TOWARDS A LOGIC FOR PRAGMATICS. ASSERTIONS AND CONJECTURES.

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Abstract. The logic for pragmatics extends classical logic in order to characterize the logical properties of the operators of *illocutionary force* such as that of *assertion* and *obligation* and of the *pragmatic connectives* which are given an intuitionistic interpretation. Here we consider the cases of *assertions* and *conjectures*: the assertion that a mathematical proposition α is true is justified by the capacity to present an actual proof of α, while the conjecture that α is true is justified by the absence of a refutation of α. We give sequent calculi of type **G3i** and **G3i**m inspired by Girard's **LU**, with subsystems characterizing intuitionistic reasoning and some forms of classical reasoning with such operators. Extending Gödel, McKinsey, Tarski and Kripke's translations of intuitionistic logic into **S4**, we show that our sequent calculi are sound and complete with respect to Kripke's semantics for **S4**.

§1. Preface. The *logic for pragmatics*, as introduced by Dalla Pozza and Garola in [7, 8] and developed in [2, 3], aims at a formal characterization of the logical properties of *illocutionary operators*: it is concerned, e.g., with the operations by which we perform the act of *asserting* a proposition as true, either on the basis of a mathematical proof or by empirical evidence or by the recognition of physical necessity, or the act of taking a proposition as an *obligation*, either on the basis of a moral principle or by inference within a normative system. [1] The discipline of pragmatics, first developed in classical texts of 20th century philosophy and philosophical logic from Austin [1] to Grice and Searle, and then resulting in a large body of linguistic literature (already conspicuous when the classical book by Levinson [14] was published) in a complex relationship with semantics and other areas of linguistics, lies at present beyond the scope of our methods. So far the logic for pragmatics has considered only

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propositional systems, thus has given no contribution to the crucial issue of the reference of individual terms. Moreover the focus of our current work is on *impersonal* acts of judgement, leaving the consideration of *speech acts* to future developments. More precisely, our present task is to characterize the abstract behaviour of a few pragmatic operators, as it is manifested in highly regimented forms of reasoniong such as mathematical discourse or the foundations of laws.

Within this range, the consideration of the impersonal operator of *assertion* in Dalla Pozza and Garola's pragmatic interpretation of intuitionistic logic [7] has given a stimulating insight in the interpretation of intuitionistic and classical connectives, briefly summarized below. In the light of their approach, this paper begins an analysis of dualities in intuitionistic logic, in so far as they can be interpreted as resulting from the relations between an impersonal operators of *assertion* and one of *conjecture*. The technical tools used here are as ancient as the **S4** translation, revisited in the light of Girard's sequent calculus **LU**, but they are perhaps enough to guess some features of the theory yet to be developed.

The viewpoint of [7] can be sketched roughly as follows. There is a logic of *propositions* and a logic of *judgements*. Propositions are entities which can be *true* of *false*, judgements are acts which can be *justified* or *unjustified*. The logic of propositions is about *truth* according to classical semantics. The logic of judgements gives conditions for the *justification* of acts of judgements. An instance of an elementary act of judgement is the *assertion* of a proposition α, which is justified by the capacity to exhibit a *proof* of it, if α is a mathematical proposition, or some kind of *empirical evidence* if α is about states of affairs[2]. It is then claimed that the justification of complex acts of judgement must be in terms of *Heyting's interpretation of intuitionistic connectives*: for instance, a *conditional judgement* where the assertion of β depends on the assertibility of α is justified by a method that transforms any justification for the assertion of α into a justification for the assertion of β.

In modern logic the distinction between propositions and judgements was established by Frege: a proposition expresses the thought

[2]In this introduction the symbol "α" stands for an arbitrary proposition, not necessarily atomic; the symbol "p" is specifically used to range among atomic propositions.

which is the content of a judgement and a judgement is the act of recognizing the truth of its content. In Frege's formalism, "$\vdash \alpha$" expresses the judgement asserting the proposition α; only truth-functional connectives and quantifiers are considered and judgements appear only at the level of the deductive system. It follows that there cannot be nested occurrences of the symbol "$\vdash$" and that truth-functional connectives cannot be applied to expressions of judgement.

The distinction between propositions and judgements has recently been taken up by Martin-Löf: in his formalism "α *prop*" expresses the assertion that α is a well-formed proposition, and "α *true*" expresses the judgement that it is known how to verify α. However, it seems that propositions are given a *verificationist* semantics, according to which in order to give meaning to a proposition we must know what counts as a verification of it. Indeed, by replacing Frege's "$\vdash \alpha$" with "α *true*" Martin-Löf seems to adopt the view that it is impossible to separate the truth of a proposition from the conditions of its verification; certainly Martin-Löf theory of types is developed in an *epoché* of classical truth, without any reference to it.

Unlike Martin-Löf and in agreement with Frege, Dalla Pozza and Garola distinguish between the truth of a proposition and the justification of a judgement, but extend Frege's framework by introducing *pragmatic connectives* and giving them Heyting's interpretation while retaining Tarski's semantics for the logic of propositions. In their *compatibilist* approach, classical logic is *extended* rather than challenged by intuitionistic pragmatics, the latter having a different *subject matter* than the former. Thus the task and the challenge for Dalla Pozza and Garola's approach is to characterize the relations between the two levels: their main tool is the **S4** interpretation, due to Gödel [10], McKinsey and Tarski [20] and Kripke [12, 13], which they regard as a *reflection* of the pragmatic level on the semantic one. In this paper we try to show that this elementary tool can be exploited to trace interesting interactions between classical and pragmatics connectives.

On one hand, Dalla Pozza and Garola's framework does appear quite close to the well-established *epistemic* approach to philosophy of mathematics, advocated by Stewart Shapiro [25, 26]. Justification of judgements depends on knowledge; Kripke's possible worlds may be regarded as possible states of knowledge and their preordering may correspond to ways our knowledge could evolve in the future. Having a proof of α *now* rules out the possibility of α being false at any *future* state of knowledge, and the possibility that α may be false at a future state of knowledge propagates the impossibility of having a proof of α backwards to all previous states of knowledge. Similarly, having now a proof that α implies β rules out the possibility that at some future state of knowledge α may be true and β false. Notice that this reading explains Gödel's, McKinsey and Tarski's and Kripke's interpretation $(\ )^M$ of intuitionistic logic, which yields $(p)^M = \Box p$ and $(A \supset B)^M = \Box(A^M \to B^M)$, and *not* the other well-known interpretation $(\ )^G$ which yields $(p)^G = p$, $(A \supset B)^G = \Box A^G \to B^G$ and $(A \cup B)^G = \Box A^G \vee \Box B^G$. The following apparently innocent remark plays an important role here: Kripke's monotonicity condition (i.e., persistence in the future of the valuations of atomic formulas) is related to the fact that in the modal translation $(\ )^M$ an intuitionistic atom p is translated as $\Box p$. The epistemic interpretation of Kripke's semantics can be given an ontological significance: some philosophers have suggested that the right standpoint of the logic for pragmatics may be a reading of Kripke's possible world semantics that would reduce intuitionistic mathematics to *classical* epistemic mathematics; presumably, the intensional notion of a proof would be explained away in an ontology of possible states of knowledge.

On the other hand, Dalla Pozza insists that the logic for pragmatics is an *intensional* logic, while Kripke's semantics for modal logics suggests an *extensional* interpretation of intensional notions. In the field of *deontic logic* Dalla Pozza has successfully applied the intensional status of the pragmatic operator of obligation, in opposition to the extensional reading of the **KD** necessity operator, by introducing a distinction between *expressive* and *descriptive* interpretations of norms, which appears to have resolved conceptual confusions [8]. Similarly, Frege's symbol "$\vdash$" may be regarded here as expressing the *intentionality* of an act of judgement, while the **S4** modality "$\Box$" would perhaps describe conditions on the states of knowledge which justify the appropriateness of such an act. Contemporary mathematical intuitionism is based on game semantics, the typed λ-calculus and categorical logic as much as on Kripke's semantics; all of these tools belong to a mathematical treatment of the logic for pragmatics. The fruitfulness of a non-reductionist philosophical view may be tested by its capacity to promote a better understanding of their relations [3].

1.1. Conjectures and assertions. If the logic of assertions is formalized and interpreted in the proof theory and model theory of mathematical intuitionism, what are the essential features of an illocutionary operator of conjecture "$\mathcal{H}$" and what shall a logic of conjectures be like, if there has to be a duality between assertions and conjectures? The following three principles seem a good starting point of our investigation:

1. the grounds that justify *asserting* a proposition α certainly suffice also for *conjecturing* it, whatever these grounds may be; in other words $\vdash \alpha \Rightarrow \mathcal{H}\, \alpha$ should be an axiom of our logic of assertions and conjectures;
2. in any situation, the grounds that justify the assertion $\vdash \alpha$ are also necessary and sufficient to regard $\mathcal{H}\,\neg\alpha$ as unjustified;
3. pragmatic connectives are operations which express ways of building up complex acts of assertion or of conjecture from elementary acts of assertion and conjecture. The justification of a complex act depends on the justification of the component acts, possibly through intensional operations.

Notice that in the formula $\mathcal{H}\,\neg\alpha$ of (2), the negation is *classical negation*, not the intuitionistic one: e.g., the conjecture $\mathcal{H}\,\neg\alpha$ may be refuted also by evidence that a certain state of affairs α does not obtain, not necessarily by a proof that there would be a contradiction assuming that α obtains. The interactions between the semantic and the pragmatic layers of Dalla Pozza's system are exploited here in an essential way.

In our pursuit we are aware of a tradition of *dual intuitionistic logic* or *Heyting-Brouwer logic*, going back to Cecylia Rauszer's *Heyting-Brouwer algebras* [22, 23], which have been taken up in categorical logic by Lawvere, Makkai, Reyes and Zolfaghari[15, 24] and more recently reconsidered by R. Gore [11] and T. Crolard [5, 6] in proof-theory and theoretical computer science, and called *subtractive* or *bi-intuitionistic* (i.e., ordinary and *dual* intuitionistic) logic. A *co-Heyting algebra* is a (distributive) lattice C such that its opposite C^{op} is a Heyting algebra. In C^{op} the operation of Heyting implication $p \to q$ is defined by the familiar adjunction, thus in the co-Heyting

[3]For instance, there is still something to say even about the different proof-theoretic properties of the $(\ )^M$ and $(\ )^G$ translations.

algebra C *co-implication* (or *subtraction*) $c \smallsetminus b$ is defined dually[4]:

$$\frac{p \wedge q \ \leq \ r}{p \ \leq \ q \to r} \qquad\qquad \frac{c \ \leq \ b \vee a}{c \smallsetminus b \ \leq \ a}$$

In this tradition the crucial move has been to consider *bi-Heyting* algebras, which have both the structure of Heyting and co-Heyting algebras. The topological models of Heyting-Brouwer logic are bi-topological spaces, but every bi-topological space consists of the final sections of some preorder; the categorical models are bi-Cartesian closed categories (with co-exponents), but unfortunately these collapse to partial orderings (see [6] for a clear summary of the matter). Since in a bi-CCC (with co-exponents) for every pair of objects A, B, $Hom(A, B)$ has at most one element, in such a categorical model of Heying-Brouwer logic it is impossible to define a sensible notion of *identity of proofs*.

In the framework of the logic for pragmatics, the identification of the objects of Heyting and co-Heyting algebras must be resisted: we look for systems which allow us to represent the interaction of the two structures while keeping them separated. Henceforth we shall write υ for formulas that are conjectures or result from conjectural connectives and ϑ for assertions or result from assertive connectives; let δ be either υ or ϑ.

In a Kripke model for bi-Heyting algebras, subtraction is interpreted in the dual preordering, thus $c \smallsetminus b$ is true at some world w if there is a world w' in the *past* of w such that c is true and b false at w' (but remember that in such a Kripke model, the valuations of atoms are still persistent in the future!). Let us write "$\boxminus$" and "$\ominus$" for the necessity and possibility operators evaluated in the past (dual preordering), as in (classical) *temporal* **S4**: thus the modal translation $(\ )^*$ into temporal **S4** yields [5]

$$(\upsilon_1 \smallsetminus \upsilon_2)^* = \ominus(\upsilon_1^* \wedge \neg\upsilon_2^*).$$

The question now arises *what shall be the modal translation of an elementary act of conjecture* $\mathcal{H}\,\alpha$? Following an "inductivist instinct", we may think that a necessary condition for conjecturing α is the fact of always having seen α true in the past up to now: if we decide

[4]The overloading of symbols here will not create confusions between meet, join, Heyting implication on one hand and the classical connectives on the other.

[5]Obviously here $\wedge$ and $\neg$ are the connectives of classical logic.

that $(\mathcal{H}\,\alpha)^* = \boxminus\,\alpha$, then $(\vdash\alpha)^* = \Box\alpha$ does not imply $(\mathcal{H}\,\alpha)^*$ and thus our principle (1) fails for this interpretation.

If by analogy with the treatment of subtraction we decide that $(\mathcal{H}\,\alpha)^* = \diamondminus\,\alpha$, then principle (1) is fulfilled; but if in a given situation $\vdash\alpha$ is justified, then there is a Kripke model in which α is true in all possible worlds *now and in the future*, but perhaps not *in the past*. As in that situation we would not be entitled to regard the conjecture $\mathcal{H}\,\neg\alpha$ as unjustified, our principle (2) fails for this choice[6].

Thus the only extension of Gödel's, McKinsey and Tarski's and Kripke's modal interpretation of intuitionistic logic which is consistent with our principles is the one which translates both assertions and conjectures in *tenseless* **S4**; let

$$(\mathcal{H}\,\alpha)^M = \Diamond\alpha.$$

In this way the truth of $(\vdash\alpha)^M = \Box\alpha$ in a given state of knowledge certainly entails that of $(\mathcal{H}\,\alpha)^M = \Diamond\alpha$, as required by principle (1); also the truth of $\Box\alpha$ in a given state of knowledge entails the falsity of $(\mathcal{H}\,\neg\alpha)^M = \Diamond\neg\alpha$, as required by (2). It should be stressed that no assumption of monotonicity or persistence in the future of the valuations is made here; however, it does follow from the definition of $(\mathcal{H}\,\alpha)^M$ that conjecturability of α is persistent in the past.

Our choice of the modal translation entails the principle that in any given situation, the grounds on which a conjecture $\mathcal{H}\,\alpha$ is regarded as unjustified are necessary and sufficient to justify the assertion $\vdash\neg\alpha$. This is an identification that some may find too strong and unintuitive. In the point of view adopted here, the duality between assertions and conjectures is related to that between necessity and possibility in **S4** and the above identification is forced upon us by the choice of the modal translation. We may write

$$(\vdash p)^\perp \;=\; \mathcal{H}\,\neg p \qquad \text{and} \qquad (\mathcal{H}\,p)^\perp \;=\; \vdash\neg p$$

It follows from our previous discussion that we have *assertive* connectives of *implication* "$\supset$", *conjunction* "$\cap$" and *disjunction* "$\cup$" with Heyting algebra structure, on one hand, and the *conjectural*

[6] Notice however that if a *proof* of $\vdash\alpha$ entails the *validity* of $\Box\alpha$ in temporal **S4** (under an assumption of sondness of the deductive system in which the proof is presented) then there is no model of temporal **S4** in which $\diamondminus\,\alpha$ is true. Hence this stricter interpretation in temporal **S4** does support a correspondence between proofs of $\vdash\alpha$ and refutations of $\mathcal{H}\,\neg\alpha$.

connectives of *subtraction* "$\smallsetminus$" (where "$\upsilon_1 \smallsetminus \upsilon_2$" is to be read "*perhaps* υ_1 *and not* υ_2"), *weak disjunction* "$\curlyvee$" and *weak conjunction* "$\curlywedge$" with co-Heyting algebra structure, on the other; to these we add an assertive *strong negation* "$\smallsetminus\!\!\smallsetminus$" and a conjectural *weak implication* "$\succ$".

The question arises, what shall be modal interpretation of our *dual* intuitionistic logic of conjectures. Here we explore the possibility of translating both assertive and conjectural connectives into to *non-temporal* (*tenseless*) **S4**, e.g., letting

$$(\upsilon_1 \smallsetminus \upsilon_2)^M = \Diamond(\upsilon_1^M \wedge \neg\upsilon_2^M)$$

In this way our modal interpretation induces a notion of duality between assertions and conjectures which is *clearly different* from that in Rauszer's Heyting-Brouwer tradition. For instance, suppose that υ_i is unjustified iff ϑ_i is justified, i.e., $\vartheta_i = (\upsilon_i)^\perp$ for $i = 1, 2$. If $\upsilon_1 \smallsetminus \upsilon_2$ is unjustified in the present, then $\upsilon_1^M \wedge \neg\upsilon_2^M$ is false in all future possible worlds and this should be the condition for justifiedly asserting $\upsilon_2^\perp \supset \upsilon_1^\perp$, i..e, $\vartheta_2 \supset \vartheta_1$.

A corollary of this choice is the existence of *orthogonal negations*. As in bi-intuitionistic logic we can define two negations, namely, the usual intuitionistic negation "$\sim$" and the weak negation "$\frown$" (*perhaps not*), namely, $\frown \delta =_{df} \bigvee \smallsetminus \delta$, (where "$\bigvee$" is an illocutionary act which is always justified) [7]; here their modal interpretation is $(\sim \delta)^M = \Box\neg\delta^M$ and $(\frown \delta)^M = \Diamond\neg\delta^M$. It follows that in our logic

$$\sim\frown \vartheta \equiv \vartheta \qquad \text{and} \qquad \frown\sim \upsilon \equiv \upsilon$$

The fact that strong and weak negation may act as inverse operations[8] is a distinctive feature of the approach adopted here with respect to Rauszer's tradition and deserves an additional technical comment.

Let $\mathcal{F} = (W, R.S)$ be a bimodal frame, where R and S are preorders. The forcing conditions for $\Box\alpha$ and $\boxminus\,\alpha$ are given by

[7]In Rauszer's tradition it is customary to use "$\neg$" for the usual intuitionistic negation and "$\sim$" for "backwards looking" weak negation. For us "$\neg$" is taken up by classical negation and, obviously, our "$\frown$" is different from Rauszer's "$\sim$"; certainly future improvements on the notations are desirable.

[8]Strictly speaking, we have *four* negations, as $\frown\vartheta$ and $\sim\upsilon$ may be regarded as orthogonalities, internalizing the (metalinguistic) maps $(\ )^\perp$. We reserve the orthogonality sign for the metalinguistic consideration of the fragments of our language which exclude mixed connectives.

$w \Vdash \Box\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that wRw';
$w \Vdash \boxminus\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that wSw'.

As R and S are preorders,

$$\Box \boxminus \Box\alpha \to \Box\alpha \qquad \text{and} \qquad \diamondminus\alpha \to \diamondminus\Diamond\diamondminus\alpha$$

are certainly valid in $\mathcal{F}$. It is easy to see that

$$(1) \qquad \Box\alpha \to \Box\boxminus\Box\alpha \qquad \text{and} \qquad (2) \qquad \diamondminus\Diamond\diamondminus\alpha \to \diamondminus\alpha$$

are valid in every Kripke model over $\mathcal{F}$ if and only if $R = S$. Indeed if S is not a subset of R then, given wSv and not wRv, we set $w' \Vdash p$ for all w' such that wRw' and $v \nVdash p$: thus (1) with $\alpha = p$ is false in w. Similarly, if R is not a subset of S then, given wRv but not wSv, we set $w' \nVdash p$ for all w' such that wSw' and $v \Vdash p$: thus (2) with $\alpha = p$ is false in w. Thus if (1) and (2) are valid in $\mathcal{F}$, then $R = S$. The converse is obvious.

The intuitionistic logic of assertions and conjectures whose modal translation is into *tenseless* **S4** (i.e., on preordered bimodal frames with $R = S$) will be called **ILP**. Thus the axioms[9]

$$(1) \qquad \vartheta \Rightarrow \sim\frown\vartheta \qquad \text{and} \qquad (2) \qquad \frown\sim\upsilon \Rightarrow \upsilon$$

characterize the logic **ILP**. The intuitionistic logic of assertions and conjectures modally interpreted over *temporal* **S4** (i.e., over bimodal frames with $S = R^{-1}$) may be called *polarized bi-intuitionistic logic* (**PBL**), in view of the fact that the modal interpretation of subtraction proper of the tradition of bi-intuitionistic logic is retained, but formulas are polarized either as *assertive* or as *conjectural*, according to our third guiding principle. The logic **PBL** could be regarded as a *pragmatic interpretation* of Reuszer's Heyting-Brouwer logic. This paper is concerned only with the system **ILP**.

Negations are the first example of *mixed* connectives, operators taking conjectures or assumption or both as arguments and yielding conjectural or assertive statement. Our consideration of mixed connectives is a preliminary recognition of an unknown territory.

As noticed above, since orthogonality maps between conjectural and assertive formulas yield $(\vdash p)^{\perp} = \mathcal{H}\neg p$ and $(\mathcal{H} p)^{\perp} = \vdash\neg p$, they relate *classical negation* with the *pragmatic operators* of conjecture and assertion. This is not the only case of interaction between the

[9] Since $\vartheta^M \equiv \Box\vartheta^M$ and $\upsilon^M \equiv \Diamond\upsilon^M$, notice that $(\sim\frown\vartheta)^M = \Box\neg\diamondminus\neg\Box\vartheta^M$ and $(\frown\sim\upsilon)^M = \diamondminus\neg\Box\neg\diamondminus\upsilon^M$.

two levels; in all the following pairs, the formulas have equivalent **S4** translation.

$$\frac{\vdash \neg\alpha}{\sim \mathcal{H}\,\alpha} \qquad \frac{\mathcal{H}\neg\alpha}{\frown\!\vdash \alpha} \qquad \frac{\mathcal{H}\,(\alpha \to \beta)}{\vdash \alpha \succ \mathcal{H}\beta} \qquad \frac{\vdash (\alpha \wedge \neg\beta)}{\vdash \alpha \mathbin{\diagdown\!\!\diagdown} \mathcal{H}\beta} \qquad \frac{\vdash (\alpha \wedge \beta)}{\vdash \alpha \cap \vdash \beta} \qquad \frac{\mathcal{H}\,(\alpha \vee \beta)}{\mathcal{H}\alpha \curlyvee \mathcal{H}\beta}$$

Therefore, for a small fragment of classical logic, reasoning which makes reference to principles of classical semantics may be lifted to intuitionistic pragmatic reasoning, and conversely, intuitionistic reasoning with pragmatic expressions may be expressed as an assertion or conjecture of a classical propositional formula (without **S4** modalities).

1.2. Potential intuitionism. Another philosophical question concerning the justification of judgements should be mentioned here, which has recently been raised by Martino and Usberti ([19], pag. 83) in a discussions of the intuitionistic philosophy of mathematics. Can we say that proofs have a *potential existence*, where "*possibility* is not understood in the traditional intuitionistic sense as knowledge of a method" to produce such a proof, but as "knowledge-independent and tenseless" possibility? Dag Prawitz accepts this possibility:

> "That we can prove A is not to be understood as meaning that it is within our practical reach to prove A, but only that it is possible in principle to prove A. ... Similarly, that there exists a proof of A does not mean that a proof of A will be constructed but only that the possibility is there for constructing a proof of A. ... I see no objection to conceiving the possibility that there is a specific method for curing cancer, which we may discover one day, but which may also remain undiscovered." ([21], pag. 153-154)

Martino and Usberti use the expression *"potential intuitionism"* to indicate the point of view of an intuitionist who believes that proofs have a potential existence independently of our present knowledge, and *"orthodox intuitionism"* for the view that there are no potential proofs. Martino and Usberti seem to hold that for an "orthodox intuitionist" intuitive proofs are nothing but acts of knowing, whose aim is to make a judgement evident and which have no ontological status, not unlike *free choice sequences*, which have no tenseless identity independently of the acts of choice constituting them.

Martino and Usberti claim that the point of view of potential intuitionism inevitably entails a compatibilist philosophy with respect to classical logic:

> "once a tenseless notion of provability has been espoused, the commitment to an objective realm of *propositions* is unavoidable. For, if the possibility to prove a proposition A is conceived as atemporal, then A itself becomes an atemporal entity." ([19], pag.84).

where "proofs and propositions have atemporal existence" means

> "the existence of a proof and of a proposition is independent of the contingent fact that in human history the proof has been found and the truth or falsity of the proposition has been recognized."

It follows that the potential intuitionist can *understand* the law of *potential excluded middle*

> "A is potentially true or A is not potentially true"

in its own framework and therefore reconstruct Tarski's truth definitions in it.

We cannot discuss Martino and Usberti's argument here. However, their characterization of *potential intuitionism* seems to fit Martin-Löf's point of view ([16, 17], see also [18]): what makes a judgement *"α true"* evident (and thus justified) is a proof t of α, where the proof is *reified*, so that it can be explicitly represented by the primitive expressions $t : \alpha$ of the formalism. It is a remarkable feature of his type theory that it axiomatizes an intuitionistic and predicative notion of what an informal proof is[10].

Is there such a thing as a *justification* of an impersonal act of conjecture $\varkappa\,\alpha$, where α is a mathematical statement, other than *the absence of a refutation* of α? If there is such a thing as a "positive" justification, can it be *inconclusive* evidence that we cannot prove the falsity of α? Even if we stick to the negative characterization, still we must explain what the "absence of a proof of the falsity of α" means in the context of the logic for pragmatics. Indeed our goal is to give a characterization of impersonal illocutionary acts in a logical theory, which should become the basis of a theory of speech acts by *relativization*.

How can we produce a *conclusive* justification of a mathematical conjecture $\varkappa\,\alpha$? This would seem very close to *proving that there can be no proof of* $\neg\alpha$, where the proof of this impossibility must also be of a mathematical nature. Now such a proof is already a

[10] It should also be mentioned that Martin-Löf does not include in his system the notion of a free choice sequence, which alone makes it possible to derive a contradiction from the law of excluded middle.

justification of the *assertion* $\sim\vdash \neg\alpha$. This notion of conjecture, which is already expressible in Dalla Pozza's original framework, is not dual to the notion of assertion, as we would like. But it is also contrary to intuition. Speaking at the very beginning of the 21st century, one is justified in conjecturing the truth or the falsity of (i) Goldbach's conjecture and also the truth or the falsity of (ii) the continuum hypothesis: *as a matter of fact*, as far as we know, nobody has produced a proof nor a refutation of (i) and also, thanks to Gödel and Cohen, we know that *there can be no proof nor any refutation* of (ii), unless we modify our current understanding of what a set is. However, it would seem odd to say that we have a *conclusive* justification for the conjectures in (ii) and *inconclusive* justification for those in (i). Indeed when a mathematical conjecture about (i) is performed, the aim of this act (its *perlocutionary* effect) is to produce the expectation that either a proof or a refutation can be found, and a conjecture about (ii) is now likely to include a proposal for new axioms of set theory. In any event, it seems that an act of conjecture should be regarded as felicitous even if the evidence in favour of its admissibility is inconclusive.

One may insist that if an *impersonal* act of conjecture $\mathcal{H}\,\alpha$ is justified, then it should remain justified when instantiated in any period of history: after all, the circumstances of the present time are relative to the persons now living. In this view to say that $\mathcal{H}\,\alpha$ is justified by the "absence of a proof" must mean that a proof of $\neg\alpha$ is nowhere to be found, either now, in the past or in the future. It seems that in this way one is taking a stand on the issue of the ontological status of potential profs.

If we take the view that potential proofs exist, we must ask whether there can be an *inconclusive* justification of $\mathcal{H}\,\alpha$. One could take the following as a definition: $\mathcal{H}\,\alpha$ is justified *inconclusively* if there is no proof of the truth of $\neg\alpha$ but also *no proof that there is no proof of* $\neg\alpha$. But in this case, conjecturing α would require that there is no proof of α, and then it would never be possible to improve our inconclusive conjecture $\mathcal{H}\,\alpha$ by giving a proof of the truth of α! Of course, this does not rule out the possibility of admitting potential proofs and at the same time give a logical status to conjecturing $\mathcal{H}\,\alpha$ with inconclusive justification: but certainly it shows that we need to avoid a way of thinking that leads to the above "definition".

A solution may come from an improved explanation of what it means to *assert* that α is true. We are inclined to say that the assertion of the truth of α is justified not merely by the *existence* of a proof of the truth of α, but *by the capacity to exhibit an actual proof* t *of* α: an act of assertion that α is true is felicitous if we can explicitly produce the pair $t : \alpha$. Our suggestion amounts to saying that a formal theory of assertions and conjectures should give formal status also to proofs. At present we cannot definitely characterize what constitutes inconclusive evidence for a justified impersonal act of conjecture; however by contrasting *conjectures* with *assertions* in the refined definition, we can conclude on one hand, that *conjecturing* is similar to *betting* and, on the other, that asserting α without knowing where to find a proof of α is very close to bad manners.

1.3. Heyting-Bouwer logic and proofs-in-time. Finally, our thought goes to Cecylia Rauszer and her tradition. There is here a notion of duality and an interesting mathematical theory which is not interpreted by our notion of duality between assertions and conjectures. Kripke models for Heyting-Brouwer logic are preorderings which cannot always be trees; thus although α may always be true in our past, present and future, there are many alternative possible histories of knowledge where α would not always be true. This picture is philosophically challenging: it may provide a more flexible framework to relativize the picture of a unique, inevitable progress to truth. Also it seems to incorporate a view which would not be incongenial to Brouwer, given that he would have denied tenseless existence to proofs. We cannot develop a *pragmatic interpretation* of Reuszer's Heyting-Brouwer logic here, but we do believe that the development suggested in this paper, i.e., keeping the structures of Heyting and co-Heyting algebras separate in their interaction, could help avoiding some shortcomings of the proof-theory and categorical logic of bi-Heyting algebras. Indeed, in this way the categorical interpretation of Heyting-Brouwer logic into bi-Cartesian closed categories becomes unnecessary, thus it becomes possible to axiomatize a sensible notion of *identity of proofs* in bi-intuitionistic logic.

§2. The pragmatic language $\mathcal{L}^P$.

DEFINITION 1. (*Syntax*) (i) The language of the *logic for pragmatics* $\mathcal{L}^P$ is based upon an infinite set of *propositional letters* p, p_0, p_1 .... The *radical formulas* are built up from propositional letters using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; the *elementary* *formulas* of the pragmatic language are obtained by prefixing a radical formula with a sign of *illocutionary force* "$\vdash$" and "$\mathcal{H}$" and also include the *elementary constants*, $\bigwedge$ and $\bigvee$, which stand for illocutionary acts which never and always justified, respectively. Finally, the *sentential formulas* of $\mathcal{L}^P$ are built from the elementary formulas, using the *pragmatic connectives* $\supset$, $\sim$, $\cap$, $\cup$, $\setminus\!\!\setminus$, $\setminus$, $\frown$, $\curlyvee$, $\curlywedge$ and $\succ$.

(ii) (*Formation Rules*) The *pragmatic* language $\mathcal{L}^P$ is the union of the sets **Rad** of *radical formulas* and **Sent** of *sentential formulas*. These sets are defined inductively by the following grammar:

$$\alpha \ := \ p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha$$

$$\delta \ := \ \vartheta \mid \upsilon$$

$$\vartheta \ := \vdash \alpha \mid \bigwedge \mid \bigvee \mid \sim \delta \mid \delta \supset \delta \mid \delta \cap \delta \mid \delta \cup \delta \mid \delta \setminus\!\!\setminus \delta$$

$$\upsilon \ := \mathcal{H}\, \alpha \mid \bigwedge \mid \bigvee \mid \frown \delta \mid \delta \setminus \delta \mid \delta \curlyvee \delta \mid \delta \curlywedge \delta \mid \delta \succ \delta$$

We use the letters α, β, α_1, ... to denote *radical* formulas, ϑ, ϑ_1, ... for *assertive expressions* and υ, υ_1, ... for *conjectural expressions*.

1. The *intuitionistic fragment* $\mathcal{L}^{IP}$ of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary sentences to those with *atomic radical* only, i.e., $\bigwedge$, $\bigvee$, $\vdash p_i$, and $\mathcal{H}\, p_i$.
2. We write $\mathcal{L}^{IP\pm}$ for the extension of $\mathcal{L}^{IP}$ with elementary formulas of the forms $\vdash \neg p_i$, $\mathcal{H}\, \neg p_i$.
3. In the language $\mathcal{L}^{IP}$ [or $\mathcal{L}^{IP\pm}$] let $\mathcal{L}^A$ [or $\mathcal{L}^{A\pm}$] be the set of expressions built up from assertive elementary ones using only assertive connectives and similarly let $\mathcal{L}^C$ [or $\mathcal{L}^{C\pm}$] be the set of expressions built up from conjectural elementary ones using only conjectural connectives.

DEFINITION 2. (*Informal Interpretation*) (i) *Radical* formulas are interpreted as propositions, with the usual classical semantics.

(ii) *Sentential expressions* ϑ and υ are interpreted as impersonal illocutionary acts of *assertion* and *conjecture*, respectively. Assertions can be *justified* or *unjustified*, conjectures can be *refuted* or *unrefuted* and we shall make the convention that conjectures are infelicitously made, i.e., *unjustified*, precisely when they are refuted, and thus conjectures are *justified* if they are unrefuted.

1. $\bigwedge$ is never *justified* and $\bigvee$ is always *justified*.
2. $\vdash \alpha$ is *justified* if and only if a proof can be exhibited that α is true. Dually, $\mathcal{H}\, \alpha$ is *refuted* is and only if a proof that α is false can be exhibited.

3. $\delta_1 \supset \delta_2$ is *justified* if and only if a proof can be exhibited that a justification of δ_1 can be transformed into a justification of δ_2; it is *unjustified*, otherwise. Dually, $\delta_1 \smallsetminus \delta_2$ is *refuted* if and only if a proof can be exhibited that a refutation of δ_2 can be transformed into a refutation of δ_2.
4. $\delta_1 \mathbin{\smallsetminus\!\!\smallsetminus} \delta_2$ is *justified* if and only if a proof can be exhibited that δ_1 is justified and δ_2 unjustified. Dually, $\delta_1 \succ \delta_2$ is *refuted* if and only if a proof can be exhibited that δ_1 is justified and δ_2 is unjustified.

The explanations of what it means for conjunctions $\vartheta_0 \cup \vartheta_1$ [$\upsilon_0 \curlywedge \upsilon_1$] and disjunctions $\vartheta_0 \cup \vartheta_1$ [$\upsilon_0 \curlyvee \upsilon_1$] to be *justified* [*refuted*] readily follow from conjunction and disjunction of clauses in the metatheory. But for a justification of $\upsilon \cup \vartheta$ a proof must be exhibited that υ is justified, together with a justification of ϑ; also for a refutation of $\upsilon \curlyvee \vartheta$ a proof must be exhibited that ϑ is unjustified, together with a refutation of υ; similarly for the other mixed cases. Finally $\sim \delta$ is defined as $\delta \supset \bigwedge$ and $\frown \delta$ is defined as $\bigvee \smallsetminus \delta$.

2.1. Topological interpretation. A mathematical model for the system $\mathcal{L}^P$ is obtained through a topological interpretation.

Definition 3. (*topological interpretation*). Let S be a set, let $\cap$, $\cup$ and $\setminus$ be the usual operations of intersection, union and (binary) complementation defined on the powerset $\wp(S)$ of S, let $(X)^C$ be $S \setminus X$ and let $\mathbf{I} : \wp(S) \to \wp(S)$ and $\mathbf{C} : \wp(S) \to \wp(S)$ be the *interior* and *closure* operators, satisfying

$$\begin{array}{rl}
\mathbf{I}(X) \subseteq X & X \subseteq \mathbf{C}(X) \\
\mathbf{I}(X) \subseteq \mathbf{I}(\mathbf{I}(X)) & \mathbf{C}(\mathbf{C}(X)) \subseteq \mathbf{C}(X) \\
X \subseteq Y \Rightarrow \mathbf{I}(X) \subseteq \mathbf{I}(Y) & X \subseteq Y \Rightarrow \mathbf{C}(X) \subseteq \mathbf{C}(Y) \\
\mathbf{C}(X) = (\mathbf{I}(X^C))^C & \mathbf{I}(X) = (\mathbf{C}(X^C))^C
\end{array}$$

A topological interpretation δ^* of the full language $\mathcal{L}^P$ is given by assigning to each atomic formula P a subset P^* of S and then by proceeding as follows:

$$\begin{array}{rcl@{\qquad}rcl}
(\bigvee)^* & =_{df} & S & (\bigwedge)^* & =_{df} & \emptyset \\
(\vdash \alpha)^* & =_{df} & \mathbf{I}(\alpha^*) & (\mathcal{H}\, \alpha)^* & =_{df} & \mathbf{C}(\alpha^*) \\
(\sim \delta)^* & =_{df} & \mathbf{I}((\delta^*)^C) & (\frown \delta)^* & =_{df} & \mathbf{C}((\delta^*)^C) \\
(\delta_1 \supset \delta_2)^* & =_{df} & \mathbf{I}((\delta_1^*)^C \cup \delta_2^*)) & (\delta_1 \smallsetminus \delta_2)^* & =_{df} & \mathbf{C}((\delta_1^*) \setminus \delta_2^*) \\
(\delta_1 \mathbin{\smallsetminus\!\!\smallsetminus} \delta_2)^* & =_{df} & \mathbf{I}((\delta_1^*) \setminus \delta_2^*)) & (\delta_1 \succ \delta_2)^* & =_{df} & \mathbf{C}((\delta_1^*)^C \cup \delta_2^*) \\
(\delta_1 \cap \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cap \mathbf{I}(\delta_2^*) & (\delta_1 \curlyvee \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cup \mathbf{C}(\delta_2^*) \\
(\delta_1 \cup \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cup \mathbf{I}(\delta_2^*) & (\delta_1 \curlywedge \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cap \mathbf{C}(\delta_2^*)
\end{array}$$

2.2. Modal interpretation. Another mathematical interpretation is obtained through an extension of Gödel, McKinsey and Tarski's

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \Diamond p & & \\
 & & & \Diamond\Box\Diamond p \arrow[ul] & \\
p \arrow[uurr] & & \Diamond\Box p \arrow[ur] & & \Box\Diamond p \arrow[ul] \\
 & & & \Box\Diamond\Box p \arrow[ul] \arrow[ur] & \\
 & & \Box p \arrow[uull] \arrow[ur] & &
\end{tikzcd}
$$

TABLE 1. The modalities of **S4**

modal translation $(\ )^{\Box}$ into the logic **S4**. The language of **S4**, Kripke's semantics and sequent calculus for it are in the Appendix. The language $\mathcal{L}^P$ is translated into **S4** as follows:

DEFINITION 4. (**S4** *translation*)

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigvee)^M & =_{df} & \top & (\bigwedge)^M & =_{df} & \bot \\
(\vdash \alpha)^M & =_{df} & \Box\alpha & (\mathcal{H}\,\alpha)^M & =_{df} & \Diamond\alpha \\
(\sim \delta)^M & =_{df} & \Box\neg\delta^M & (\frown \delta)^M & =_{df} & \Diamond\neg\delta^M \\
(\delta_1 \supset \delta_2)^M & =_{df} & \Box(\delta_1^M \to \delta_2^M) & (\delta_1 \smallsetminus \delta_2)^M & =_{df} & \Diamond(\delta_1^M \wedge \neg\delta_2^M) \\
(\delta_1 \mathbin{\setminus\!\!\setminus} \delta_2)^M & =_{df} & \Box(\delta_1^M \wedge \neg\delta_2^M) & (\delta_1 \succ \delta_2)^M & =_{df} & \Diamond(\delta_1^M \to \delta_2^M) \\
(\vartheta_1 \cap \vartheta_2)^M & =_{df} & \vartheta_1^M \wedge \vartheta_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & =_{df} & \upsilon_1^M \vee \upsilon_2^M \\
(\vartheta_1 \cup \vartheta_2)^M & =_{df} & \vartheta_1^M \vee \vartheta_2^M & (\upsilon_1 \curlywedge \upsilon_2)^M & =_{df} & \upsilon_1^M \wedge \upsilon_2^M \\
\text{If } \delta_i \text{ is an } \upsilon, & \text{for} & i = 1 \text{ or } 2, \text{ then} & \text{If } \epsilon_i \text{ is a } \vartheta & \text{for} & i = 1 \text{ or } 2, \text{ then} \\
(\delta_1 \cap \delta_2)^M & =_{df} & \Box\delta_1^M \wedge \Box\delta_2^M & (\epsilon_1 \curlyvee \epsilon_2)^M & =_{df} & \Diamond\epsilon_1^M \vee \Diamond\epsilon_2^M \\
(\delta_1 \cup \delta_2)^M & =_{df} & \Box\delta_1^M \vee \Box\delta_2^M & (\epsilon_1 \curlywedge \epsilon_2)^M & =_{df} & \Diamond\epsilon_1^M \wedge \Diamond\epsilon_2^M
\end{array}
$$

2.3. Dualities. Notice that of the seven modalities of **S4** (see Table 1) only three are expressible in usual intuitionistic logic, namely

$$(\vdash p)^{\Box} \ = \ \Box p \qquad (\sim\sim \vdash p)^{\Box} \ = \ \Box\Diamond\Box p \qquad (\sim \vdash \neg p)^{\Box} \ = \ \Box\Diamond p$$

In $\mathcal{L}^{IP\pm}$ there is a counterpart to three other modalities of **S4**:

$$(\mathcal{H}\,p)^M \ = \ \Diamond p \qquad (\frown\frown \mathcal{H}\,p)^M \ = \ \Diamond\Box\Diamond p \qquad (\frown \mathcal{H}\,\neg p)^M \ = \ \Diamond\Box p$$

The above topological and modal interpretations suggest the definition of the following involutory maps between $\mathcal{L}^{A\pm}$ and $\mathcal{L}^{C\pm}$. (Remember that these are the assertive and the conjectural part of the pragmatic language $\mathcal{L}^{IP\pm}$, the fragment of $\mathcal{L}^P$ where the radical parts are negated or non-negated atoms.)

DEFINITION 5. (*Duality*)

$$
\begin{array}{ccccc}
 & & \mathcal{H}p & & \\
 & \nearrow & & \nwarrow & \\
 & & & & \frown\frown\mathcal{H}\,p \\
 & & & \nearrow & & \nwarrow \\
p & & & \frown\mathcal{H}\,\neg p & & \sim\vdash\neg p \\
 & & & \nwarrow & & \nearrow \\
 & \nwarrow & & & \sim\sim\vdash p \\
 & & & \nearrow & \\
 & & \vdash p & &
\end{array}
$$

TABLE 2. Asserting and conjecturing

$$
\begin{array}{lllllll}
(\vdash p)^{\perp} & =_{df} & \mathcal{H}\neg p & \qquad & (\mathcal{H}\,p)^{\perp} & =_{df} & \vdash\neg p \\
(\bigvee)^{\perp} & =_{df} & \bigwedge & & (\bigwedge)^{\perp} & =_{df} & \bigvee \\
(\vartheta_1 \supset \vartheta_2)^{\perp} & =_{df} & \vartheta_2^{\perp} \smallsetminus \vartheta_1^{\perp} & & (\upsilon_1 \smallsetminus \upsilon_2)^{\perp} & =_{df} & \upsilon_2^{\perp} \supset \upsilon_1^{\perp} \\
(\vartheta_1 \lessdot \vartheta_2)^{\perp} & =_{df} & \vartheta_2^{\perp} \succ \vartheta_1^{\perp} & & (\upsilon_1 \succ \upsilon_2)^{\perp} & =_{df} & \upsilon_2^{\perp} \lessdot \upsilon_1^{\perp} \\
(\vartheta_1 \cap \vartheta_2)^{\perp} & =_{df} & \vartheta_1^{\perp} \curlyvee \vartheta_2^{\perp} & & (\upsilon_1 \curlyvee \upsilon_2)^{M} & =_{df} & \upsilon_1^{\perp} \cap \upsilon_2^{\perp} \\
(\vartheta_1 \cup \vartheta_2)^{\perp} & =_{df} & \vartheta_1^{\perp} \curlywedge \vartheta_2^{\perp} & & (\upsilon_1 \curlywedge \upsilon_2)^{\perp} & =_{df} & \upsilon_1^{\perp} \cup \upsilon_2^{\perp}
\end{array}
$$

LEMMA 1. *For all $\vartheta \in \mathcal{L}^{A\pm}$ and all $\upsilon \in \mathcal{L}^{C\pm}$, we have*

$$(\vartheta)^{\perp M} = \neg(\vartheta^M) \qquad \textit{and} \qquad (\upsilon)^{\perp M} = \neg(\upsilon^M).$$

Clearly $(\vdash p)^{\perp M} = \Diamond\neg p = \neg\Box p = \neg(\vdash p)^M$. Assuming $\vartheta_i^{\perp M} = \neg\vartheta_i^M$, we have $(\vartheta_1 \supset \vartheta_2)^{\perp M} = (\vartheta_2^{\perp} \smallsetminus \vartheta_1^{\perp})^M = \Diamond(\vartheta_2^{\perp M} \wedge \neg\vartheta_1^{\perp M}) = \Diamond(\vartheta_1^M \wedge \neg\vartheta_2^M) = \neg\Box(\vartheta_1^M \to \vartheta_2^M) = \neg(\vartheta_1 \supset \vartheta_2)^M$, and similarly for all inductive cases.

§3. Sequent calculus for the logic of pragmatics. The sequent calculus for the logic of pragmatics presented here is gigantic; it includes two very familiar systems, a couple of intriguing fragments and also a wilder bunch of *mixed* assertive and conjectural rules.

1. Technically, the classical logic of proposition seems to live a life of its own in the underlying *radical part* of the system. For a "descriptive interpretation" of assertive and conjectural expressions we need to extend classical logic to an **S4** modal system, which is formalized by a standard Gentzen system with a complete semantic tableax procedure, briefly summarized in Appendix I. Modal expressions of **S4** are not considered here as radical parts of other assertive and conjectural expressions, although this would certainly be possible.

2. The intuitionistic logic of assertions and conjectures is formalized by a sequent calculus **ILP**, which contains only rules for the pragmatic connectives, leaving the radical part constant.
3. The interaction between the radical part (logic of propositions) and the pragmatic level (logic of judgement) is studied for a fragment of the classical language in a system which we shall call *polarized classical* **CLP**. The calculus is classical in the sense that it expresses interactions of the pragmatic level with the underlying classical logic. Classical propositions α (possibly containing classical connectives) are *polarized* positively or negatively, as follows: positive formulas are conjunctions of positive formulas and negative formulas are disjunctions of negative formulas, starting from positive and negative atoms, while negation exchanges polarity. If a classical α formula can be polarized positively, then there is an intuitionistic assertive formula ϑ such that $\Rightarrow \vdash \alpha;$ is provable in **CLP** if and only if $\Rightarrow \vartheta;$ is provable in **ILP**; similarly, if α can be polarized negatively then there exists a conjectural υ such that $\Rightarrow; \mathcal{H}\, \alpha$ is provable in **CLP** if and only if $\Rightarrow; \upsilon$ is provable in **ILP**.

The calculus **ILP** for the Intuitionistic Logic for Pragmatics is a system of type **G3i** in the classification of Gentzen systems by Troestra and Schwichtenberg [27], where the rules of weakening and contraction are *implicit*. Gentzen's familiar restriction for intuitionistic sequents is generalized, by using sequents with privileged areas in the antecedent and in the succedent and by requiring that each sequent must contain at most one privileged formula. The inspiration here is Girard's *unitary* system [9], which contains fragments for the formalization of classical and intuitionistic reasoning, although our motivations are very different. The cut-elimination theorem for **ILP** has been proved by [4], where the standard proofs of admissibility of depth-preserving contraction, depth-preserving weakening and context-sharing cut for **G3i** systems (cf. [27], chapters 3 and 4) are extended to **ILP**.

Within the intuitionistic sequent calculus **ILP** in (2) we may distinguish the following:

(i) its restriction to the assertive fragment $\mathcal{L}^A$ of the pragmatic language $\mathcal{L}^P$, which is just the intuitionistic propositional calculus;
(ii) its restriction to the conjectural fragment $\mathcal{L}^C$, which is part of a proof-theoretic account of co-Heyting algebras.

(iii) The remaining part of **ILP** (*mixed* assertive and conjectural) attempts a more general characterization of forms of reasoning combining assertions and conjectures.

DEFINITION 6. All the sequents S are of the form

$$\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon$$

where

- Θ is a sequence of assertive formulas $\vartheta_1, \ldots, \vartheta_m$;
- Υ is a sequence of conjectural formulas $\upsilon_1, \ldots, \upsilon_n$;
- ϵ is conjectural and ϵ' is assertive and at most one of ϵ, ϵ' occurs in S.

The rules of **ILP** are given in the Appendix II. The main result of this paper is the following theorem:

THEOREM 1. *The intuitionistic sequent calculus* **ILP** *without the rules of cut is sound and complete for Kripke's semantics over preordered frames (i.e., the modal interpretation in* **S4** *is sound and faithful). The finite model property holds for* **ILP**.

In order to prove the completeness theorem for **ILP**, we reduce the problem to the completeness of **S4** and use the "semantic tableaux" procedure for **S4** given in Appendix I. More precisely, given an **ILP** sequent S of the form $\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon$ we consider its modal translation S^M, namely $\Theta^M, \epsilon^M \;\Rightarrow\; \epsilon'^M, \Upsilon^M$, and apply the "semantic tableaux procedure" to S^M. If S^M is falsifiable, in a finite number of steps the procedure yields a Kripke model $\mathcal{M}$ on a preordered frame which falsifies S^M, and it is regarded as a countermodel for S. Otherwise, S^M is derivable in the sequent calculus for **S4** and we must show that S is derivable in **ILP**. We find it convenient to introduce an auxiliary system **FILP** equivalent to **ILP** and to prove that if S^M is derivable in the sequent calculus for **S4** then S is derivable in **FILP**.

§4. FILP. The auxiliary system **FILP** of Full Intuitionistic Logic of Pragmatics generalizes intuitionistic sequent calculi with multiple succedent, such as the systems **G3i**m in [27] or the logic **FILL** (*Full Intuitionistic Linear Logic*) by De Paiva and others (from which we take the acronym). As **FILL** relaxes the intuitionistic restriction on the succedent, so in **FILP** the distinction between two areas in the antecedent and sucedent of sequents is removed and the restriction on the pair ϵ, ϵ' is relaxed whenever this is possible from a logical point of view. In this way, **FILP** retains exactly those restrictions on the sequent-premises S of its rules which are needed for S^M to preserve the restrictions on the modal inferences $\Box$-R and $\Diamond$-L of **S4**. Because of its closeness to sequent calculus for **S4**, the system **FILP** may have an independent interest in the logic for pragmatics.

The rules of the sequent calculus **FILP** are given in Appendix III. In our tables for **ILP** and **FILP**, the rules marked with an asterisk **(*)** are those for which it is not possible to relax the restriction on the sequent premises.

LEMMA 2. *A sequent* $\Theta, \Upsilon' \Rightarrow \Theta', \Upsilon$ *is derivable in* **FILP** *(without cut) if and only if* $\vdash_{S4} \Theta^M, \Upsilon'^M \Rightarrow \Theta'^M, \Upsilon^M$ *is derivable in* **S4** *(without cut).*

The *"only if"* part is left to the reader. To prove the *"if"* part, let d be a derivation in **S4** of a sequent S^M, where S is a **FILP** sequent. Given a sequent derivation d and a formula-occurrence α in a sequent S in d we can define the notion of *ancestor* [*descendant*] of α in d as usual and so it is clear what it means to say that a formula β in a sequent S is traceable to a formula α in a sequent S', when S' occurs above S. To simplify the proof we make some assumptions on the structure of d which are summarized in the following proposition.

PROPOSITION 1. *Let S be a* **FILP** *sequent. If S^M is derivable in the sequent calculus for* **S4***, then there exists a derivation d of S^M with the following properties:*

(a) *Let $\mathcal{I}$ be an application of $\vee$-L [$\wedge$-R]. If the principal formula of $\mathcal{I}$ is $\Box\gamma_1 \vee \Box\gamma_2$ [$\Diamond\gamma_1 \wedge \Diamond\gamma_2$], then the inference immediately above $\mathcal{I}$ on both branches is $\Box$-L [$\Diamond$-R] with principal formula the active formula of $\mathcal{I}$.*
Similarly, let $\mathcal{I}$ be an application of $\wedge$-L [$\vee$-R]. If the principal formula of $\mathcal{I}$ is $\Box\gamma_1 \wedge \Box\gamma_2$ [$\Diamond\gamma_1 \vee \Diamond\gamma_2$], then the two inferences immediately above $\mathcal{I}$ are applications of $\Box$-L [$\Diamond$-R] and descendants of their principal formulas are active in $\mathcal{I}$.

(b) *Let $\mathcal{I}$ be an application of $\Box$-L [$\Diamond$-R] and let $\beta = \neg\gamma$ or $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$. If the principal formula of $\mathcal{I}$ is $\Box\beta$ [$\Diamond\beta$], then the inference $\mathcal{I}'$ immediately above $\mathcal{I}$ is an application of $\neg$-L or $\to$-L or $\wedge$-L immediately below an inference $\neg$-L [$\neg$-R or $\to$-R or $\wedge$-R immediately below a $\neg$-L] respectively, and the principal formula of $\mathcal{I}'$ is the active formula β of $\mathcal{I}$.*

(c) *Let $\mathcal{I}$ be an application of $\Box$-R [$\Diamond$-L] and let $\beta = \neg\gamma$ or $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$. If the principal formula of $\mathcal{I}$ is $\Box\beta$ [$\Diamond\beta$], then the inference $\mathcal{I}'$ immediately above $\mathcal{I}$ is an application of $\neg - R$ or $\to$-R or $\wedge$-R immediately below an inference $\neg - L$ [$\neg$-L or $\to$-L or $\wedge$-L immediately below a $\neg$-L] respectively, and the principal formula of $\mathcal{I}'$ is the active formula β of $\mathcal{I}$.*

(d) *Let $\mathcal{I}$ be an application of $\vee$-R, $\wedge$-R, $\wedge$-L, $\vee$-L with principal formula β of the form*
(I) $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ *in the antecedent or* $\Box\gamma_0 \vee \Box\gamma_1$ *in the succedent;*
(II) $\Diamond\gamma_0 \vee \Diamond\gamma_1$ *in the antecedent or* $\Box\gamma_0 \wedge \Box\gamma_1$ *in the succedent.*
Then the sequent-conclusion of $\mathcal{I}$ has the form

$$\Pi, \underline{\Box\Gamma}, \Diamond\Delta', \Lambda \;\Rightarrow\; \Lambda', \Box\Gamma', \underline{\Diamond\Delta}, \Pi'$$

where Π, Π' are pairwise disjoint sequences of atoms and where Λ and Λ' are sequences of formulas of the form (I) or (II).

The proof of the proposition can be obtained by implementing conditions (a), (b), (c) and (d) as a search-strategy in the "semantic tableaux" procedure.

If d is a sequent derivation, the *size* $s(d)$ of d is 1 plus the number of inferences in d (not counting exchange and weakening rules). The proof of the lemma is by induction on the size of the given derivation d of S^M in **S4**, assumed to satisfy conditions (a), (b) (c) and (d) of the Proposition; in the proof we construct a **FILP** derivation d^- of S. We consider the last inference of d, having classified the inferences in four cases, we indicate how to prove the inductive step in each case and give all details only for some example.

Case 0. If a sequent S^M is an axiom of one of the forms

$$\Gamma, \Box\alpha \Rightarrow \Box\alpha, \Delta \quad \text{or} \quad \Gamma, \Diamond\alpha \Rightarrow \Diamond\alpha, \Delta \quad \text{or} \quad \Gamma, \bot \Rightarrow \Delta \quad \text{or} \quad \Gamma \Rightarrow \Delta, \top$$

where Γ and Δ are translations of $\mathcal{L}^P$ formulas, then S is a logical axiom or an absurdity or validity axiom, respectively, of **FILP**. If S^M has the form $\Gamma, \Box\alpha \Rightarrow \Diamond\alpha, \Delta$ then S is an assumption-conjecture axiom of **FILP**.

Otherwise the derivation d has size greater than 1 and we consider the last inference $\mathcal{I}$ of d. There are four cases:

Case 1. *Propositional* **S4** *rules corresponding to invertible pragmatic rules.* This case excludes inferences with principal formula $\Box\gamma_0 \vee \Box\gamma_1$ or $\Box\gamma_0 \wedge \Box\gamma_1$ *in the succedent* or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ or $\Diamond\gamma_0 \vee \Diamond\gamma_1$ *in the antecedent*: for instance, the rule corresponding to an inference $\vee$-R with principal formula $\Box\gamma_0 \vee \Box\gamma_1$ is a right mixed assertive disjunction $\cup$-R which is non-invertible.

Subcase 1.1. If the last inference $\mathcal{I}$ has principal formula $\vartheta_0^M \wedge \vartheta_1^M$, $\vartheta_0^M \vee \vartheta_1^M$ $\upsilon_0^M \wedge \upsilon_1^M$ or $\upsilon_0^M \vee \upsilon_1^M$, then the sequent-premises are also translations of a **FILP** sequent and we build the derivation d^- by applying

- either an assertive rule $\cap$-R, $\cap$-L, $\cup$-R, $\cup$-L;
- or a conjectural rule $\curlywedge$-R, $\curlywedge$-L, $\curlyvee$-R, $\curlyvee$-L.

Subcase 1.2. Suppose the last inference $\mathcal{I}$ has principal formula $\Box\gamma_0 \wedge \Box\gamma_1$ or $\Box\gamma_0 \vee \Box\gamma_1$ in the antecedent $\Diamond\gamma_0 \vee \Diamond\gamma_1$ or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ in the succedent.

If the last inference $\mathcal{I}$ is $\vee$-L, then by clause (a) of the Proposition d has the form

$$\dfrac{\Box\text{-L}\,\dfrac{\overset{d_{1,1}}{\Theta^M, \Upsilon'^M, \gamma_0, \Box\gamma_0 \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_0 \Rightarrow \Upsilon^M} \qquad \Box\text{-L}\,\dfrac{\overset{d_{2,1}}{\Theta^M, \Upsilon'^M, \gamma_1, \Box\gamma_1, \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_1, \Rightarrow \Upsilon^M}}{\Theta^M, \Upsilon'^M, \Box\gamma_0 \vee \Box\gamma_1, \Rightarrow \Upsilon^M}\,\vee\text{-L}$$

Let d_1 and d_2 the immediate subderivations of d. By applying $\vee$-L to the sequent-conclusions of $d_{1,1}$ and d_2 we derive a sequent which is translation of

$$S_1 : \ \Theta, \Upsilon', \delta_0, \delta_0 \cup \delta_1 \Rightarrow \Upsilon$$

letting $\delta_i^M = \gamma_i$. Moreover $s(d_{1,1}) + s(d_2) + 1 < s(d_1) + s(d_2) + 1 = s(d)$ thus we may apply the induction hypothesis and obtain a derivation of S_1. In a similar way we obtain a derivation of

$$S_2 : \ \Theta, \Upsilon', \delta_1, \delta_0 \cup \delta_1 \Rightarrow \Upsilon$$

We build the derivation d^- by applying

- a mixed assertive rule $\cup$-L.

The cases when $\mathcal{I}$ is a $\wedge$-L with principal formula $\Box\gamma_0 \wedge \Box\gamma_1$ or a $\vee$-R [or $\wedge$-R] with principal formula $\Diamond\gamma_0 \vee \Diamond\gamma_1$ [or $\Diamond\gamma_0 \wedge \Diamond\gamma_1$] are similar and dealt with by an application of

- a mixed assertive rule $\cap$-L,
- a mixed conjectural rule $\curlywedge$-R [or $\curlyvee$-R].

Case 2. *Modal rules $\Box L$ or $\Diamond R$ corresponding to invertible pragmatic rules.* The principal formula of such an inference $\mathcal{I}$ is either $\Box\beta$ in the antecedent or $\Diamond\beta$ in the succedent, where β is $\neg\gamma$, $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$ and where γ, γ_1 and γ_2 are translations of $\mathcal{L}^P$ formulas.

Suppose $\Diamond\beta = \Diamond(\delta_1^M \wedge \neg\delta_2^M)$. By clause (b) in the Proposition, d has the form

$$\wedge\text{-R}\ \dfrac{\begin{array}{c} d_{1,1} \\ \Gamma \Rightarrow \delta_1^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta \end{array} \qquad \neg\text{-R}\ \dfrac{\begin{array}{c} d_{1,2} \\ \delta_2^M, \Gamma, \Rightarrow \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta \end{array}}{\Gamma, \Rightarrow \neg\delta_2^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}}{\Diamond\text{-R}\ \dfrac{\Gamma \Rightarrow \delta_1^M \wedge \neg\delta_2^M, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}{\Gamma \Rightarrow \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Delta}}$$

where $\Gamma = \Theta^M$, Υ'^M and $\Delta = \Theta'^M$, Υ^M. The endsequents of $d_{1,1}$ and of $d_{1,2}$ are translations of **FILP** sequents and $s(d_{1,1}) < s(d)$, $s(d_{1,2} < s(d)$ hence we can apply the inductive hypothesis and obtain the desired derivation d^- by applying $\setminus$-R.

If the principal formula of $\mathcal{I}$ has another form $\Box\beta$ to the left or $\Diamond\beta$ to the right, we proceed in a similar way, using

- either the assertive rules $\sim$-L, $\supset$-L, $\setminus\!\!\setminus$-L;
- or the conjectural rules $\frown$-R, $\succ$-R, $\setminus$-R;
- or the mixed assertive rules $\sim$-L, $\supset$-L, $\setminus\!\!\setminus$-L;
- or the mixed conjectural rules $\frown$-R, $\sim$-R, $\setminus$-R.

Case 3. *Modal rules* $\Box$-R *or* $\Diamond$-L *corresponding to non-invertible pragmatic rules.* The principal formula of such an inference $\mathcal{I}$ is either $\Box\beta$ in the succedent or $\Diamond\beta$ in the antecedent, where β is $\neg\gamma$, $\gamma_1 \to \gamma_2$ or $\gamma_1 \wedge \neg\gamma_2$ and where γ, γ_1 and γ_2 are translations of $\mathcal{L}^P$ formulas.

Let $\diamond\beta = \Diamond(\delta_1^M \wedge \neg\delta_2^M)$. By clause (c) in the Proposition , the derivation d has the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} d_{1,1} \\ \Box\Gamma, \delta_1^M \Rightarrow \delta_2^M, \Diamond\Delta \end{array}}{\Box\Gamma, \delta_1^M, \neg\delta_2^M \Rightarrow \Diamond\Delta}\ \neg\text{-L}}{\Box\Gamma, \delta_1^M \wedge \neg\delta_2^M \Rightarrow \Diamond\Delta}\ \wedge\text{-L}}{\Box\Gamma, \Diamond(\delta_1^M \wedge \neg\delta_2^M), \Rightarrow \Diamond\Delta}\ \Diamond\text{-L}$$

where $\Box\Gamma = \Theta^M$ and $\Diamond\Delta = \Upsilon^M$ and the desired derivation d^- is

$$\dfrac{\begin{array}{c} d_{1,1}^- \\ \Theta, \delta_1 \Rightarrow \delta_2, \Upsilon \end{array}}{\Theta, \delta_1 \setminus \delta_2, \Rightarrow \Upsilon}\ \setminus\text{-L}$$

If the principal formula of $\mathcal{I}$ has another form $\Box\beta$ in the succedent or $\Diamond\beta$ in the antecedent we proceed in a similar way, by applying one of the following rules:

- the assertive rule $\sim$-R or $\supset$-R or $\setminus\!\!\setminus$-R;

- the conjectural rule $\frown$-L or $\succ$-R or $\smallsetminus$-L;
- the mixed assertive rule $\sim$-R or $\supset$-R or $\smallsetminus\!\!\smallsetminus$-R;
- the mixed conjectural rule $\frown$-L or $\succ$-L or $\smallsetminus$-L.

Case 4. *Propositional rules corresponding to non-invertible pragmatic rules.* The remaining cases are those of inferences whose principal formula β has one of the following forms:

(I) $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ in the antecedent or $\Box\gamma_0 \vee \Box\gamma_1$ in the succedent;
(II) $\Diamond\gamma_0 \vee \Diamond\gamma_1$ in the antecedent or $\Box\gamma_0 \wedge \Box\gamma_1$ in the succedent.

where γ_0 and γ_1 are translations of $\mathcal{L}^P$ formulas. By clause (d) of the Proposition, we may assume that the endsequent S of d has the form

$$\Pi, \underline{\Box\Gamma}, \Diamond\Delta', \Lambda \;\Rightarrow\; \Lambda', \Box\Gamma', \underline{\Diamond\Delta}, \Pi'$$

where Π, Π' are pairwise disjoint sequences of atoms and where Λ, Λ' are sequences of formulas of the form (I) or (II). We consider the part $\overline{d}$ of d which is below all applications of $\Box$-R or $\Diamond$-L; thus $\overline{d}$ is a tree whose leaves are either axioms, or sequents of the form

$$\overline{S_\ell}: \qquad \Box\Gamma, \Diamond\alpha_\ell \;\Rightarrow\; \Diamond\Delta \qquad \text{or} \qquad \Box\Gamma \Rightarrow \Box\alpha_\ell, \Diamond\Delta$$

In each branch $\mathcal{B}$ of $\overline{d}$ below an $\overline{S_\ell}$ we find an application of weakening with conclusion S^+_ℓ and then a sequence $\mathcal{I}_1, \ldots, \mathcal{I}_k$ of applications of $\vee$-R, $\wedge$-R, $\vee$-L, or $\wedge$-L, whose principal formula β is (an ancestor of a formula) in Λ or in Λ'. Among these **S4** inferences we are searching for one which may be *relevant* for our desired **FILP** derivation. We consider the inferences $\mathcal{I}_j$, of a branch $\mathcal{B}$ starting with $j = 1$. Let $S_{j,0}$ [and $S_{j,1}$] be the sequent-premises of $\mathcal{I}_j$. We have the following cases:

(a) $\beta = \Box\gamma_0 \vee \Box\gamma_1$ and β is not traceable to α_ℓ, i.e., α_ℓ is an ancestor neither of $\Box\gamma_0$ nor of $\Box\gamma_1$. In this case we remove $S_{j,0}$ and the inference $\mathcal{I}_j$ and replace β for the pair $\Box\gamma_0$, $\Box\gamma_1$ in S^+_ℓ. Similarly, if β is $\Diamond\gamma_0 \wedge \Diamond\gamma_1$ and is not traceable to α_ℓ.
(b) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is not traceable to α_ℓ. We remove the inference $\mathcal{I}_j$ and replace β for the $\Box\gamma_i$ which occurs in S^+_ℓ, for $i = 0$ or 1. Similarly, if β is $\Diamond\gamma_0 \vee \Diamond\gamma_1$ and is not traceable to α_ℓ.
(c) $\beta = \Box\gamma_0 \vee \Box\gamma_1$ and β is traceable to α_ℓ. In this case we say that the search has found a *relevant* inference.
(d) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is traceable to α_ℓ through the active formula $\Box\gamma_i$ and also the active formula $\Box\gamma_{1-i}$ is traceable to some $\alpha_{\ell'}$ in some other leaf $S_{\ell'}$ of $\overline{d}$. In this case also we have found a *relevant* inference and we consider (nondeterministically) a

branch $\mathcal{B}'$ which starts from such $S_{\ell'}$. Similarly, in the case of $\beta = \Diamond\gamma_0 \vee \Diamond\gamma_1$ and $\Diamond\gamma_0$ and $\Diamond\gamma_1$ are traceable to α_ℓ, $\alpha_{\ell'}$.

(e) $\beta = \Box\gamma_0 \wedge \Box\gamma_1$ is traceable to α_ℓ through the active formula $\Box\gamma_i$ but the active formula $\Box\gamma_{1-i}$ is not traceable to the $\alpha_{\ell'}$ in any other leaf $S_{\ell'}$ of $\overline{d}$. In this case we consider (nondeterministically) a branch $\mathcal{B}'$ which starts from such a $S_{\ell'}$.

Notice that in each branch $\mathcal{B}$ the search may find a relevant inference only once, and also that steps (a), (b), (e) reduce the size of $\overline{d}$: thus in the end any branch contains at most one inference $\mathcal{I}_j$ and the resulting derivation d' has size not greater than d.

We apply the induction hypothesis to the premises of the $\Box$-R or $\Diamond$-L occurring in the remaining branches of $\overline{d}$. We have three cases:

(i) Case (c) succeeds: the desired derivation d^- is obtained by an application of $\cup$-R or of $\curlywedge$-L;

(ii) Case (d) succeeds: the desired derivation d^- is obtained by an application of $\cap$-R or $\curlyvee$-L.

(iii) otherwise: since α is an ancestor of a formula in $\Diamond\Delta'$ or $\Box\Gamma'$ we are back to **Case 3**.

This concludes the proof of the Lemma.

4.1. Equivalence of ILP and FILP. If $\Theta = \vartheta_1, \ldots, \vartheta_m$, we write $\cup\Theta$ for $\vartheta_1 \cup \ldots \cup \vartheta_m$; similarly, we write $\curlywedge\Upsilon$ for $\upsilon_1 \curlywedge \ldots \curlywedge \upsilon_n$; notice that generalized associativity holds for both $\cup$ and $\curlywedge$.

LEMMA 3. *If*

$$\Theta, \Upsilon' \Rightarrow \Theta', \Upsilon$$

is derivable in the sequent calculus for **FILP**, *then*

$$\Theta\ ; \Rightarrow \curlywedge\Upsilon' \supset \cup\Theta'; \Upsilon$$

is derivable in the sequent calculus for **ILP**.

The proof is by induction on the length of the given **FILP** derivation d. It is a lengthy exercise, whose details can be found in [4], including proofs of admissibility of depth-preserving contraction, depth-preserving weakening and context-sharing cut for **ILP**. Here we consider only one case.

Let d end with an application of the $\cap$-L rule of type $\vartheta \times \upsilon \to \vartheta$, corresponding to the **ACA.4** rule

$$\frac{\Theta, \vartheta \cap \upsilon, \vartheta, \upsilon, \Upsilon' \Rightarrow \Theta', \Upsilon}{\Theta, \vartheta \cap \upsilon\ , \Upsilon' \Rightarrow \Theta', \Upsilon}$$

By inductive hypothesis we have an **ILP** derivation d_1^* of

$$\Theta, \vartheta \cap \upsilon, \vartheta; \;\Rightarrow\; (\Yup \Upsilon' \Yup \upsilon) \supset \cup\Theta'; \Upsilon$$

In **ILP** we have the following derivation d^*

$$\dfrac{\dfrac{\dfrac{\dfrac{; \Yup \Upsilon' \Rightarrow ; \Yup \Upsilon'}{; \Rightarrow \Yup \Upsilon' \supset \cup\Theta' ; \Yup \Upsilon'}}{; \Rightarrow \upsilon \supset (\Yup \Upsilon' \supset \cup\Theta'); \Yup \Upsilon'} \qquad \dfrac{; \upsilon \Rightarrow ; \upsilon}{; \Rightarrow \upsilon \supset (\Yup \Upsilon' \supset \cup\Theta'); \upsilon}}{; \Rightarrow \upsilon \supset (\Yup \Upsilon' \supset \cup\Theta'); \Yup \Upsilon' \Yup \upsilon} \qquad \dfrac{\dfrac{\cup\Theta'; \Rightarrow \cup\Theta';}{\cup\Theta'; \Rightarrow \Yup \Upsilon' \supset \cup\Theta';}}{\cup\Theta'; \Rightarrow \upsilon \supset (\Yup \Upsilon' \supset \cup\Theta');}}{(\Yup \Upsilon' \Yup \upsilon) \supset \cup\Theta'; \Rightarrow \upsilon \supset (\Yup \Upsilon' \supset \cup\Theta');}$$

Writing $\delta = \Yup \Upsilon' \supset \cup\Theta$, and applying cut_1 to d_1^* and d^* we obtain a derivation d_1^{**} of $\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \upsilon \supset \delta; \Upsilon$. Hence we obtain the following **ILP** derivation:

$$\dfrac{\dfrac{\begin{array}{c} d_1^{**} \\ \Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \upsilon \supset \delta; \Upsilon \end{array} \qquad \dfrac{\text{ACA.4}\,\dfrac{\Theta, \vartheta \cap \upsilon, \vartheta, \vartheta; \upsilon \Rightarrow ; \Upsilon, \upsilon}{\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \delta; \Upsilon, \upsilon} \qquad \delta, \Theta, \vartheta \cap \upsilon\, \vartheta; \Rightarrow \delta; \Upsilon}{\delta, \Theta, \vartheta \cap \upsilon\, \vartheta; \Rightarrow \delta; \Upsilon}}{\Theta, \vartheta \cap \upsilon, \vartheta; \Rightarrow \delta; \Upsilon}\, cut_1}{\Theta, \vartheta \cap \upsilon; \Rightarrow \delta; \Upsilon}\,\text{ACA.5}$$

§5. Sequent calculus for polarized classical $\mathcal{L}^P$. We would like to characterize interactions between the *radical part* and the *pragmatic part* of our logic for pragmatics. Such extensions of the intuitionistic logic of pragmatics may or may not extend the expressive power of the logical system, but certainly exhibit interesting properties of our logic.

We are looking for an extended system **CLP**, which like **ILP** is sound and complete with respect the semantics of **S4**, with a set of inference rules that modify the radical part of pragmatic sentential expressions in a *compositional way*, inferring formulas with a more complex radical part from simpler ones. In the system *polarized* **CLP** presented here we consider only cases in which the **S4** translation remains invariant with respect to the interactions between classical and pragmatic connectives. More precisely, we look for illocutionary operators $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ and a pair of connectives $\circ$ and $\bullet$, where $\circ$ is classical and $\bullet$ is pragmatic, such that

$$(\mathcal{O}(\alpha_1 \circ \alpha_2))^M \equiv (\mathcal{O}'\alpha_1)^M \bullet (\mathcal{O}''\alpha_2)^M$$

When such a relation holds, we are able to establish a strict correspondence between these forms of "classical" reasoning and their intuitionistic counterparts. As pointed out in the introduction, this holds in the following cases[11]:

$$\frac{\vdash \neg\alpha}{\sim \mathcal{H}\,\alpha} \qquad \frac{\mathcal{H}\,\neg\alpha}{\frown\!\vdash \alpha} \qquad \frac{\mathcal{H}\,(\alpha \to \beta)}{\vdash \alpha \succ \mathcal{H}\,\beta} \qquad \frac{\vdash (\alpha \wedge \neg\beta)}{\vdash \alpha \smallsetminus \mathcal{H}\,\beta} \qquad \frac{\vdash (\alpha \wedge \beta)}{\vdash \alpha \cap \vdash \beta} \qquad \frac{\mathcal{H}\,(\alpha \vee \beta)}{\mathcal{H}\,\alpha \curlyvee \mathcal{H}\,\beta}$$

In the sequent calculus *polarized* **CLP** of table 3, the *left* and *right* rules *preserve validity* and are *semantically invertible* in the **S4** translation; therefore the soundness and completeness theorem with respect to the semantic interpretation in **S4** are particularly easy. Here we shall deal only with a cut-free sequent calculus for our classical fragment.

identity rules

logical axiom: $\quad {}^{\vdash}p, \Theta \,;\, \Rightarrow {}^{\vdash}p \,;\, \Upsilon$

logical axiom: $\quad \Theta \,;\, \mathcal{H}\,p \Rightarrow \,;\, \Upsilon, \mathcal{H}\,p$

structural rules

exchange:
$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon}$$

exchange:
$$\frac{\Theta, \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

logical rules

right assert-negation:
$$\frac{\Theta \,;\, \mathcal{H}\,\alpha \Rightarrow \,;\, \Upsilon}{\Theta \,;\, \Rightarrow {}^{\vdash}\neg\alpha \,;\, \Upsilon}$$

left assert-negation
$$\frac{\Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon, \mathcal{H}\,\alpha}{{}^{\vdash}\neg\alpha, \Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\Upsilon}$$

right hyp-negation:
$$\frac{\Theta, {}^{\vdash}\alpha \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon}{\Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon, \mathcal{H}\,\neg\alpha}$$

left hyp-negation
$$\frac{\Theta \,;\, \Rightarrow {}^{\vdash}\alpha \,;\, \Upsilon}{\Theta \,;\, \mathcal{H}\,\neg\alpha \Rightarrow \,;\, \Upsilon}$$

right assert-and:
$$\frac{\Theta \,;\, \Rightarrow {}^{\vdash}\alpha \,;\, \Upsilon \qquad \Theta \,;\, \Rightarrow {}^{\vdash}\beta \,;\, \Upsilon}{\Theta \,;\, \Rightarrow {}^{\vdash}(\alpha \wedge \beta) \,;\, \Upsilon}$$

left assert-and:
$$\frac{\Theta, {}^{\vdash}\alpha, {}^{\vdash}\beta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon}{\Theta, {}^{\vdash}(\alpha \wedge \beta) \,;\, \epsilon \Rightarrow \epsilon' \,;\, \Upsilon}$$

right hyp-or:
$$\frac{\Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \mathcal{H}\,\alpha, \mathcal{H}\,\beta, \Upsilon}{\Theta \,;\, \epsilon \Rightarrow \epsilon' \,;\, \mathcal{H}\,(\alpha \vee \beta), \Upsilon}$$

left hyp-or
$$\frac{\Theta \,;\, \mathcal{H}\,\alpha \Rightarrow \,;\, \Upsilon \quad \Theta \,;\, \mathcal{H}\,\beta \Rightarrow \,;\, \Upsilon}{\Theta \,;\, \mathcal{H}\,(\alpha \vee \beta) \Rightarrow \,;\, \Upsilon}$$

TABLE 3. Polarized classical sequent calculus

[11] Here we leave out the cases of weak implication and strong subtraction, as their classical counterparts are definable in terms of conjunction, disjunction and negation.

DEFINITION 7. Let us consider two copies of our infinite set of propositional letters, one positive $p_0^+, p_1^+, \ldots$, the other negative p_0^-, $p_1^-, \ldots$.

(i) Consider the following grammar for *radical formulas*

$$\begin{array}{lcl|c|c} \mathbf{P} & := & p^+ & \neg\mathbf{N} & \mathbf{P} \wedge \mathbf{P} \\ \mathbf{N} & := & p^- & \neg\mathbf{P} & \mathbf{N} \vee \mathbf{N} \end{array}$$

(ii) Consider the sublanguage of $\mathcal{L}^P$ where elementary pragmatic expressions are generated by the following rules:

$$\vartheta := \vdash \mathbf{P} \qquad\qquad \upsilon := \mathcal{H}\,\mathbf{N}.$$

Let us call such a language *polarized classical language*.

(iii) The *polarized classical sequent calculus* **CLP** is system of sequent calculus for classical logic where sequents are restricted to elementary formulas in the polarized classical language, i.e., sequents have one of the forms

$$\vdash \alpha_1, \ldots, \vdash \alpha_m \; ; \; \Rightarrow \; \vdash \alpha \; ; \; \mathcal{H}\,\beta_1, \ldots, \mathcal{H}\,\beta_n$$

$$\vdash \alpha_1, \ldots, \vdash \alpha_m \; ; \; \mathcal{H}\,\beta \; \Rightarrow \; ; \; \mathcal{H}\,\beta_1, \ldots, \mathcal{H}\,\beta_n$$

where the α_i, α are of the form $\mathbf{P}$ and the β_j, β are of the form $\mathbf{N}$.

THEOREM 2. *The polarized classical sequent calculus* **CLP** *is sound and complete with respect to the modal interpretation in* **S4**.

To prove the theorem, notice that in the semantics of **S4** there is a countermodel to the translation of the sequent-conclusion if and only if there is a countermodel to the translation of at least one sequent-premise. Notice also that when a rule is inverted an obvious measure of complexity of the sequents always decreases from the sequent-conclusion to the sequent-premises. For **CLP** sequents consisting of elementary formulas whose radical is in the polarized classical language, there is always a rule in the sequent calculus which can be applied, until we reach an axiom with atomic radical or a sequent falsifiable in the **S4** interpretation where all elementary formulas have atomic radicals. Therefore the proof-search procedure and the completeness theorem in the case of *polarized* **CLP** are completely straightforward.

Consider the following translation $(\;)^I$, which maps the *radical part* of an expression $\vdash \mathbf{P}$ or $\mathcal{H}\,\mathbf{N}$ to a formula in $\mathcal{L}^{IP}$:

$$\begin{array}{llllll}
(p^+)^I & =_{df} & \vdash p; & & & \\
(p^-)^I & =_{df} & \mathcal{H} p; & & & \\
(\neg \mathbf{N})^I & =_{df} & \sim (\mathbf{N}^I) & (\mathbf{P} \wedge \mathbf{P})^I & =_{df} & \mathbf{P}^I \cap \mathbf{P}^I; \\
(\neg \mathbf{P})^I & =_{df} & \frown (\mathbf{P}^I) & (\mathbf{N} \vee \mathbf{N})^I & =_{df} & \mathbf{N}^I \curlyvee \mathbf{N}^I.
\end{array}$$

THEOREM 3. *Let S be a sequent consisting of elementary formulas in the polarized classical language. Then S is derivable (without cut) in polarized* **CLP** *if and only if S^I is derivable (without cut) in the* **ILP**.

Given a (cut free) derivation d in polarized **CLP**, if d is a logical axiom, then its principal formulas have atomic radical part which are also principal formulas of axioms of **ILP** so d can be transformed into the required **ILP** axiom by choosing the context which is required by the $(\ )^I$ translation. Suppose the last inference of d is a right [or left] rule for *assert-negation*: since the active formula has the form $\mathcal{H}\, \alpha$, by definition 7(ii) α must a (possibly molecular) negative formula and the translation of the radical part of the principal formula is $(\neg\alpha)^I =\sim (\alpha)^I$; thus the inductive step is concluded by applying a right [or left] rule for $\sim$ to the derivation given by inductive hypothesis. Suppose the last inference of d is a right [or left] rule for *hyp-negation*: since the active formula has the form $\vdash \alpha$, α must be positive and we have $(\neg\alpha)^I = \frown(\alpha^I)$ and we conclude the inductive step by applying a right [or left] rule for $\frown$. The cases of assertive conjunction and conjectural disjunction are easy.

Conversely, notice that if a sequent has the form S^I, then its formulas will only contain elementary formulas of the form $\vdash p$ (where p is positive) or $\mathcal{H} p$ (where p is negative), conjectural negation $\frown$ and assertive conjunction $\cap$ applied to translations of positive formulas, and negation $\sim$ and conjectural negation applied to translations of negative formulas. Moreover, a cut-free proof d of S^I will only contain rules for these connectives. Thus by replacing a CA.1 or CA.2 rule with right or left *assert-negation*, AC.1 or AC.2 with right or left *hyp-negation*, the rules A.5 or A.6 with right or left *assert-and* and, finally, the rules C.9 and C.10 with right or left *hyp-or* we succeed in any case to perform the inductive step as required.

§6. Conclusion: work in progress. Finally, after wading through a long stretch of *ancient-style* proof-theory, it is rewarding to catch at least a glimpse into forthcoming work on the *modern* proof-theory of the logic for pragmatics, namely, a natural deduction system with term assignment for our logic **ILP** of assertions and conjectures. To this aim it is convenient to develop a natural deduction system for polarized bi-intuitionistic logic **PBL** and to look into a pragmatic interpretation of Heyting-Brouwer's logic. We are concerned only with the fragment of the language $\mathcal{L}^P$ given by the grammar

$$\delta \; := \; \vartheta \mid \upsilon \mid$$

$$\vartheta \; := \vdash p \mid \bigvee \mid \vartheta \supset \vartheta \mid \vartheta \cap \vartheta \mid \; \sim \upsilon \mid$$

$$\upsilon \; := \; \mathcal{H} p \mid \bigwedge \mid \upsilon \smallsetminus \upsilon \mid \upsilon \curlyvee \upsilon \mid \; \frown \vartheta \mid$$

In the case of **ILP**, the duality $(\;)^{\perp}$ can be extended from formulas to proof-terms, but can also be taken as an explicit orthogonality operator. For instance, in the case of a *weak negation left* we can write

$$\frac{\Theta; \;\Rightarrow\; t : \vartheta \;;\; \Upsilon}{\Theta \;;\; t^{\perp} : \frown \vartheta \;\Rightarrow\; ;\; \Upsilon}$$

Terms and co-terms are defined from variables x and covariables a according to the grammar

$$t \; := \; x \mid \star \mid \lambda x.t \mid tt \mid \; < t, t > \; \mid \pi_0 t \mid \pi_1 t \mid c^{\perp}$$

$$c \; := \; a \mid * \mid \lambda a.c \mid cc \mid \; < c, c > \; \mid \pi_0 c \mid \pi_1 c \mid t^{\perp}$$

To the usual α*-equivalence*, β*-reductions*, η*-reductions* we need to add only the equations

$$t^{\perp\perp} = t, \qquad c^{\perp\perp} = c.$$

Using orthogonalities, every sequent $\Theta \;;\; \Rightarrow \; \vartheta \;;\; \Upsilon$ can be labelled either as

$$\sim \Upsilon, \Theta \;;\; \Rightarrow \; t : \vartheta \qquad \text{or} \qquad ;\; t^{\perp} : \frown \vartheta \Rightarrow \;;\; \Upsilon, \frown \Theta$$

It does not seem that this computational interpretation of **ILP** could lead us into a new land.

The case of **PBL** is more challenging. In a sequent calculus for this system the *right* $\supset$ and *left* $\smallsetminus$ rules must be restricted

$$\frac{\Theta, \; \vartheta_1 \;;\; \Rightarrow \; \vartheta_2 \;;}{\Theta \;;\; \Rightarrow \; \vartheta_1 \supset \vartheta_2 \;;\; \Upsilon} \qquad \text{and} \qquad \frac{;\; \upsilon_1 \;\Rightarrow\; ;\; \upsilon_2, \; \Upsilon}{\Theta \;;\; \upsilon_1 \smallsetminus \upsilon_2 \;\Rightarrow\; ;\; \Upsilon}$$

i.e., Υ $[\Theta]$ cannot occur in the sequent-premise of a *right* $\supset$ [*left* $\smallsetminus$] rule. Here negations cannot be dealt with as orthogonalities and the task of defining proof-terms for the *dual intuitionistic* fragment is non-trivial. We need a term assignment for a *multiple-conclusions one-premise* natural deduction system. This requires

(i) a treatment of *contraction right*, with an assignment of a *lists of terms* ℓ to each conclusion υ;
(ii) the use of the `inl` ( ), `inr` ( ) constructors and of the `casel` ( ), `caser` ( ) destructors for terms resulting from $\curlyvee$ introduction and $\curlyvee$-elimination, respectively;
(iii) a use of *continuations* to deal with the introduction of a new conclusion resulting from the rule $\smallsetminus$-*introduction*;
(iv) the introduction of *control terms* of the form `postpone` $(x :: \ell')$ `until` (ℓ) to deal with the removal of a conclusion in an application of the rule $\smallsetminus$-*elimination*.

The term assignment to the rules for subtraction is as follows:

$$\frac{y:\epsilon\ \vdash\ \overline{t}:\Upsilon,\ \ell:\upsilon_1 \qquad\qquad z:\upsilon_2\ \vdash\ \overline{t'}:\Upsilon}{y:\epsilon\ \vdash\ \overline{t}:\Upsilon,\overline{t'}:\Upsilon,\ \texttt{continue from}(z)\ \texttt{using}(\ell):\upsilon_1\smallsetminus\upsilon_2}\ \smallsetminus\text{-}intro$$

$$\frac{y:\epsilon\vdash\overline{u}:\Upsilon,\ell:\upsilon_1\smallsetminus\upsilon_2 \qquad\qquad x:\upsilon_1\ \vdash\ \overline{u'}:\Upsilon,\ell':\upsilon_2}{y:\epsilon\vdash\ \overline{u}:\Upsilon,\overline{u'}:\Upsilon,\ \texttt{postpone}(x::\ell')\ \texttt{until}(\ell):\bullet}\ \smallsetminus\text{-}elim$$

Notice that a *redex* is a control term of the form

$$\texttt{postpone}\ (x :: \ell')\ \texttt{until}\ (\texttt{continue from}\ (z)\ \texttt{using}\ (\ell)) : \bullet$$

It is reduced by substituting each term t_i of the list $\ell : \upsilon_1$ for x in each term r_j of lists $\ell' : \upsilon_2$ and $\overline{u'} : \Upsilon$ and then substituting each term $r_j[t_i/x] : \upsilon_2$ thus obtained for z in each term in $\overline{t} : \Upsilon$.

We conjecture that in this way a *purely intuitionistic* calculus of continuations can be obtained that is *isomorphic* to the simply typed lambda calculus: to any *refutation* $x : \upsilon \vdash \ell : \bullet$ there corresponds a *proof* $\vdash (\ell)^{\perp} :\sim \upsilon$, to every reduction sequence $\beta = \ell, \ell_1, \ldots, \ell_n$ there corresponds a reduction sequence $\beta^{\perp} = \ell^{\perp}, \ell_1^{\perp}, \ldots, \ell_n^{\perp}$ and β terminates in a normal form if and only if $\beta^{\perp}$ does. Details of these technical results are contained in a forthcoming sequel to the present paper.

In conclusion, we are now in position to compare the systems **ILP** and **PBL** in a new light. It seems that we have a reasonable philosophical account of **ILP** as a logic of the illocutionary operators of assertion and of conjectures. From a mathematical point of view, the computational interpretation of **PBL** promises interesting and fruitful results. Since **ILP** can also be formalized in **PBL** by adding the axioms

$$\vartheta \Rightarrow \sim\frown \vartheta \qquad \text{and} \qquad \frown\sim \upsilon \Rightarrow \upsilon$$

these mathematical results can also be exploited in the study of **ILP**. Thus any progress in the understanding of the Heyting-Brouwer logic may contribute to the development of the logics for pragmatics.

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§7. APPENDIX I.The modal language and the semantics for K and S4.

DEFINITION 8. (*Syntax*) (i) The language $\mathcal{L}^m$ is built from an infinite set **Atoms** of *propositional letters* p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; and the modal operators $\Box$ and $\Diamond$.

(ii) (*Formation Rules*) The expressions of the language $\mathcal{L}^m$ are given by the following grammar, where p ranges over **Atoms**:

$$\alpha \ := \ p \mid \bot \mid \top \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \Box\alpha \mid \Diamond\alpha \mid$$

7.1. Frames and Kripke models.

DEFINITION 9. (*Frames and Kripke models*) (i) A *frame* is a pair $\mathcal{F} = (W, \sqsubseteq)$ where

- W is a set (of *"possible worlds"*);
- $\sqsubseteq \ \subset \ W \times W$ is a relation (the *"accessibility relation"* between possible worlds).

(ii) A *Kripke model* is a triple $\mathcal{M} = (W, \sqsubseteq, \Vdash)$ where $\mathcal{F} = (W, \sqsubseteq)$ is a frame and $\Vdash \subseteq W \times$ **Atoms** is the *forcing relation*, usually written in infix notation: $w \Vdash p$ means *"p is true in the possible world w"* and $w \not\Vdash p$ means *"p is false in the possible world w"*.

(iii) The relation $\Vdash$ is extended to a relation $\Vdash \quad \subset \quad W \times \mathcal{L}^m$ according to the following rules:

1. $w \not\Vdash \bot$ and $w \Vdash \top$, for all $w \in W$;
2. $w \Vdash \neg\alpha$ iff $w \not\Vdash \alpha$;
3. $w \Vdash (\alpha \wedge \beta) = V$ iff $w \Vdash \alpha$ and $w \Vdash \beta$;
4. $w \Vdash (\alpha \vee \beta)$ iff $w \Vdash \alpha$ or $w \Vdash \beta$;
5. $w \Vdash (\alpha \to \beta)$ iff either $w \not\Vdash \alpha$ or $w \Vdash B$;
6. $w \Vdash \Box\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that $w' \sqsubseteq w$;
7. $w \Vdash \Diamond\alpha$ iff $w' \Vdash \alpha$ for some $w' \in W$ such that $w' \sqsubseteq w$.

If Γ and Δ are sequences of formulas in $\mathcal{L}^m$, then the *sequent* $\Gamma \Rightarrow \Delta$ is *true in* $w \in W$ iff $w \Vdash (\bigwedge \Gamma \to \bigvee \Delta)$.

(iv) We say that a formula α is *valid in a model* $\mathcal{M} = (W, \sqsubseteq, \Vdash)$, in symbols $\models_{\mathcal{M}} \alpha$, iff for every $w \in W$ we have $w \Vdash \alpha$. Similarly, given a *sequent* $S = \Gamma \Rightarrow \Delta$ we say that S is *valid in* $\mathcal{M}$ iff for every $w \in W$, S is true in w.

(v) We say that a formula α is *valid in a frame* $\mathcal{F}$ iff for every $\mathcal{M}$ over $\mathcal{F}$ we have $\models_{\mathcal{M}} \alpha$. Similarly, a sequent S is *valid in a frame* $\mathcal{F}$ iff it is valid in every Kripke model over $\mathcal{F}$.

(vi) A formula α [a sequent S] is *valid* in the system **K** iff α [S] it is valid in all Kripke models $\mathcal{M}$.

(vii) A formula α [a sequent S] is *valid* in the system **S4** iff α [S] is valid in all preordered frames, i.e., all frames where the accessibility relation $\sqsubseteq$ is reflexive and transitive.

7.2. Sequent calculi G3c, K and S4. Gentzen-Kleene's sequent calculus **G3c** for classical propositional logic (cfr.[27], p. 77) is given by the sequent-axioms and rules of inference in Table 4. Notice that the rules of *weakening* and *contraction* are implicit.

DEFINITION 10. (i) Given a notion of semantic validity, a rule of the sequent calculus $\dfrac{S_1, \ldots, S_n}{S}$ *preserves validity* if for every instance of the rule, the sequent conclusion S is valid whenever the sequent-premises $S_1, \ldots, S_n$ are all valid; a rule is *semantically invertible* if for every instance of the rule the sequent-premises are all valid whenever the sequent-conclusion is valid.

PROPOSITION 2. *(i) The rules of the system* **G3c** *preserve validity and are semantically invertible for any modal semantics;*
(ii) the modal rules for the systems **K** *and* **S4** *preserve validity and are semantically invertible in the semantics of the system* **K** *and* **S4***, respectively;*
(iii) the rules of weakening *preserve validity but are not semantically invertible.*

7.2.1. *Semantic Tableaux procedure for* **K**. The "semantic tableaux" procedure decides whether a sequent S is valid in the semantics for **K** by building a *refutation tree* labelled with sequents and with S at the root; if S is valid, then it return a derivation of S in the sequent calculus for **K**; if S not valid, it returns a counterexample $\mathcal{M}$ which refutes S.

DEFINITION 11. *(semantic tableaux procedure)* Start with tree τ_0 consisting of the root S; at stage $n+1$, for every leaf S' of the tree τ_n check whether the sequent S' matches the conclusion of a rule of inference (in some given order, e.g., checking the one-premise rules first). If yes, invert that rule; otherwise, the leaf in question is a sequent of the form

$$p_1, \dots, p_k, \Box\Gamma, \Diamond\alpha_1, \dots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \dots, \Box\beta_n, \Diamond\Delta, q_1, \dots, q_\ell \qquad (\dagger)$$

Rewrite the sequent (†) as a *hypersequent* as follows:

$$\Rightarrow [p_1, \dots, p_k \Rightarrow q_1, \dots, q_\ell] \dots [\Box\Gamma, \Diamond\alpha_i \Rightarrow \Diamond\Delta] \dots [\Box\Gamma, \Rightarrow \Box\beta_j, \Diamond\Delta] \dots \qquad (\ddagger)$$

We call this step a *disjunctive ramification*. Now there are three cases:

(a) the sequent $p_1, \dots, p_k \Rightarrow q_1, \dots, q_\ell$ is valid, because $p_i = q_j$ for some $i \leq k$, $j \leq \ell$ or because $p_i = \bot$ for some $i \leq k$ or $q_j = \top$ for some $j \leq \ell$: in this case the sequent (†) is a *logical axiom* or a *falsity axiom* or a *truth axiom* and the procedure halts on this branch, which is *closed*.
(b) otherwise, if (†) is not an axiom and $m = 0 = n$, then the procedure halts on this branch leaving it *open*;
(c) otherwise, (†) is not an axiom and $m + n > 0$: in this case the procedures *branches* by inverting the $\Diamond$-L or $\Box$-R rules in the remaining $m + n$ sequents of the hypersequent.

DEFINITION 12. We define inductively what it means for a refutation tree τ to be *closed* (starting from the *leaves*):

- a *logical axiom*, a *falsity axiom* or a *truth axiom* is closed;

- if τ results from τ_0 by a *one-premise* inference rule, then τ is closed iff τ_0 is closed;
- if τ results from τ_0 and τ_1 by a *two-premises* inference rule, then τ is closed iff τ_0 and τ_1 are both closed;
- if τ ends with a hypersequent and results from $\tau_1, \ldots, \tau_{m+n}$ by a *disjunctive ramification*, then τ is closed iff at least one τ_i is closed, for $i \leq m+n$.

Fact 1: *The semantic tableax procedure for* **K** *terminates.*

Fact 2: *If a refutation tree τ with conclusion S is closed, then we can obtain a derivation of S in the sequent calculus for* **K** *as follows:*

- for each disjunctive ramification branching from a sequent of the form (‡) with subtrees $\tau_1, \ldots, \tau_{m+n}$, first we *prune* τ by selecting a *closed* subtree τ_k, by removing the others and the hypersequent notation; the endsequent of τ_k has the form $\Box\Gamma, \Diamond\alpha \ \Rightarrow \ \Diamond\Delta$ or $\Box\Gamma \ \Rightarrow \ \Box\alpha, \Diamond\Delta$ and now we apply *weakening* to obtain the sequent (†).

Fact 3: *If a refutation tree τ with conclusion S is open, the we can construct a Kripke model $\mathcal{M}$ which refutes S*:

- for every two-premises logical rule, if the sequent-conclusion is open, then we select one of the sequent-premises which is open. In this way we eventually obtain a tree τ' where all branches are open.
- Consider all fragments of branches $\beta_1, \ldots, \beta_z$ obtained from τ' by removing every hypersequent and every conclusion of a modal inference;
  (i) identify β_i with a possible world w_i;
  (ii) put $w_i \sqsubseteq w_j$ if and only if the lowermost sequent of β_i is the premise of a **KR** occurring immediately above a sequent S^* of the form (†) and S^* is the uppermost sequent of β_j;
  (iii) let $w_i \Vdash p_i$ if and only if p_i occurs in the antecedent of a sequent S^* of the form (†) and S^* is the uppermost sequent of β_i.

From facts 1-3 we obtain the following theorem:

THEOREM 4. *The semantic tableaux procedure for* **K** *is sound and complete with respect to the semantics of* **K**. *The system* **K** *has the finite model property.*

7.2.2. *Semantic Tableaux procedure for* **S4**. In the case of **S4** the procedure is modified by inverting the $\Box$-left and $\Diamond$-right in the same way as the propositional rules, but we must deal with the fact that in this way the procedure may enter infinite loops. The first problem is that the $\Box$-left and $\Diamond$-right rules could be iterated forever with the same principal formula. It is enough to mark the modal formula which is principal formula of such an inference and remove the mark later when some $\Box$-right or $\Diamond$-left rule is inverted; in other words we take modal rules of the forms

$$\Box\ \textit{left}: \quad \frac{\alpha, \Gamma, \underline{\Box\alpha}, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}}{\Box\alpha, \Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}} \qquad \Box\ \textit{right}: \quad \frac{\Box\Gamma \Rightarrow \alpha, \Diamond\Delta}{\underline{\Box\Gamma} \Rightarrow \Box\alpha, \underline{\Diamond\Delta}}$$

$$\Diamond\ \textit{left}: \quad \frac{\Box\Gamma, \alpha \Rightarrow \Diamond\Delta}{\underline{\Box\Gamma}, \Diamond\alpha \Rightarrow \underline{\Diamond\Delta}} \qquad \Diamond\ \textit{right}: \quad \frac{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \alpha, \underline{\Diamond\alpha}, \underline{\Diamond\Lambda}}{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \Diamond\alpha, \underline{\Diamond\Lambda}}$$

A *disjunctive branching* in **S4** has the form

$$\frac{\Rightarrow [\Pi \Rightarrow \Pi'], \dots, \frac{\Box\Gamma, \alpha_i \Rightarrow \Diamond\Delta}{[\underline{\Box\Gamma}, \Diamond\alpha_i \Rightarrow \underline{\Diamond\Delta}]}, \dots, \frac{\Box\Gamma \Rightarrow \beta_j, \Diamond\Delta}{[\underline{\Box\Gamma}, \Rightarrow \Box\beta_j, \underline{\Diamond\Delta}]}, \forall i \leq m, \forall j \leq n}{\Pi, \underline{\Box\Gamma}, \Diamond\alpha_1, \dots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \dots, \Box\beta_n, \underline{\Diamond\Delta}, \Pi'}$$

where $\Pi = p_1, \dots, p_k$ and $\Pi' = q_1, \dots, q_\ell$.

The second source of non-termination is the fact that in general an inversion of the $\Box$-left and of the $\Diamond$-right rules increases the logical complexity of the sequent instead of reducing it. However, since the procedure satisfies the *subformula property* and there is only a finite number of modal subformulas in any given sequent, eventually on any branch the procedure must invert a $\Box$-right or $\Diamond$-left rule with a sequent-conclusion S such that the same rule with the same sequent-conclusion S had already inverted at some point below in the refutation tree (here we consider sequents S *modulo* exchange and contraction). Let $< \mathcal{I}, \mathcal{I}' >$ be such a pair of inferences, where $\mathcal{I}'$ occurs above $\mathcal{I}$. In this case we *identify* the sequent-premise of $\mathcal{I}'$ with the sequent premise of $\mathcal{I}$ and the procedure stops on that branch. Notice that as a consequence of such a *gluing* there will be a loop in the transitive closure of the accessibility relation $\sqsubseteq$ of the countermodel constructed in **Fact 3**. Other details are left to the reader. It follows that

THEOREM 5. *The semantic tableaux procedure for* **S4** *is sound and complete with respect to the semantics of* **S4**. *The system* **S4** *has the finite model property.*

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SEQUENT CALCULUS G3c FOR CLASSICAL LOGIC

axioms: $p, \Gamma \Rightarrow \Delta, p$ *falsity axioms:* $\bot, \Gamma \Rightarrow \Delta$ *truth axioms:* $\Gamma \Rightarrow \Delta, \top$

right exchange:
$$\dfrac{\Gamma \Rightarrow \Delta, \alpha, \beta, \Delta'}{\Gamma \Rightarrow \Delta, \beta, \alpha, \Delta'}$$

left exchange:
$$\dfrac{\Gamma, \alpha, \beta, \Gamma' \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Gamma' \Rightarrow \Delta}$$

right $\neg$*:*
$$\dfrac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

left $\neg$*:*
$$\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}$$

right $\wedge$*:*
$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

left $\wedge$*:*
$$\dfrac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$$

right $\rightarrow$*:*
$$\dfrac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \rightarrow \beta, \Delta}$$

left $\rightarrow$*:*
$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta}$$

right $\vee$*:*
$$\dfrac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

left $\vee$*:*
$$\dfrac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}$$

EXTENSION TO MODAL SYSTEMS

weakenings

$$\dfrac{\Box\Gamma, \Diamond\alpha \ \Rightarrow \ \Diamond\Delta}{\Pi, \Box\Gamma, \Diamond\alpha, \Diamond\Delta' \ \Rightarrow \ \Box\Gamma', \Diamond\Delta, \Pi'} \qquad \dfrac{\Box\Gamma \ \Rightarrow \ \Box\alpha, \Diamond\Delta}{\Pi, \Box\Gamma, \Diamond\Delta' \ \Rightarrow \ \Box\alpha, \Box\Gamma', \Diamond\Delta, \Pi'}$$

where Π, Π' are sequences of atoms.

modal rules for K

K-$\Box$-*rule*:
$$\dfrac{\Gamma \Rightarrow \alpha, \Delta}{\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta}$$

K-$\Diamond$-*rule*:
$$\dfrac{\Gamma, \alpha \Rightarrow \Delta}{\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta}$$

modal rules for S4

$\Box$ *left*:
$$\dfrac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}$$

$\Box$ *right*:
$$\dfrac{\Box\Gamma \ \Rightarrow \ \alpha, \Diamond\Delta}{\Box\Gamma \ \Rightarrow \ \Box\alpha, \Diamond\Delta}$$

$\Diamond$ *left*:
$$\dfrac{\Box\Gamma, \alpha \ \Rightarrow \ \Diamond\Delta}{\Box\Gamma, \Diamond\alpha \ \Rightarrow \ \Diamond\Delta}$$

$\Diamond$ *right*:
$$\dfrac{\Gamma \Rightarrow \Delta, \Diamond\alpha, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}$$

TABLE 4. Sequent calculi for **K** and **S4**

APPENDIX II: The rules of ILP

identity rules

S.1: logical axiom:
$$\vartheta, \Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon$$

S.2: logical axiom:
$$\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon, \upsilon$$

S.3: absurdity axiom:
$$\Theta, \bigwedge \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon$$

S.4: absurdity axiom:
$$\Theta \; ; \; \bigwedge \; \Rightarrow \; ; \; \Upsilon$$

S.5: assertion-conjecture:
$$\vdash \alpha, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \mathcal{H}\, \alpha$$

S.6: validity axiom:
$$\Theta \; ; \; \Rightarrow \; \bigvee \; ; \; \Upsilon$$

S.7: validity axiom:
$$\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \bigvee$$

S.8: cut$_1$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon \qquad \vartheta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

S.9: cut$_2$:
$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \qquad \Theta' \; ; \; \upsilon \; \Rightarrow \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

structural rules

S.10: exchange:
$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

S.11: exchange:
$$\frac{\Theta, \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

TABLE 5. **ILP**, identity and structural rules

ASSERTIVE LOGICAL RULES

connective of type $\vartheta \to \vartheta$

(*) *A.1: right negation:*

$$\frac{\Theta, \vartheta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \sim \vartheta \; ; \; \Upsilon}$$

A.2: left negation:

$$\frac{\sim \vartheta, \Theta ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}{\sim \vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

connectives of type $\vartheta \times \vartheta \to \vartheta$

(*) *A.3: right $\supset$:*

$$\frac{\Theta, \vartheta_1 \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \supset \vartheta_2 \; ; \; \Upsilon}$$

A.4: left $\supset$:

$$\frac{\vartheta_1 \supset \vartheta_2, \Theta ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \vartheta_2, \Theta \; ; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.5: right $\cap$:

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \Theta \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \cap \vartheta_2 \; ; \; \Upsilon}$$

A.6: left $\cap$:

$$\frac{\vartheta_0, \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_0 \cap \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.7,8: right $\cup^i$:

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_i \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_0 \cup \vartheta_1 \; ; \; \Upsilon}$$

for $i = 0, 1$.

A.9: left $\cup$:

$$\frac{\vartheta_0, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon \qquad \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_0 \cup \vartheta_1, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

(*) *A.10: right $\smallsetminus$:*

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta_1 \; ; \; \Upsilon \qquad \vartheta_2, \Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta_1 \smallsetminus \vartheta_2 \; ; \; \Upsilon}$$

A.11: left $\smallsetminus$:

$$\frac{\vartheta_1 \smallsetminus \vartheta_2, \Theta, \vartheta_1 \; ; \; \Rightarrow \; \vartheta_2 \; ; \; \Upsilon}{\vartheta_1 \smallsetminus \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

A.12: left $\smallsetminus$:

$$\frac{\vartheta_1 \smallsetminus \vartheta_2, \Theta, \vartheta_1 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta_1 \smallsetminus \vartheta_2, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

TABLE 6. Sequent calculus for **ILP**, the standard fragment

CONJECTURAL RULES

connective of type $\upsilon \to \upsilon$

C.1: right $\frown$:
$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\; ;\, \Upsilon, \frown \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \frown \upsilon}$$

(*) *C.2: left* $\frown$:
$$\frac{\Theta; \;\Rightarrow\; ;\, \Upsilon, \upsilon}{\Theta \,;\, \frown \upsilon \;\Rightarrow\; ;\, \Upsilon}$$

connectives of type $\upsilon \times \upsilon \to \upsilon$

C.3: right $\succ$:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \upsilon_2, \Upsilon, \upsilon_1 \succ \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.4: right $\succ$:
$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \succ \upsilon_2}$$

(*) *C.5: left* $\succ$:
$$\frac{\Theta \,;\, \Rightarrow\; ;\, \Upsilon, \upsilon_1 \qquad \Theta \,; \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_1 \succ \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

C.6: right $\curlywedge$:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_0 \qquad \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_0 \curlywedge \upsilon_1}$$

C.7,8: left $\curlywedge$:
$$\frac{\Theta \,;\, \upsilon_i \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_0 \curlywedge \upsilon_1 \;\Rightarrow\; ;\, \Upsilon}$$
for $i = 0, 1$.

C.9: right $\curlyvee$:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1, \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \curlyvee \upsilon_2}$$

C.10: left $\curlyvee$:
$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon \qquad \Theta \,;\, \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \upsilon_1 \curlyvee \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

C.11: right $\smallsetminus$:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \qquad \Theta \,;\, \upsilon_2 \;\Rightarrow\; ;\, \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

(*)*C.12: left* $\smallsetminus$:
$$\frac{\Theta \,;\, \upsilon_1 \;\Rightarrow\; ;\, \Upsilon, \upsilon_2}{\Theta \,;\, \upsilon_1 \smallsetminus \upsilon_2 \;\Rightarrow\; ;\, \Upsilon}$$

TABLE 7. Sequent calculus for **ILP**, the dual fragment

MIXED-TYPE NEGATIONS

connective of type $\upsilon \to \vartheta$

(*) *CA.1: right* $\sim$:
$$\dfrac{\Theta \,;\, \upsilon \;\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \Rightarrow\, \sim\upsilon \,;\, \Upsilon}$$

CA.2: left negation$_1$
$$\dfrac{\Theta ;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon}{\sim\upsilon, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

connective of type $\vartheta \to \upsilon$

AC.1: right $\frown$:
$$\dfrac{\Theta, \vartheta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \frown\vartheta}$$

(*)*AC.2: left* $\frown$:
$$\dfrac{\Theta ;\, \Rightarrow\; \vartheta \,;\, \Upsilon}{\Theta \,;\, \frown\vartheta \;\Rightarrow\, ;\, \Upsilon}$$

MIXED-TYPE SUBTRACTIONS

connective of type $\vartheta \times \upsilon \to \vartheta$

(*) *ACA.9: right* $\bbslash$:
$$\dfrac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon \qquad \Theta \,;\, \upsilon \;\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \Rightarrow\; \vartheta \bbslash \upsilon \,;\, \Upsilon}$$

ACA.10: left $\bbslash$:
$$\dfrac{\vartheta, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon}{\vartheta \bbslash \upsilon, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \vartheta$

(*) *CAA.10: right* $\bbslash$:
$$\dfrac{\Theta \,;\, \Rightarrow\, ;\, \Upsilon, \upsilon \qquad \vartheta, \Theta \,;\, \Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \Rightarrow\; \upsilon \bbslash \vartheta \,;\, \Upsilon}$$

CAA.11: left $\bbslash$:
$$\dfrac{\upsilon \bbslash \vartheta, \Theta \,;\, \upsilon \;\Rightarrow\, ;\, \Upsilon}{\upsilon \bbslash \vartheta, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

CAA.12: left $\bbslash$:
$$\dfrac{\upsilon \bbslash \vartheta, \Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon}{\upsilon \bbslash \vartheta, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

connective of type $\upsilon \times \upsilon \to \vartheta$

(*) *CCA.9: right* $\bbslash$:
$$\dfrac{\Theta \,;\, \Rightarrow\, ;\, \Upsilon, \upsilon_1 \qquad \Theta \,;\, \upsilon_2 \;\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \Rightarrow\; \upsilon_1 \bbslash \upsilon_2 \,;\, \Upsilon}$$

CCA.10: left $\bbslash$:
$$\dfrac{\upsilon_1 \bbslash \upsilon_2, \Theta \,;\, \upsilon_1 \;\Rightarrow\, ;\, \Upsilon, \upsilon_2}{\upsilon_1 \bbslash \upsilon_2, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

CCA.11: left $\bbslash$:
$$\dfrac{\upsilon_1 \bbslash \upsilon_2, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon_2}{\upsilon_1 \bbslash \upsilon_2, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

connective of type $\vartheta \times \upsilon \to \upsilon$

ACC.9: right $\setminus$:
$$\dfrac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \vartheta \setminus \upsilon \qquad \Theta \,;\, \upsilon \;\Rightarrow\, ;\, \Upsilon, \vartheta \setminus \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta \setminus \upsilon}$$

(*) *ACC.10: left* $\setminus$:
$$\dfrac{\Theta, \vartheta \,;\, \Rightarrow\, ;\, \Upsilon, \upsilon}{\Theta \,;\, \vartheta \setminus \upsilon \;\Rightarrow\, ;\, \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \upsilon$

CAC.10: right $\setminus$:
$$\dfrac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon \qquad \vartheta, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon \setminus \vartheta}$$

(*) *CAC.11: left* $\setminus$:
$$\dfrac{\Theta \,;\, \upsilon \;\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \upsilon \setminus \vartheta \;\Rightarrow\, ;\, \Upsilon}$$

(*) *CAC.12: left* $\setminus$:
$$\dfrac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon}{\Theta \,;\, \upsilon \setminus \vartheta \;\Rightarrow\, ;\, \Upsilon}$$

connective of type $\vartheta \times \vartheta \to \upsilon$

AAC.10: right $\setminus$:
$$\dfrac{\Theta \,;\, \Rightarrow\; \vartheta_1 \,;\, \Upsilon, \vartheta_1 \setminus \vartheta_2 \qquad \vartheta_2, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon,}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta_1 \setminus \vartheta_2}$$

(*) *AAC.11: left* $\setminus$:
$$\dfrac{\Theta, \vartheta_1 \,;\, \Rightarrow\; \vartheta_2 \,;\, \Upsilon}{\Theta \,;\, \vartheta_1 \setminus \vartheta_2 \;\Rightarrow\, ;\, \Upsilon}$$

TABLE 8. Mixed-type negations and subtractions

MIXED-TYPE ASSERTIVE LOGICAL RULES

connectives of type $\vartheta \times \upsilon \to \vartheta$

(*) *ACA.1: right* $\supset$:
$$\frac{\Theta, \vartheta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \supset \upsilon\ ;\ \Upsilon}$$

ACA.2: left $\supset$:
$$\frac{\vartheta \supset \upsilon, \Theta;\ \Rightarrow \vartheta\ ;\ \Upsilon \qquad \vartheta \supset \upsilon, \Theta\ ; \upsilon\ \Rightarrow\ ;\ \Upsilon}{\vartheta \supset \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

(*) *ACA.3: right* $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cap \upsilon\ ;\ \Upsilon}$$

ACA.4: left $\cap$:
$$\frac{\vartheta \cap \upsilon, \Theta, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\vartheta \cap \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

ACA.5: left $\cap$:
$$\frac{\vartheta \cap \upsilon, \Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\vartheta \cap \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

(*) *ACA.6: right* $\cup$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

(*) *ACA.7: right* $\cup$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

ACA.8: left $\cup$:
$$\frac{\vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon \qquad \vartheta \cup \upsilon, \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\vartheta \cup \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

connectives of type $\upsilon \times \vartheta \to \vartheta$

(*) *CAA.1: right* $\supset$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

(*) *CAA.2: right* $\supset$:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

CAA.3: left $\supset$:
$$\frac{\Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon \qquad \vartheta, \Theta\ ; \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\upsilon \supset \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

(*) *CAA.4: right* $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cap \vartheta\ ;\ \Upsilon}$$

CAA.5: left $\cap$:
$$\frac{\upsilon \cap \vartheta, \Theta, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\upsilon \cap \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.6: left $\cap$:
$$\frac{\upsilon \cap \vartheta, \Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\upsilon \cap \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

(*) *CAA.7: right* $\cup$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

(*) *CAA.8: right* $\cup$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

CAA.9: left $\cup$:
$$\frac{\upsilon \cup \vartheta, \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon \qquad \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\upsilon \cup \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

connectives of type $\upsilon \times \upsilon \to \vartheta$

(*) *CCA.1: right* $\supset$:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon, \upsilon_2}{\Theta\ ;\ \Rightarrow\ \upsilon_1 \supset \upsilon_2\ ;\ \Upsilon}$$

CCA.2: left $\supset$:
$$\frac{\Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \qquad \upsilon_1 \supset \upsilon_2, \Theta\ ; \upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\upsilon_1 \supset \upsilon_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon';\ \Upsilon}$$

(*) *CCA.3: right* $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_0, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon_1, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cap \upsilon_1\ ;\ \Upsilon}$$

CCA.4,5: left $\cap$:
$$\frac{\Theta, \upsilon_0 \cap \upsilon_1\ ;\ \upsilon_i\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon_0 \cap \upsilon_1\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$
for $i = 0, 1$.

(*) *CCA.6, 7: right* $\cup$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_i, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cup \upsilon_1\ ;\ \Upsilon}$$

CCA.8: left $\cup$:
$$\frac{\upsilon_0 \cup \upsilon_1, \Theta\ ;\ \upsilon_0\ \Rightarrow\ ;\ \Upsilon \qquad \upsilon_0 \cup \upsilon_1, \Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon}{\upsilon_0 \cup \upsilon_1, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$
for $i = 0, 1$.

TABLE 9. Mixed-type assertive logical rules

MIXED-TYPE CONJECTURAL LOGICAL RULES

connective of type $\vartheta \times \upsilon \to \upsilon$

ACC.1: right $\succ$:
$$\frac{\Theta, \vartheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta \succ \upsilon}$$

(*) *ACC.2: left $\succ$:*
$$\frac{\Theta ; \;\Rightarrow \vartheta \;;\; \Upsilon \qquad \Theta ; \upsilon \;\Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta \succ \upsilon \;\Rightarrow ;\; \Upsilon}$$

ACC.3: right $\curlywedge$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon, \vartheta \curlywedge \upsilon \qquad \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta \curlywedge \upsilon}$$

(*) *ACC.4: left $\curlywedge$:*
$$\frac{\Theta, \vartheta \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta \curlywedge \upsilon \;\Rightarrow ;\; \Upsilon}$$

(*) *ACC.5: left $\curlywedge$:*
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta \curlywedge \upsilon \;\Rightarrow ;\; \Upsilon}$$

ACC.6: right $\curlyvee$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \upsilon, \Upsilon, \vartheta \curlyvee \upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta \curlyvee \upsilon}$$

ACC.7: right $\curlyvee$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \upsilon, \Upsilon, \vartheta \curlyvee \upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta \curlyvee \upsilon}$$

(*)*ACC.8: left $\curlyvee$:*
$$\frac{\Theta, \vartheta \;;\; \Rightarrow ;\; \Upsilon \qquad \Theta \;;\; \upsilon \;\Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta \curlyvee \upsilon \;\Rightarrow ;\; \Upsilon}$$

connective of type $\upsilon \times \vartheta \to \upsilon$

CAC.1: right $\succ$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon, \upsilon \succ \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon \succ \vartheta}$$

CAC.2: right $\succ$:
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow ;\; \Upsilon, \upsilon \succ \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon \succ \vartheta}$$

(*) *CAC.3: left $\succ$:*
$$\frac{\Theta ; \;\Rightarrow ;\; \Upsilon, \upsilon \qquad \vartheta, \Theta \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \upsilon \succ \vartheta \;\Rightarrow ;\; \Upsilon}$$

CAC.4: right $\curlywedge$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon, \qquad \Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon, \upsilon \curlywedge \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon \curlywedge \vartheta}$$

(*) *CAC.5: left $\curlywedge$:*
$$\frac{\Theta ; \; \upsilon \;\Rightarrow ;\; \Upsilon}{\Theta \;;\; \upsilon \curlywedge \vartheta \;\Rightarrow ;\; \Upsilon}$$

(*)*CAC.6: left $\curlywedge$:*
$$\frac{\Theta, \vartheta \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \upsilon \curlywedge \vartheta \;\Rightarrow ;\; \Upsilon}$$

CAC.7: right $\curlyvee$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon \curlyvee \vartheta}$$

CAC.8: right $\curlyvee$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \upsilon \curlyvee \vartheta}$$

(*) *CAC.9: left $\curlyvee$:*
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow ;\; \Upsilon \qquad \Theta, \vartheta \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \upsilon \curlyvee \vartheta \;\Rightarrow ;\; \Upsilon}$$

connective of type $\vartheta \times \vartheta \to \upsilon$

AAC.1: right $\succ$:
$$\frac{\Theta, \vartheta_1 \;;\; \Rightarrow\; \vartheta_2 \;;\; \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta_1 \succ \vartheta_2}$$

AAC.2: right $\succ$:
$$\frac{\Theta, \vartheta_1 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta_1 \succ \vartheta_2}$$

(*) *AAC.3: left $\succ$:*
$$\frac{\Theta ; \;\Rightarrow \vartheta_1 \;;\; \Upsilon \qquad \vartheta_2, \Theta \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta_1 \succ \vartheta_2 \;\Rightarrow ;\; \Upsilon}$$

AAC.4: right $\curlywedge$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta_0 \;;\; \Upsilon, \vartheta_0 \curlywedge \vartheta_1 \qquad \Theta \;;\; \Rightarrow\; \vartheta_1 ;\; \Upsilon, \vartheta_0 \curlywedge \vartheta_1}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta_0 \curlywedge \vartheta_1}$$

(*) *AAC.5, 6: left $\curlywedge$:*
$$\frac{\Theta, \vartheta_i \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta_0 \curlywedge \vartheta_1 \;\Rightarrow ;\; \Upsilon}$$
for $i = 0, 1$.

AAC.7,8: right $\curlyvee$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta_i \;;\; \Upsilon, \vartheta_0 \curlyvee \vartheta_1,}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \vartheta_0 \curlyvee \vartheta_1}$$
for $i = 0, 1$.

(*) *AAC.9: left $\curlyvee$:*
$$\frac{\Theta, \vartheta_0 \;;\; \Rightarrow ;\; \Upsilon \qquad \Theta, \vartheta_1 \;;\; \Rightarrow ;\; \Upsilon}{\Theta \;;\; \vartheta_0 \curlyvee \vartheta_1 \;\Rightarrow ;\; \Upsilon}$$

TABLE 10. Mixed-type conjectural logical rules

APPENDIX III: The rules of FILP

identity and pragmatic axioms

logical axiom:
$$\delta, \Theta, \Upsilon' \;\Rightarrow\; \delta, \Theta', \Upsilon$$

absurdity axiom:
$$\Theta, \bigwedge, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon$$

assertion-conjecture:
$$\vdash \alpha, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \mathcal{H}\,\alpha$$

validity axiom:
$$\Theta, \Upsilon' \;\Rightarrow\; \Theta', \bigvee, \Upsilon$$

structural rules

left exchange:
$$\frac{\Theta_0, \vartheta_0, \vartheta_1, \Theta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta_0, \vartheta_1, \vartheta_0, \Theta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

right exchange:
$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon_0, \upsilon_0, \upsilon_1, \Upsilon_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon_0, \upsilon_1, \upsilon_0, \Upsilon_1}$$

TABLE 11. **FILP**, identity and structural rules

ASSERTIVE LOGICAL RULES

connective of type $\vartheta \to \vartheta$

(*) $\sim$-R:

$$\frac{\Theta, \vartheta \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \Rightarrow \sim \vartheta, \Theta', \Upsilon}$$

$\sim$-L:

$$\frac{\sim \vartheta, \Theta, \Upsilon' \;\Rightarrow\; \vartheta, \Theta', \Upsilon}{\sim \vartheta, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\vartheta \times \vartheta \to \vartheta$

(*) $\supset$-R:

$$\frac{\Theta, \vartheta_1 \;\Rightarrow\; \vartheta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \supset \vartheta_2, \Theta', \Upsilon}$$

$\supset$-L:

$$\frac{\vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon \qquad \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\cap$-R:

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon \qquad \Theta, \Upsilon' \;\Rightarrow\; \vartheta_2, \Theta', \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \cap \vartheta_2, \Theta', \Upsilon}$$

$\cap$-R:

$$\frac{\vartheta_0, \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_0 \cap \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

$\cup$-R:

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_0, \vartheta_1, \Theta', \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_0 \cup \vartheta_1, \Theta', \Upsilon}$$

$\cup$-L:

$$\frac{\vartheta_0, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\vartheta_0 \cup \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\fatbslash$-R:

$$\frac{\Theta \;\Rightarrow\; \vartheta_1, \Upsilon \qquad \vartheta_2, \Theta, \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \vartheta_1 \fatbslash \vartheta_2, \Theta', \Upsilon}$$

$\fatbslash$-L:

$$\frac{\vartheta_0 \fatbslash \vartheta_1, \vartheta_0, \Theta, \Upsilon' \;\Rightarrow\; \vartheta_1, \Theta', \Upsilon}{\vartheta_0 \fatbslash \vartheta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

TABLE 12. Sequent calculus for **FILP**, the standard fragment

CONJECTURAL RULES

connective of type $\upsilon \to \upsilon$

$$\frac{\Theta, \Upsilon', \upsilon \;\Rightarrow\; \Theta', \Upsilon, \frown \upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \frown \upsilon}\;\frown\text{-R:} \qquad \frac{\Theta \;\Rightarrow\; \Upsilon, \upsilon}{\Theta, \frown \upsilon, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\;\textbf{(*)}\;\frown\text{-L:}$$

connectives of type $\upsilon \times \upsilon \to \upsilon$

$$\frac{\Theta, \Upsilon', \upsilon_1 \;\Rightarrow\; \Theta', \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \succ \upsilon_2}\;\succ\text{-R:} \qquad \frac{\Theta, \;\Rightarrow\; \Upsilon, \upsilon_1 \qquad \Theta, \upsilon_2 \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon', \upsilon_1 \succ \upsilon_2 \;\Rightarrow\; \Theta' \Upsilon}\;\textbf{(*)}\;\succ\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_0 \qquad \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_0 \curlywedge \upsilon_1}\;\curlywedge\text{-R:} \qquad \frac{\Theta, \upsilon_0, \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta, \upsilon_0 \curlywedge \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta' \Upsilon}\;\curlywedge\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1, \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \curlyvee \upsilon_2}\;\curlyvee\text{-R:} \qquad \frac{\Theta, \upsilon_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \Theta, \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\Theta, \upsilon_1 \curlyvee \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\;\curlyvee\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1, \upsilon_1 \smallsetminus \upsilon_2 \qquad \Theta, \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}\;\smallsetminus\text{-R:} \qquad \frac{\Theta, \upsilon_1 \;\Rightarrow\; \Upsilon, \upsilon_2}{\Theta, \upsilon_1 \smallsetminus \upsilon_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\;\textbf{(*)}\;\smallsetminus\text{-L:}$$

TABLE 13. Sequent calculus for **FILP**, the dual fragment

MIXED ASSERTIVE RULES

connective of type $\upsilon \to \vartheta$

(*) $\sim$-R:

$$\frac{\Theta, \upsilon \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \Rightarrow \;\sim \upsilon, \Theta', \Upsilon}$$

$\sim$-L:

$$\frac{\sim \upsilon, \Theta, \Upsilon' \;\Rightarrow\; \upsilon, \Theta', \Upsilon}{\sim \upsilon, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

connectives of type $\vartheta \times \upsilon \to \vartheta, \upsilon \times \vartheta \to \vartheta, \upsilon \times \upsilon \to \vartheta$

(*) $\supset$-R:

$$\frac{\Theta, \delta_1 \;\Rightarrow\; \delta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \supset \delta_2, \Theta', \Upsilon}$$

$\supset$-L:

$$\frac{\delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \delta_1, \Upsilon \qquad \delta_2, \delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_1 \supset \delta_2, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\cap$-R:

$$\frac{\Theta \;\Rightarrow\; \delta_1, \Upsilon \qquad \Theta \;\Rightarrow\; \delta_2, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \cap \delta_2, \Theta', \Upsilon}$$

$\cap$-L:

$$\frac{\delta_0, \delta_1, \delta_0 \cap \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_0 \cap \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\cup^i$-R:

$$\frac{\Theta \;\Rightarrow\; \delta_i, \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_0 \cup \delta_1, \Theta', \Upsilon}$$

$\cup$-L:

$$\frac{\delta_0, \delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon \qquad \delta_1, \delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}{\delta_0 \cup \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

(*) $\setminus\!\!\setminus$-R:

$$\frac{\Theta \;\Rightarrow\; \delta_1, \Upsilon \qquad \delta_2, \Theta \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon' \;\Rightarrow\; \delta_1 \setminus\!\!\setminus \delta_2, \Theta', \Upsilon}$$

$\setminus\!\!\setminus$-L:

$$\frac{\delta_0 \setminus\!\!\setminus \delta_1, \Theta, \delta_0, \Upsilon' \;\Rightarrow\; \delta_1, \Theta', \Upsilon}{\delta_0 \setminus\!\!\setminus \delta_1, \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}$$

TABLE 14. **FILP**, mixed assertive rules

MIXED CONJECTURAL RULES

connective of type $\vartheta \to \upsilon$

$$\frac{\Theta, \Upsilon', \vartheta \;\Rightarrow\; \Theta', \Upsilon, \frown \vartheta}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \frown \vartheta}\ \frown\text{-R:} \qquad \frac{\Theta \;\Rightarrow\; \Upsilon, \vartheta}{\Theta, \frown \vartheta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\ (*)\ \frown\text{-L}$$

connectives of type $\vartheta \times \upsilon \to \upsilon, \upsilon \times \vartheta \to \upsilon, \vartheta \times \vartheta \to \upsilon,$

$$\frac{\Theta, \delta_1, \Upsilon' \;\Rightarrow\; \Theta', \delta_2, \Upsilon, \delta_1 \succ \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \succ \delta_2}\ \succ\text{-R:} \qquad \frac{\Theta, \;\Rightarrow\; \delta_1, \Upsilon \qquad \Theta, \delta_2 \;\Rightarrow\; \Upsilon}{\Theta, \Upsilon', \delta_1 \succ \delta_2 \;\Rightarrow\; \Theta', \Upsilon}\ (*)\ \succ\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1, \delta_0 \qquad \Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1, \delta_1}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_0 \curlywedge \delta_1}\ \curlywedge\text{-R:} \qquad \frac{\Theta, \delta_i \;\Rightarrow\; \Upsilon}{\Theta, \delta_0 \curlywedge \delta_1, \Upsilon' \;\Rightarrow\; \Theta' \Upsilon}\ (*)\ \curlywedge^i\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \curlyvee \delta_2, \delta_1, \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \curlyvee \delta_2}\ \curlyvee\text{-R:} \qquad \frac{\Theta, \delta_1 \;\Rightarrow\; \Upsilon \qquad \Theta, \delta_2 \;\Rightarrow\; \Upsilon}{\Theta, \delta_1 \curlyvee \delta_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\ (*)\ \curlyvee\text{-L:}$$

$$\frac{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1, \delta_1 \searrow \delta_2 \qquad \Theta, \delta_1, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \searrow \delta_2}{\Theta, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon, \delta_1 \searrow \delta_2}\ \searrow\text{-R:} \qquad \frac{\Theta, \delta_1 \;\Rightarrow\; \Upsilon, \delta_2}{\Theta, \delta_1 \searrow \delta_2, \Upsilon' \;\Rightarrow\; \Theta', \Upsilon}\ (*)\ \searrow\text{-L:}$$

TABLE 15. **FILP**, mixed conjectural rules