $1 \sqsubseteq w$
3. $w \Vdash \neg\beta$ if and only if $w \not\Vdash \beta$
4. $w \Vdash \beta_1 \wedge \beta_2$ if and only if $w \Vdash \beta_1$ and $w \Vdash \beta_2$
5. $w \Vdash \beta_1 \vee \beta_2$ if and only if $w \Vdash \beta_1$ or $w \Vdash \beta_2$
6. $w \Vdash \beta_1 \rightarrow \beta_2$ if and only if $w \not\Vdash \beta_1$ or $w \Vdash \beta_2$
7. $w \Vdash \Box\beta$ if and only if $w' \Vdash \beta$ for all $w' \in W$ such that $w \sqsubseteq w'$
8. $w \Vdash \circ\beta$ if and only if $v \Vdash \beta$ for all $v \in V$ such that $w \prec v$
9. $w \Vdash \beta_1 \otimes \beta_2$ if and only if there exists $w_1, w_2 \in W$ such that $w_1 \Vdash \beta_1$, $w_2 \Vdash \beta_2$ and $w_1 \cdot w_2 \sqsubseteq w$
10. $w \Vdash \beta_1 \multimap \beta_2$ if and only if
    (a) $w \cdot w' \Vdash \beta_2$ for all $w' \in W$ such that $w' \Vdash \beta_1$
    (b) $w \cdot v \Vdash \beta_2$ for all $v \in V$ such that $v \Vdash \beta_1$
11. $v \Vdash \Box\beta$ if and only if $v \Vdash \beta$
12. $v \Vdash \beta_1 \otimes \beta_2$ if and only if there exists $v_1, v_2 \in V$ such that $v_1 \Vdash \beta_1$, $v_2 \Vdash \beta_2$ and $v_1 \cdot v_2 = v$
13. $v \Vdash \beta_1 \multimap \beta_2$ if and only if
    (a) $w \cdot v \Vdash \beta_2$ for all $w \in W$ such that $w \Vdash \beta_1$
    (b) $v \cdot v' \Vdash \beta_2$ for all $v' \in V$ such that $v' \Vdash \beta_1$

Figure 3: The forcing relation $\Vdash$ for extended radical formulas

A Kripke frame is finite if both sets W and V are finite. A Kripke model $\mathcal{M}$ is a pair $(M, \Vdash)$ where $\Vdash \subseteq (W \cup V) \times At$ is the forcing relation for atomic formulas. The forcing relation is defined inductively for extended radical formulas by the clauses given in figure 3. A Kripke model $\mathcal{M}$ is finite if its Kripke frame is finite.

W is called the set of possible worlds and V the set of "virtuous" worlds. Note that the monoidal operation of $(W, \cdot, 1, \sqsubseteq)$ is increasing, i.e. for all $w_1, w_1', w_2, w_2' \in W$, if $w_1 \sqsubseteq w_1'$ and $w_2 \sqsubseteq w_2'$ then $w_1 \cdot w_2 \sqsubseteq w_1' \cdot w_2'$. Furthermore, note that a semigroup (monoid) with an commutative and idempotent operation is a semilattice. In particular we have that the preorder on the monoid $(M, \cdot, 1)$ is the one induced by the semilattice operation $\cdot$, i.e. $w \sqsubseteq w'$ if and only if $w \cdot w' = w'$. Anyway, we prefer to keep the notation of the above definition in order to distinguish between the semantics of relevant and of modal logic.

Suppose that a formula $\delta^\star$ is obtained by the extended modal interpretation of sentential formulas. By induction on the complexity of the formula $\delta^\star$ we can show that the property of Krikpe monotonicity holds.

Lemma 3.2 *For all $w, w' \in W$, if $w \Vdash \delta^\star$ and $w \sqsubseteq w'$ then $w' \Vdash \delta^\star$.*

Furthermore, as an immediate consequence of property (3b) of the definition of Kripke model we have the following

Lemma 3.3 *For all $w \in W$, $w \Vdash \Box\circ\beta$ if and only if $w \Vdash \circ\beta$.*

3.2 Validity and soundness

Suppose that $\Gamma = \delta_1, \ldots, \delta_n$ and $\Delta = \delta_1', \ldots, \delta_m'$. Let $\Gamma^\star$ and $\Delta^\star$ denote $\delta_1^\star \otimes \ldots \otimes \delta_n^\star$ and $(\delta_1')^\star \wedge \ldots \wedge (\delta_m')^\star \wedge \mathbf{1}$, respectively. A sequent $\Gamma; \Delta \Rightarrow \delta$ can then be seen as $\Gamma^\star \otimes \Delta^\star \Rightarrow \delta^\star$ i.e. the semicolon and the commas in the relevant area are interpreted as tensor products, whereas the commas in the pragmatic area are interpreted as classical conjunctions.

Definition 3.4 (validity) *A sequent $\Gamma; \Delta \Rightarrow \delta$ is valid if for all Kripke models $\mathcal{M}$ and for all $w \in W$, such that $w \Vdash \Gamma^\star \otimes \Delta^\star$, we have that $w \Vdash \delta^\star$. If the sequent $\Gamma; \Delta \Rightarrow \delta$ is valid then we write $\Gamma; \Delta \models \delta$.*

Theorem 3.5 (soundness) *If a sequent $\Gamma; \Delta \Rightarrow \delta$ is provable in* **ILP** *then $\Gamma; \Delta \models \delta$.*

Proof The proof is by induction on the structure of the proofs of **ILP**.

1. If the proof consists of a logical axiom Ax, then the property follows from the fact that by the definition of forcing there exists $w, w_1, w_2 \in W$, such that $w_1 \Vdash \delta^\star$, $w_2 \Vdash \mathbf{1}$ and $w_1 \cdot w_2 \sqsubseteq w$, and from the fact that $w_1 = w_1 \cdot 1 \sqsubseteq w_1 \cdot w_2$.

2. Suppose that the last rule of the proof is a Cut_r rule. The induction hypothesis is that

   (a) for all $w' \in W$, such that $w' \Vdash \Gamma^\star \otimes \Delta^\star$, we have that $w' \Vdash \delta^\star$,

   (b) for all $w'' \in W$, such that $w'' \Vdash \delta^\star \otimes (\Gamma')^\star \otimes \Delta^\star$, we have that $w'' \Vdash (\delta')^\star$.

   We have to show that for all $w \in W$, such that $w \Vdash \Gamma^\star \otimes (\Gamma')^\star \otimes \Delta^\star$, it holds that $w \Vdash (\delta')^\star$.

   By the definition of forcing there exists $w_1, w_2, w_3 \in W$ such that $w_1 \cdot w_2 \cdot w_3 \sqsubseteq w$, $w_1 \Vdash \Gamma^\star$, $w_2 \Vdash (\Gamma')^\star$ and $w_3 \Vdash \Delta^\star$. By the idempotence of the monoidal operation $w_1 \cdot w_2 \cdot w_3 = w_1 \cdot w_2 \cdot w_3 \cdot w_3$. Thus by the definition of forcing $w_1 \cdot w_3 \Vdash \Gamma^\star \otimes \Delta^\star$ and $w_2 \cdot w_3 \Vdash (\Gamma')^\star \otimes \Delta^\star$. By the first clause of the induction hypothesis $w_1 \cdot w_3 \Vdash \delta^\star$. Thus by the second clause of the induction hypothesis $w \Vdash (\delta')^\star$.

3. Suppose that the last rule of the proof is a p rule. The induction hypothesis is that for all $w' \in W$, such that $w' \Vdash \Gamma^\star \otimes \delta^\star \otimes \Delta^\star$, we have that $w' \Vdash (\delta')^\star$. We have to show that for all $w \in W$, such that $w \Vdash \Gamma^\star \otimes (\delta^\star \wedge \Delta^\star)$, it holds that $w \Vdash (\delta')^\star$.

   By the definition of forcing there exists $w_1, w_2 \in W$ such that $w_1 \cdot w_2 \sqsubseteq w$, $w_1 \Vdash \Gamma^\star$ and $w_2 \Vdash \delta^\star \wedge \Delta^\star$. By the definition of forcing $w_2 \Vdash \delta^\star$ and $w_2 \Vdash \Delta^\star$ and by the idempotence of the monoidal operation $w_1 \cdot w_2 = w_1 \cdot w_2 \cdot w_2$. Thus by the induction hypothesis $w \Vdash (\delta')^\star$.

4. Suppose that the last rule of the proof is a $R \supset$ rule. The induction hypothesis is that for all $w' \in W$, such that $w' \Vdash \Gamma^\star \otimes (\Box\alpha) \otimes \mathbf{1}$, we have that $w' \Vdash \zeta^\star$. We have to show that for all $w \in W$, such that $w \Vdash \Gamma^\star \otimes \mathbf{1}$, it holds that $w \Vdash \Box\alpha \multimap \zeta^\star$.

   Suppose that $w'' \in W$ is such that $w'' \Vdash \Box\alpha$. By the definition of forcing $w \cdot w'' \Vdash \Gamma^\star \otimes (\Box\alpha) \otimes \mathbf{1}$. By the induction hypothesis $w \cdot w'' \Vdash \zeta^\star$. Thus by the definition of forcing $w \Vdash \Box\alpha \multimap \zeta^\star$.

5. Suppose that the last rule of the proof is a $L \supset$ rule. The induction hypothesis is that

   (a) for all $w' \in W$, such that $w' \Vdash \Gamma^\star \otimes \mathbf{1}$, we have that $w' \Vdash \Box\alpha$,

(b) for all $w'' \in W$, such that $w'' \Vdash \zeta^\star \otimes (\Gamma')^\star \otimes \Delta^\star$, we have that $w'' \Vdash \delta^\star$.

We have to show that for all $w \in W$, such that $w \Vdash (\Box\alpha \multimap \zeta^\star) \otimes \Gamma^\star \otimes (\Gamma')^\star \otimes \Delta^\star$, it holds that $w \Vdash \delta^\star$.

By the definition of forcing there exists $w_1, w_2, w_3 \in W$ such that $w_1 \cdot w_2 \cdot w_3 \sqsubseteq w$, $w_1 \Vdash \Box\alpha \multimap \zeta^\star$, $w_2 \Vdash \Gamma^\star$ and $w_3 \Vdash (\Gamma')^\star \otimes \Delta^\star$. By the definition of Kripke model $w_2 = w_2 \cdot 1$ and thus by the definition of forcing $w_2 \Vdash \Gamma^\star \otimes \mathbf{1}$. By the first clause of the induction hypothesis $w_2 \Vdash \Box\alpha$ and by the definition of forcing $w_1 \cdot w_2 \Vdash \zeta^\star$. Thus by the second clause of the induction hypothesis $w \Vdash \delta^\star$.

6. Suppose that the last rule of the proof is a $\supset/\multimapinv$ rule. The induction hypothesis is that for all $w' \in W$, such that $w' \Vdash (\vdash\mathbf{A})^\star \otimes \mathbf{Z}^\star \otimes \mathbf{1}$, we have that $w' \Vdash \Box\alpha$. We have to show that for all $w \in W$, such that $w \Vdash (\multimapinv\mathbf{A})^\star \otimes \mathbf{Z}^\star \otimes \mathbf{1}$, it holds that $w \Vdash \Box\circ\alpha$.

   By the definition of forcing there exists $w_0, w'' \in W$ such that $w_0 \cdot w'' \sqsubseteq w$, $w_0 \Vdash (\multimapinv\mathbf{A})^\star$ and $w'' \Vdash \mathbf{Z}^\star \otimes \mathbf{1}$. By the induction hypothesis and by the definition of forcing $w'' \Vdash (\vdash\mathbf{A})^\star \multimap \Box\alpha$.

   By the definition of forcing there exists $w_1, \dots, w_n \in W$ such that $w_1 \cdot \dots \cdot w_n \sqsubseteq w_0$ and $w_1 \Vdash \Box\circ\alpha_1, \dots, w_n \Vdash \Box\circ\alpha_n$. Suppose that $i = 1, \dots, n$. By the definition of forcing $v_i \Vdash \alpha_i$, i.e. $v_i \Vdash \Box\alpha_i$, for every $v_i \in V$ such that $w_i \prec v_i$. Thus by the definition of forcing $v_1 \cdot \dots \cdot v_n \Vdash (\vdash\mathbf{A})^\star$. Furthermore, by the property (3c) of the definition of Kripke model every $v \in V$, such that $w_0 \prec v$, is of the form $v_1 \cdot \dots \cdot v_n$.

   By the definition of forcing $w'' \cdot v_1 \cdot \dots \cdot v_n \Vdash \Box\alpha$. By the property (3d) of the definition of Kripke model, every $v \in V$ such that $w_0 \cdot w'' \prec v$ is of the form $w'' \cdot v_1 \cdot \dots \cdot v_n$ and thus $v \Vdash \Box\alpha$, i.e. $v \Vdash \alpha$. Thus by the definition of forcing $w \Vdash \Box\circ\alpha$.

The other cases are similiar and left to the reader. $\Box$

4 Completeness

4.1 Proof–search procedure

The following remarks are an immediate consequence of the properties of the sequent calculus **ILP** given in section 2.2.

1. If the left premise of the $L \supset$ rule is an instance of the logical axiom Ax then we never need to split the context when we consider a $L \supset$ rule in the proof–search procedure.

2. If we deal only with proofs in quasi–canonical form then we can consider the sequent calculus **ILP** as an intuitionistic sequent calculus where the atomic formulas are of the form $\vdash \alpha$ or ζ. Thus we can apply one of the well known proof–search procedures for intuitionistic propositional logic and need only to consider its extension to the relevant case. Note that we can absorb the structural rules in the pragmatic area of **ILP** in order to obtain a sequent calculus with the same properties as **G3i** [16] for which we have immediately a proof–search procedure because the rules can be read bottom–up. Furthermore we can use the technique described in [7] to make the proof–search procedure loop–free.

3. If we deal only with proofs in canonical form then the proof–search procedure for the extension to the relevant case is determined by the order in which the rules are applied in the **ILP**–proof.

Following these guidelines we describe a proof–search procedure which determines whether there exists a proof in canonical form. As explained above we can apply a terminating proof–search procedure for intuitionistic propositional logic which works only in the pragmatic area and in the succedent of the sequent. If this procedure fails in finding a proof then we are left with a certain number of sequents $\Gamma; \Delta \Rightarrow \delta$ which occur at the leaves of the search tree and which are not of the form $;\delta, \Delta \Rightarrow \delta$ or $\delta; \Delta \Rightarrow \delta$ or $\Gamma; \bigwedge, \Delta \Rightarrow \delta$. We call the sequents at the leaves *initial sequents* of the extension to the relevant case. Note that in an initial sequent δ is of the form $\vdash \alpha$ or ζ. We can now apply to every initial sequent $\Gamma; \Delta \Rightarrow \delta$ the following procedure which determines if it is provable or not.

We denote with Δ' the multiset Δ from which we have deleted all formulas with pragmatic connectives and consider the set $\wp^*(\Delta')$ of all multisets (including the empty one) of formulas contained in Δ'. Because we don't know which formulas should introduced in the pragmatic area by the permeability rule p in order to achieve provability, we consider the sequents $\Gamma, \Delta'', [] \Rightarrow \delta$ for each $\Delta'' \in \wp^*(\Delta')$. If at least one of these sequents is provable then the initial sequent is provable. Note that if a sequent is provable with respect to a certain multiset Δ'' then the formulas which occur in Δ but not in Δ'' can be introduced in the pragmatic area by the weakening rule w. Furthermore, we use the square brackets to collect the analyzed formulas in order to detect hidden contraction rules (see steps (PS2c) and (PS2d) and example 4.1 below).

PS1. (a) If the sequent is of the form $\Gamma, [] \Rightarrow \vdash \alpha \mathbin{\supset\!\!\!\cdot\,} \zeta$ then we consider the sequent $\Gamma, \vdash \alpha, [] \Rightarrow \zeta$ and repeat this step until ζ is of the form $\vdash \alpha'$.

(b) If the sequent is of the form $\circ\!\!-\mathbf{A}, \mathbf{Z}, [] \Rightarrow \circ\!\!-\alpha$ then we consider the sequent $\vdash\mathbf{A}, \mathbf{Z}, [] \Rightarrow \vdash \alpha$. If $\circ\!\!-\mathbf{A}$ or $\mathbf{Z}$ is empty, then we don't apply the rule and stop the construction of the search tree for this sequent.

PS2. We construct now a search tree starting from the sequent obtained by step (PS1). Suppose that after a certain number of steps in the construction the search tree has n sequents of the form $\Gamma, [\Delta] \Rightarrow \vdash \alpha$ as leaves. For every leaf we can extend the search tree in the following way.

(a) If there is a sequent $\Gamma', [\Delta'] \Rightarrow \vdash \alpha$ in the path from the leaf to the root such that $\Gamma' = \Gamma$ and $Set(\Delta') = Set(\Delta)$[2] then we have a *loop* and we don't consider this leaf of the search tree anymore.

(b) Otherwise, if two formulas $\vdash \alpha'$ and $\vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta$ occur in Γ then we denote with Γ' the multiset Γ from which we have deleted $\vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta$ and $\vdash \alpha'$ and add the sequent $\Gamma', \zeta, [\Delta, \vdash \alpha', \vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta] \Rightarrow \vdash \alpha$ as a new leaf to the search tree.

(c) Otherwise, if a formula $\vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta$ occurs in Γ and a formula $\vdash \alpha'$ occurs in Δ then we denote with Γ' the multiset Γ from which we have deleted $\vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta$ and add the sequent $\Gamma', \zeta, [\Delta, \vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta] \Rightarrow \vdash \alpha$ as a new leaf to the search tree.

(d) Otherwise, if a formula $\vdash \alpha'$ occurs in Γ and m formulas $\vdash \alpha' \mathbin{\supset\!\!\!\cdot\,} \zeta_i$ $(i = 1, \ldots, m)$ occur in Δ then we denote with Γ' the multiset Γ from which we have deleted $\vdash \alpha'$ and add the m sequents $\Gamma', \zeta_i, [\Delta, \vdash \alpha'] \Rightarrow \vdash \alpha$ as new leaves to the search tree. If at least one of these sequents is provable then the initial sequent is provable.

(e) Otherwise, if no one of the cases above is applicable to the leaf then we don't consider this leaf of the search tree anymore.

If we reach a leaf of the form $\vdash \alpha, [\Delta] \Rightarrow \vdash \alpha$ then the initial sequent is provable and we stop the proof–search for this initial sequent.

[2] $Set(\Delta)$ denotes the multiset Δ regarded as set, i.e. $Set(\Delta) = \{\delta \mid \delta \text{ occurs in } \Delta\}$.

$$
\dfrac{\dfrac{\vdash\alpha_1;\Rightarrow\vdash\alpha_1 \qquad \dfrac{\vdash\alpha_1;\Rightarrow\vdash\alpha_1 \qquad \dfrac{\vdash\alpha_2;\Rightarrow\vdash\alpha_2 \qquad \dfrac{\vdash\alpha_1;\Rightarrow\vdash\alpha_1 \quad \vdash\alpha_2;\Rightarrow\vdash\alpha_2}{\vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}\,L\Supset}{\vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}\,L\Supset}{\dfrac{\vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}{\vdash\alpha_1,\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}\,c_r}\,L\Supset}{\vdash\alpha_1,\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}\,L\Supset}{a_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2;\Rightarrow\vdash\alpha_2}\,c_r
$$

Figure 4: Example of a canonical proof

Example 4.1 *Consider the sequent*

$$\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_2 \Supset \vdash\alpha_1;\Rightarrow\vdash\alpha_2$$

which can be obtained by the proof given in figure 4. Note that the final sequent is obtained by two occurrences of the contraction rule c_r. Suppose that $\Gamma \equiv \vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2$. Because the pragmatic area is empty we need only to apply the proof–search procedure to the sequent $\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2,[\,] \Rightarrow \vdash\alpha_2$. From the procedure we obtain the following complete search tree where the left branch determines that the sequent is provable, whereas in the right branch we have a loop. Note that the number of occurrences of step (PS2c) *and of step* (PS2d) *in the left branch coincides exactly with the number of occurrences of the contraction rule in the proof. The use of square brackets permits to determine contractions but it is also the source of looping.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\times}{\vdash\alpha_2,[\ldots]\Rightarrow\vdash\alpha_2} \qquad \dfrac{\dfrac{\textit{loop}}{\vdash\alpha_1,[\Gamma,\vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1,\vdash\alpha_2 \Supset \vdash\alpha_1]\Rightarrow\vdash\alpha_2}\,\text{(PS2a)}}{\vdash\alpha_2 \Supset \vdash\alpha_1,[\Gamma,\vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1]\Rightarrow\vdash\alpha_2}\,\text{(PS2c)}}{\vdash\alpha_1,[\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2,\vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1]\Rightarrow\vdash\alpha_2}\,\text{(PS2d)}}{\vdash\alpha_2,\vdash\alpha_2 \Supset \vdash\alpha_1,[\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2]\Rightarrow\vdash\alpha_2}\,\text{(PS2b)}}{\vdash\alpha_2 \Supset \vdash\alpha_1,\vdash\alpha_1 \Supset \vdash\alpha_2,[\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1)]\Rightarrow\vdash\alpha_2}\,\text{(PS2c)}}{\vdash\alpha_1,\vdash\alpha_1 \Supset (\vdash\alpha_2 \Supset \vdash\alpha_1),\vdash\alpha_1 \Supset \vdash\alpha_2,[\,]\Rightarrow\vdash\alpha_2}\,\text{(PS2a)}
$$

Example 4.2 *Consider the following sequent which is not provable in* **ILP**.

$$\multimap\alpha_2;\Rightarrow(\multimap\alpha_1 \supset (\vdash\alpha_1 \Supset \vdash\alpha_2)) \cup ((\vdash\alpha_1 \Supset \vdash\alpha_2) \supset \multimap\alpha_1)$$

In order to show that the sequent is not provable we have to consider two cases, i.e. we have to show that neither $\circ\!\!-\alpha_2; \Rightarrow \circ\!\!-\alpha_1 \supset (\vdash \alpha_1 \Supset \vdash \alpha_2)$ *nor* $\circ\!\!-\alpha_2; \Rightarrow (\vdash \alpha_1 \Supset \vdash \alpha_2) \supset \circ\!\!-\alpha_1$ *is provable. In the first case we have to consider the initial sequent* $\circ\!\!-\alpha_2; \circ\!\!-\alpha_1 \Rightarrow \vdash \alpha_1 \Supset \vdash \alpha_2$ *and to apply the extension of the proof–search procedure to the sequents* $\circ\!\!-\alpha_2, [] \Rightarrow \vdash \alpha_1 \Supset \vdash \alpha_2$ *and* $\circ\!\!-\alpha_2, \circ\!\!-\alpha_1, [] \Rightarrow \vdash \alpha_1 \Supset \vdash \alpha_2$*, but the procedure terminates after step* (PS1a) *in both cases. In the second case we have to consider the initial sequent* $\circ\!\!-\alpha_2; \vdash \alpha_1 \Supset \vdash \alpha_2 \Rightarrow \circ\!\!-\alpha_1$ *and to apply the extension of the proof–search procedure to the sequents* $\circ\!\!-\alpha_2, [] \Rightarrow \circ\!\!-\alpha_1$ *and* $\circ\!\!-\alpha_2, \vdash \alpha_1 \Supset \vdash \alpha_2, [] \Rightarrow \circ\!\!-\alpha_1$*, but the procedure terminates immediately and after step* (PS1a)*, respectively.*

The procedure for the initial sequents explained above considers all possible cases. Thus we are able to decide if an initial sequent is provable or not. We can show by induction on the cardinality $|\Gamma|$ of the multiset Γ that the procedure is also terminating. The crucial step in the proof is that if the cardinality of Γ doesn't decrease, i.e. $|\Gamma| = c$ for some constant c, then after a certain number of steps there will be a loop because there is only a finite number of subformulas of formulas in Γ and in Δ and the procedure continues to analyze the same formulas.

So we can use the information obtained by the procedure for the initial sequents of the extension to the relevant case in the proof–search procedure for propositional intuitionistic logic. By the decidability of propositional intuitionistic logic [4] we have the following

Theorem 4.3 (decidability) *To determine whether there exists a proof in canonical form of a sequent* $\Gamma; \Delta \Rightarrow \delta$ *is decidable.*

4.2 Counter–models

If the proof–search procedure explained in the previous section determines that there is no proof in canonical form of the sequent $\Gamma; \Delta \Rightarrow \delta$ then we can construct a counter–model for this sequent. The following facts explain the basic ideas and semantical motivations which occur in the construction of the counter–model.

F1. In order to refute $\Gamma^\star \otimes \Delta^\star \Rightarrow \delta^\star$ we need a possible world w_0 such that $w_0 \Vdash \Delta^\star$ and n different possible worlds x_i such that $x_i \Vdash \delta_i^\star$ for each formula δ_i in the multiset Γ. It follows that $x_1 \cdot \ldots \cdot x_n \cdot w_0 \Vdash \Gamma^\star \otimes \Delta^\star$ by the reflexivity of the accessibility relation $\sqsubseteq$. In order to ensure that $x_1 \cdot \ldots \cdot x_n \cdot w_0 \nVdash \delta^\star$ we need k different possible worlds y_j such that $y_j \Vdash (\vdash \alpha_j)^\star$ for each formula $\vdash \alpha_j$ added to the relevant area by step

(PS1a) of the extension of the proof–search procedure. Otherwise it could happen that causal implications in δ are satisfied vacuously.

F2. Each possible world w of the counter–model which forces a formula $\Delta^\star$ comprising formulas occuring in the pragmatic area has to be such that $1 \sqsubseteq w$, where 1 is the identity of the preordered monoid, because $\mathbf{1}$ is a subformula of $\Delta^\star$.

F3. If we consider the right implication rule of intuitionistic propositional logic

$$\frac{\Delta, \delta_1 \Rightarrow \delta_2}{\Delta \Rightarrow \delta_1 \supset \delta_2}$$

then the following property holds: if a possible world w refutes $\Delta^\star \wedge \delta_1^\star \Rightarrow \delta_2^\star$ then each possible world $w' \sqsubseteq w$ refutes $\Delta^\star \Rightarrow (\delta_1 \supset \delta_2)^\star$. In order to have that the same holds for the $R\supset$ rule

$$\frac{\Gamma; \Delta, \delta_1 \Rightarrow \delta_2}{\Gamma; \Delta \Rightarrow \delta_1 \supset \delta_2} \, R\supset$$

of **ILP** we have to impose the following condition: if the possible worlds $\bar{x}$ and w are such that $\bar{x} \Vdash \Gamma^\star$, $w \Vdash \Delta^\star \wedge \delta_1^\star$ and $w \nVdash \delta_2^\star$ then $\bar{x} \cdot w \sqsubseteq w$. Thus each possible world $\bar{x} \cdot w' \sqsubseteq \bar{x} \cdot w$ refutes $\Gamma^\star \otimes \Delta^\star \Rightarrow (\delta_1 \supset \delta_2)^\star$.

F4. Consider the preliminary step of the extension of the proof–search procedure which can be expressed by the following compact notation.

$$\frac{\Gamma, \Delta'', [] \Rightarrow \delta}{\Gamma; \Delta \Rightarrow \delta}$$

Note that the multiset $\Delta'' \subseteq \Delta$ is shifted from the pragmatic to the relevant area, thus there exists a possible world w such that $1 \sqsubseteq w$ and $w \Vdash \delta_i^\star$ $(i = 1, \ldots, m')$ for each $\delta_i \in \Delta''$. Therefore Δ'' can equally be interpreted[3] as $(\delta_1^\star \wedge \mathbf{1}) \otimes \ldots \otimes (\delta_{m'}^\star \wedge \mathbf{1})$ or as $\delta_1^\star \wedge \ldots \wedge \delta_{m'}^\star \wedge \mathbf{1}$. Suppose that the possible worlds $\bar{x}$ and w are such that $\bar{x} \Vdash \Gamma^\star$ and $w \Vdash (\Delta'')^\star$, respectively. We have that $\bar{x} \cdot w$ refutes $\Gamma^\star \otimes (\Delta'')^\star \Rightarrow \delta^\star$ if and only if $\bar{x} \cdot w$ refutes $\Gamma^\star \otimes \Delta^\star \Rightarrow \delta^\star$.

[3] In a system like Girard's **LU** [10] where sequents have mixed linear and intuitionistic contexts the permeability rule is based upon the fundamental principle of linear logic $!\delta_1 \otimes !\delta_2 \equiv !(\delta_1 \& \delta_2)$. In sequents with relevant and intuitionistic contexts where the contraction rule is available everywhere it suffices to have $(\mathbf{1} \wedge \delta_1) \otimes (\mathbf{1} \wedge \delta_2) \equiv \mathbf{1} \wedge \delta_1 \wedge \delta_2$.

F5. Consider step (PS1a) of the extension of the proof–search procedure.

$$\frac{\Gamma, \vdash \alpha, [] \Rightarrow \zeta}{\Gamma, [] \Rightarrow \vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta}$$

Suppose that the possible worlds $\bar{x}$ and y are such that $\bar{x} \Vdash \Gamma^\star$ and $y \Vdash (\vdash \alpha)^\star$, respectively. We have that $\bar{x} \cdot y$ refutes $\Gamma^\star \otimes (\vdash \alpha)^\star$ if and only if $\bar{x}$ refutes $\Gamma^\star \Rightarrow (\vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta)^\star$.

F6. Consider step (PS1b) of the extension of the proof–search procedure.

$$\frac{\vdash \mathbf{A}, \mathbf{Z}, [] \Rightarrow \vdash \alpha}{\multimapinv \mathbf{A}, \mathbf{Z}, [] \Rightarrow \multimapinv \alpha}$$

Suppose that for each virtuous world $\bar{v}$ such that $\bar{x} \prec \bar{v}$ it holds that $\bar{v} \Vdash (\vdash \mathbf{A})^\star$ and that the possible world $\bar{x}'$ is such that $\bar{x}' \Vdash \mathbf{Z}^\star$. We have that $\bar{x}' \cdot \bar{v}$ refutes $(\vdash \mathbf{A})^\star \otimes \mathbf{Z}^\star \Rightarrow (\vdash \alpha)^\star$ if and only if $\bar{x} \cdot \bar{x}'$ refutes $(\multimapinv \mathbf{A})^\star \otimes \mathbf{Z}^\star \Rightarrow (\multimapinv \alpha)^\star$.

F7. Consider step (PS2b) of the extension of the proof–search procedure.

$$\frac{\zeta, \Gamma, [\Delta, \vdash \alpha, \vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta] \Rightarrow \vdash \alpha'}{\vdash \alpha, \vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta, \Gamma, [\Delta] \Rightarrow \vdash \alpha'}$$

Note that there exists possible worlds x_1 and x_2 such that $x_1 \Vdash (\vdash \alpha)^\star$, $x_2 \Vdash (\vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta)^\star$ and $x_1 \cdot x_2 \Vdash \zeta^\star$. Suppose that $\bar{x} \Vdash \Gamma^\star$. We have that $x_1 \cdot x_2 \cdot \bar{x}$ refutes $\zeta^\star \otimes \Gamma^\star \Rightarrow (\vdash \alpha')^\star$ if and only if $x_1 \cdot x_2 \cdot \bar{x}$ refutes $(\vdash \alpha)^\star \otimes (\vdash \alpha \mathbin{c\!\!\!\!\supset} \zeta)^\star \otimes \Gamma^\star \Rightarrow (\vdash \alpha')^\star$. The same argument applies also to step (PS2c) and (PS2d).

Starting from these facts we construct now a finite Kripke frame and define then a forcing relation such that we obtain a finite counter–model of the sequent $\Gamma; \Delta \Rightarrow \delta$.

KF1. Suppose that we have a set of possible worlds $W_{\mathcal{I}}$ together with an accessibility relation $\sqsubseteq \subseteq W_{\mathcal{I}} \times W_{\mathcal{I}}$ obtained from the proof–search procedure for intuitionistic propositional logic by the usual way. We know that intuitionistic propositional logic satisfies the finite model property [4], thus we suppose that $W_{\mathcal{I}}$ is finite. Let w_0 denote the root of the intuitionistic refutation tree.

We define the set W' of possible worlds of the Kripke frame as $W_{\mathcal{I}} \cup \{1\}$ and extend the accessibility relation $\sqsubseteq$ on $W_{\mathcal{I}}$ with $1 \sqsubseteq w_0$ and its reflexive and transitive closure to an accessibility relation on W'. Furthermore, we endow the preordered set W' with an idempotent and commutative operation $\cdot$ and extend it to a monoid $(W', \cdot, 1)$: if $w, w' \in W'$ then $w \cdot w' \in W'$.

Definition 4.4 *If $w \not\sqsubseteq w'$ and $w' \not\sqsubseteq w$ then we extend the accessibility relation $\sqsubseteq$ with $w \sqsubseteq w \cdot w'$, $w' \sqsubseteq w \cdot w'$ and its reflexive and transitive closure. Furthermore, if $w_1 \sqsubseteq w_2$ and $w_1' \sqsubseteq w_2'$ then we extend $\sqsubseteq$ with $w_1 \cdot w_2 \sqsubseteq w_1' \cdot w_2'$ and its reflexive and transitive closure.*

Note that as an immediate consequence of this definition we have that $(W', \cdot, 1, \sqsubseteq)$ is a preordered monoid. Furthermore, there exists an equivalence relation between the elements of the preordered monoid. Thus they can be divided into equivalence classes.

Lemma 4.5 *If $w \sqsubseteq w'$ then $w' = w \cdot w'$.*

Proof It follows from idempotence and from $1 \sqsubseteq w$ that $w' \sqsubseteq w \cdot w'$ by definition 4.4. Similarly we have that idempotence and $w \sqsubseteq w'$ imply $w \cdot w' \sqsubseteq w'$. □

KF2. We consider the set $U = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_k\}$ of possible worlds of fact (F1), endow it with an idempotent and commutative operation $\cdot$ and extend it to a semigroup $(U, \cdot)$: if $u, u' \in U$ then $u \cdot u' \in U$.

We consider the set $\bar{W} = \{w \cdot u | w \in W', u \in U\} \cup \{u \cdot w | w \in W', u \in U\}$ where we impose that $1 \cdot u = u = u \cdot 1$ and $u \cdot w = w \cdot u$ for each $w \in W'$ and $u \in U$. Let W denote $W' \cup \bar{W}$. Note that $(W, \cdot, 1)$ is a monoid.

Definition 4.6 *For each possible world w such that $w_0 \sqsubseteq w$ and $w \neq w_0$ we extend the accessibility relation $\sqsubseteq$ with $x_1 \cdot \ldots \cdot x_n \cdot w \sqsubseteq w$ and its reflexive and transitive closure. Furthermore, if $w_1, w_1', w_2, w_2' \in W$, $w_1 \sqsubseteq w_1'$ and $w_2 \sqsubseteq w_2'$ then we extend $\sqsubseteq$ with $w_1 \cdot w_2 \sqsubseteq w_1' \cdot w_2'$ and its reflexive and transitive closure.*

Note, in particular, that the equivalence relation on W' extends to an equivalence relation on $\bar{W}$. Furthermore, as an immediate consequence of this definition we have that $(W, \cdot, 1, \sqsubseteq)$ is a preordered monoid.

KF3. We define the set V of virtuous worlds and the relation $\prec \subseteq W \times V$ in the following way: for each $w \in W$ we add exactly one virtuous world v such that $w \prec v$ to V. Furthermore, for each $w' \in W$ such that $w' \sqsubseteq w$ we impose that $w \prec v$. We endow the set V with an idempotent and commutative operation $\cdot$ and extend it to a semigroup $(V, \cdot)$: if $v, v' \in V$ then $v \cdot v' \in V$.

Definition 4.7 *If $w \prec v$, $w' \prec v'$ and $w \cdot w' \prec v''$ then $v'' = v \cdot v'$.*

Note that as an immediate consequence of this definition it holds that if $w \sqsubseteq w'$ then $v' = v \cdot v'$.

Definition 4.8 *We define an operation $\cdot : W \times V \to V$ in the following way. If $w \sqsubseteq w'$ and $w' \prec v'$ then $w \cdot v' = v'$. Furthermore, if $w \not\sqsubseteq w'$, $w' \not\sqsubseteq w$, $w \prec v$ and $w' \prec v'$ then $w \cdot v' = w' \cdot v = v \cdot v'$.*

Note that as an immediate consequence of these definitions we have that the properties (3a), (3b), (3c) and (3d) of the definition of Kripke model are satisfied.

Lemma 4.9 *Every finitely generated commutative semigroup $(S, \cdot)$ with an idempotent operation $\cdot$ is finite.*

Proof By commutativity and idempotence every element of S has the form $s_1 \cdot \ldots \cdot s_k$ where $s_1, \ldots, s_k$ are generators. $\Box$

Lemma 4.10 *The preordered monoid $\mathcal{W} = (W, \cdot, 1, \sqsubseteq)$, the semigroup $\mathcal{V} = (V, \cdot)$, the operation $\cdot : W \times V \to V$ and the relation $\prec \subseteq W \times V$ defined above form a Kripke frame $(\mathcal{W}, \mathcal{V}, \cdot, \prec)$. Furthermore, the Kripke frame is finite.*

Proof That $(\mathcal{W}, \mathcal{V}, \cdot, \prec)$ is a Kripke frame follows immediately from the above definitions. $\mathcal{W}$ and $\mathcal{V}$ are finitely generated. Thus they are finite by lemma 4.9. $\Box$

Example 4.11 *Let us consider the following refutation tree obtained from the intuitionistic proof–search procedure applied to the sequent of example 4.2.*

$$\begin{array}{ccc} w_1 & & w_2 \\ & \diagdown \quad \diagup & \\ & w_0 & \end{array}$$

From the extension of the proof–search procedure to the relevant case we know that we have to consider also the sets of possible worlds $\{x\}$ and $\{y\}$. Figure 5 shows the Kripke frame obtained from these possible worlds and the accessibility relation of the intuitionistic refutation tree. Note that the nodes of the Kripke frame are equivalence classes of possible or virtuous worlds. In particular the following property holds. Suppose that the possible world w is such that $w_0 \sqsubseteq w$ and $w \neq w_0$ and that $w \prec v$ and $x \prec v'$. If $x \cdot w \prec v''$ then $v'' = v' \cdot v = v$ by definition 4.7. Furthermore, note that $y \cdot w \prec y \cdot v$, $y \cdot x \prec y \cdot v'$ and $y \cdot x \cdot w \prec y \cdot v''$ by definition 4.8. Thus $y \cdot v'' = y \cdot v' \cdot v = y \cdot v$ by definition 4.7.

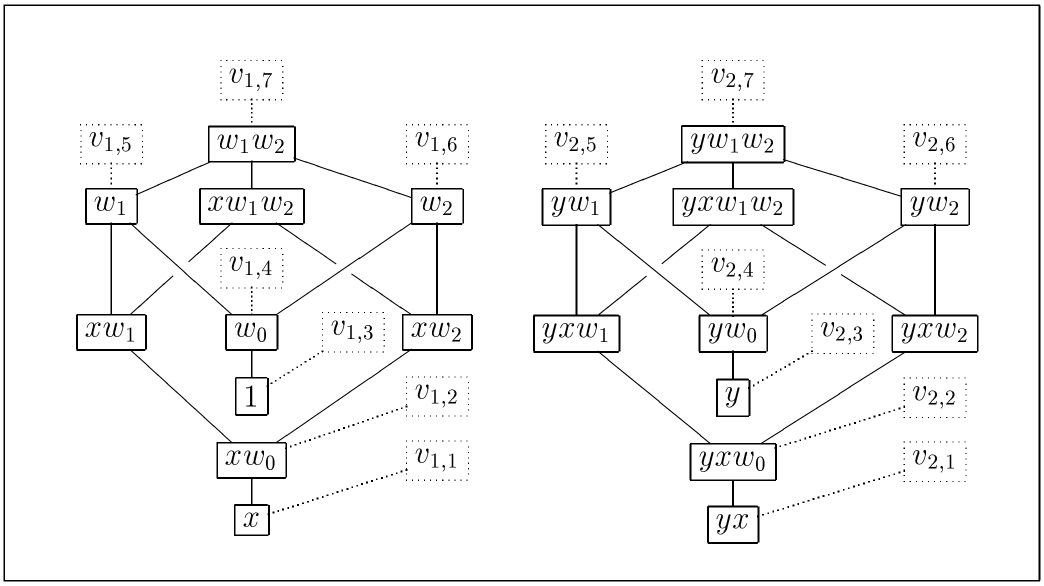

Figure 5: Example of a Kripke frame

Before we define the forcing relation $\Vdash$ for atomic formulas we associate to each possible world $w \in W$ and to each virtuous world $v \in V$ a set Ξ_w of formulas of the form $\mathbf{1}$, $\multimapinv\alpha$ and ζ and a set Ξ_v of formulas of the form ζ, respectively.

Definition 4.12 *$\Xi_{(-)}$ is defined inductively by the following clauses.*

1. *Suppose that w is a possible world of the counter–model $(W_{\mathcal{I}}, \sqsubseteq, \Vdash)$ obtained from the proof–search procedure for intuitionistic propositional logic. Ξ_w is the set of formulas of the form $\multimapinv\alpha$ and ζ which interpretations $(\multimapinv\alpha)^\star$ and $\zeta^\star$, respectively, are forced at w by the intuitionistic forcing relation. Furthermore, we have that $\Xi_1 = \{\mathbf{1}\}$.*

2. *For each possible world x_i $(i = 1, \ldots, n)$ of fact* F1 *we have that $\Xi_{x_i} = \{\delta_i\}$ where δ is of the form $\multimapinv\alpha$ or ζ.*

3. *For each possible world y_j $(j = 1, \ldots, k)$ of fact* F1 *we have that $\Xi_{y_j} = \{\vdash\alpha_j\}$.*

4. *If the formula δ is of the form $\mathbf{1}$, $\multimapinv\alpha$ or ζ and the possible worlds $w, w' \in W$ are such that $\delta \in \Xi_w$ and $w \sqsubseteq w'$ then we set $\Xi_{w'}$ to $\Xi_{w'} \cup \{\delta\}$.*

5. *If $\multimapinv\alpha \in \Xi_w$ then for each virtuous world v such that $w \prec v$, we set Ξ_v to $\Xi_v \cup \{\vdash\alpha\}$.*

6. *If* $\vdash\alpha \supset_c \zeta \in \Xi_w$ *then for each possible world* w' *such that* $\vdash\alpha \in \Xi_{w'}$ *and for each virtuous world* v *such that* $\vdash\alpha \in \Xi_v$*, we set* $\Xi_{w\cdot w'}$ *to* $\Xi_{w\cdot w'} \cup \{\zeta\}$ *and* $\Xi_{w\cdot v}$ *to* $\Xi_{w\cdot v} \cup \{\zeta\}$*, respectively.*

7. *If* $\vdash\alpha \supset_c \zeta \in \Xi_v$ *then for each possible world* w *such that* $\vdash\alpha \in \Xi_w$ *and for each virtuous world* v' *such that* $\vdash\alpha \in \Xi_{v'}$*, we set* $\Xi_{w\cdot v}$ *to* $\Xi_{w\cdot v} \cup \{\zeta\}$ *and* $\Xi_{v\cdot v'}$ *to* $\Xi_{v\cdot v'} \cup \{\zeta\}$*, respectively.*

The set of formulas Ξ_w and Ξ_v associated to each possible world w and to each virtuous world v, respectively, permits to define the forcing relation $\Vdash \subseteq (W \cup V) \times At$ for atomic formulas. Note, in particular, that without $\Xi_{(-)}$ it wouldn't be possible to determine which atomic formulas must be forced in order to obtain a counter–model because it could happen that the decision procedure doesn't analyze all formulas of the form $\multimapinv\alpha$ and ζ (see example 4.2).

Definition 4.13 *For each possible world* $w \in W$ *and each virtuous world* $v \in V$*, if* $\vdash\alpha \in \Xi_w$ *or* $\vdash\alpha \in \Xi_v$ *then* $w \Vdash \alpha$ *or* $v \Vdash \alpha$*, respectively.*

As an immediate consequence of the clauses of the forcing relation for extended radical formulas given in figure 3 and of the facts F1, …F7 given at the beginning of this section we have the following

Lemma 4.14 $x_1 \cdot \ldots \cdot x_n \cdot w_0 \Vdash \Gamma^\star \otimes \Delta^\star$*, but* $x_1 \cdot \ldots \cdot x_n \cdot w_0 \nVdash \delta^\star$*.*

Example 4.15 *We complete the examples 4.2 and 4.11 with the definition of the forcing relation for atomic formulas and show that this model refutes*

$$\Box\circ\alpha_2 \otimes \mathbf{1} \Rightarrow \Box(\Box\circ\alpha_1 \to (\Box\alpha_1 \multimap \Box\alpha_2)) \vee \Box((\Box\alpha_1 \multimap \Box\alpha_2) \to \Box\circ\alpha_1)$$

The base cases of the defintion of $\Xi_{(-)}$ *are the following:* $\Xi_{w_0} = \emptyset$*,* $\Xi_{w_1} = \{\multimapinv\alpha_1\}$*,* $\Xi_{w_2} = \{\vdash\alpha_1 \supset_c \vdash\alpha_2\}$*,* $\Xi_1 = \{\mathbf{1}\}$*,* $\Xi_x = \{\multimapinv\alpha_2\}$ *and* $\Xi_y = \{\vdash\alpha_1\}$*. If we apply the inductive clauses of definition 4.12 then we obtain the following table for possible worlds*

$- \cdot -$	1	x	y	$y \cdot x$
1	$\mathbf{1}$	$\multimapinv\alpha_2$	$\vdash\alpha_1$	$\emptyset$
w_0	$\mathbf{1}$	$\multimapinv\alpha_2$	$\vdash\alpha_1$	$\emptyset$
w_1	$\mathbf{1}, \multimapinv\alpha_1, \multimapinv\alpha_2$	$\multimapinv\alpha_2$	$\vdash\alpha_1$	$\emptyset$
w_2	$\mathbf{1}, \multimapinv\alpha_2, \vdash\alpha_1 \supset_c \vdash\alpha_2$	$\multimapinv\alpha_2$	$\vdash\alpha_1, \vdash\alpha_2$	$\emptyset$
$w_1 \cdot w_2$	$\mathbf{1}, \multimapinv\alpha_1, \multimapinv\alpha_2, \vdash\alpha_1 \supset_c \vdash\alpha_2$	$\multimapinv\alpha_2$	$\vdash\alpha_1, \vdash\alpha_2$	$\emptyset$

and the following table for virtuous worlds

$v_{-,-}$	1	2	3	4	5	6	7
1	$\vdash\alpha_2$	$\vdash\alpha_2$	$\emptyset$	$\emptyset$	$\vdash\alpha_1, \vdash\alpha_2$	$\vdash\alpha_2$	$\vdash\alpha_1, \vdash\alpha_2$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

By definition 4.13 it follows that the atomic formula α_1 is forced at the worlds y, $y \cdot w_0$, $y \cdot w_1$, $y \cdot w_2$, $y \cdot w_1 \cdot w_2$, $v_{1,5}$ and $v_{1,7}$ and that the atomic formula α_2 at the worlds $y \cdot w_1$, $y \cdot w_1 \cdot w_2$, $v_{1,1}$, $v_{1,2}$, $v_{1,5}$, $v_{1,6}$ and $v_{1,7}$.

For each v such that $x \prec v$ it holds that $v \Vdash \alpha_2$. Thus $x \Vdash \circ\alpha_2$ by the definition of forcing and $x \Vdash \Box\circ\alpha_2$ by lemma 3.3. From $1 \sqsubseteq w_0$ it follows that $w_0 \Vdash \mathbf{1}$ and therefore that $x \cdot w_0 \Vdash \Box\circ\alpha_2 \otimes \mathbf{1}$ by the definition of forcing.

For each v such that $w_1 \prec v$ it holds that $v \Vdash \alpha_1$. Thus $w_1 \Vdash \circ\alpha_1$ by the definition of forcing and $w_1 \Vdash \Box\circ\alpha_1$ by lemma 3.3. From $w \Vdash \alpha_1$ for each w such that $y \sqsubseteq w$ and from $y \cdot w_1 \not\Vdash \alpha_2$ it follows that $y \Vdash \Box\alpha_1$ and $w_1 \cdot y \not\Vdash \Box\alpha_2$, respectively, by the definition of forcing. Thus $w_1 \not\Vdash \Box\alpha_1 \multimap \Box\alpha_2$ and $w_1 \not\Vdash \Box\circ\alpha_1 \to (\Box\alpha_1 \multimap \Box\alpha_2)$ by the definition of forcing. From $x \cdot w_o \sqsubseteq w_1$ it follows that $x \cdot w_0 \not\Vdash \Box(\Box\circ\alpha_1 \to (\Box\alpha_1 \multimap \Box\alpha_2))$ by the definition of forcing.

For each w such that $y \cdot w_2 \sqsubseteq w$ it holds that $w \Vdash \alpha_1$ and $w \Vdash \alpha_2$. Thus $w \Vdash \Box\alpha_1$ and $w \Vdash \Box\alpha_2$ by the definition of forcing. Note that $w_2 \cdot w = w$ by idempotence and thus $w_2 \Vdash \Box\alpha_1 \multimap \Box\alpha_2$ by the definition of forcing. From $w_2 \prec v_{1,6}$ and $v_{1,6} \not\Vdash \alpha_1$ it follows that $w_2 \not\Vdash \circ\alpha_1$ by the definition of forcing and $w_2 \not\Vdash \Box\circ\alpha_1$ by lemma 3.3. Thus $w_2 \not\Vdash (\Box\alpha_1 \multimap \Box\alpha_2) \to \Box\circ\alpha_1$ by the definition of forcing. From $x \cdot w_o \sqsubseteq w_2$ it follows that $x \cdot w_0 \not\Vdash \Box((\Box\alpha_1 \multimap \Box\alpha_2) \to \Box\circ\alpha_1)$ by the definition of forcing.

Thus $x \cdot w_0 \not\Vdash \Box(\Box\circ\alpha_1 \to (\Box\alpha_1 \multimap \Box\alpha_2)) \vee \Box((\Box\alpha_1 \multimap \Box\alpha_2) \to \Box\circ\alpha_1)$ by the definition of forcing.

As an immediate consequence of lemma 4.10 and lemma 4.14 we have the following

Theorem 4.16 (finite model property) *If there doesn't exist a proof in canonical form of the sequent $\Gamma; \Delta \Rightarrow \delta$ then it has a finite counter–model.*

4.3 Completeness and cut elimination

Theorem 4.17 (completeness) *The sequent $\Gamma; \Delta \Rightarrow \delta$ is provable in* **ILP** *if and only if $\Gamma; \Delta \models \delta$.*

Proof Suppose that $\Gamma; \Delta \models \delta$ and that the decision procedure determines that the sequent $\Gamma; \Delta \Rightarrow \delta$ is not provable in **ILP**. Then we can construct a counter–model as explained above which contradicts the hypothesis that $\Gamma; \Delta \Rightarrow \delta$ is valid. The other direction holds by the soundness theorem. $\Box$

Remember that proofs in canonical form are cut–free proofs. Thus as a consequence of the fact that the proof–search procedure determines whether there exists a proof in canonical form and the fact that the cut rules are sound with respect to the given Kripke–style semantics we obtain an alternative proof (which doesn't provide a procedure to transform proofs with cuts in cut–free proofs) of cut elimination with respect to the one given in [2].

Theorem 4.18 (cut elimination) *If a sequent* $\Gamma; \Delta \Rightarrow \delta$ *is provable in* **ILP** *then it is provable without cuts.*

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