TOWARDS A FORMAL PRAGMATICS: AN INTUITIONISTIC THEORY OF ASSERTIVE AND CONJECTURAL JUDGEMENTS WITH AN EXTENSION OF GÖDEL, MCKINSEY AND TARSKI'S S4 TRANSLATION.

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Abstract. Formal pragmatics extends classical logic to characterize the logical properties of the operators of *illocutionary force* such as those expressing *assertions* and *obligations* [3, 8, 9] and of the *pragmatic connectives* which are given an intuitionistic interpretation. Here we consider the cases of *assertions* and *conjectures*: for a mathematical proposition α, the act of asserting of α is justified by the availability of a proof of α, while the act of conjecturing α is justified the absence of a refutation of α. We give a unitary sequent calculus with subsystems characterizing intuitionistic and a fragment of classical reasoning with such operators. Extending Gödel's and McKinsey and A. Tarski's translations of intuitionistic logic into **S4**, we prove soundness and completenss of our sequent calculus with respect to the **S4** semantics.

§1. Preface. Formal pragmatics, as introduced by Dalla Pozza and Garola in [8, 9] and developed in [3, 22], aims at a characterization of the logical properties of *illocutionary operators*: it is concerned, e.g., with the operations by which we performs the act of *asserting* a proposition as true, either on the basis of a mathematical proof or by empirical evidence or by the recognition of physical necessity, or the act of taking a proposition as an *obligation*, either on the basis of a moral principle or by inference within a normative system.

Although some classical texts of 20th century philosophy and philosophical logic (e.g., by Austin and Grice) are very appropriately characterized as contributions to *pragmatics*, the word "pragmatics" is often used in a general way to indicate what lies beyond syntax and semantics in the study of language. Given such a vague and ambiguous usage, it is very hard to provide arguments for the claim that pragmatics could or should be given a formal treatment; it is much easier, and a better strategy, to produce a formal axiomatization of the behaviour of some pragmatic operator and to show that such a formal system can be used to clarify some confusion and to improve our understanding of linguistic behaviour. In our view, this is what Dalla Pozza and Garola's pragmatic interpretation of intuitionistic logic has done with respect to the interpretation of intuitionistic and classical connectives, with potential benefits for related controversies in the theory of meaning.

The viewpoint of [8] could be sketched roughly as follows. There is a logic of *propositions* and a logic of *judgements*. The former is about *truth* according to classical Tarskian semantics. The latter gives conditions for the *justification* of complex acts of judgement from elementary ones, such as *assertions*. The assertion of a mathematical proposition is justified by a *proof* of it, while some kind of *empirical evidence* is needed to justify assertions about states of affairs. In any case the justification of complex acts of judgement should be in terms of *Heyting's interpretation of intuitionistic connectives*: for instance, a conditional judgement "if α is assertible, then β is assertible" is justified by a method that transforms any justification for the assertion of α into a justification for the assertion of β.

In modern logic the distinction between propositions and judgements was established in Frege's system. According to Frege propositions express the thoughts which are the content of judgements. The formal expression $\vdash \alpha$ expresses the judgement asserting the proposition $-\alpha$: such a judgement is the act of recognizing the truth of its content. Only truth-functional connectives and quantifiers are considered by Frege, which belong to the logic of propositions; judgements appear only at the level of the deductive system; there are no connectives to form complex judgements (though the status of hypothetic judgements may be debatable). The distinction between propositions and judgements has recently been taken up in Martin-Löf's theory of types, which formalizes the informal notion of a proof as a collection of *judgements*. But here *propositions* are given an intuitionistic semantics, not a classical one, so that Frege's expression of judgement $\vdash \alpha$ is replaced by Martin-Löf with α *true*: the truth of α cannot be separated from the conditions justifying the assertion of α. Concerning the logical constants, their meaning is ultimately determined by the introduction and elimination rules for each constant: these rules of inference relate *judgments* about propositions involving the logical constant in question [14, 15].

Thus Martin-Löf theory of types is developed without any reference to the classical notion of truth: there is an *epoché* of classical truth, rather than an argument against it. Perhaps it should be remembered that in his system Martin-Löf has not introduced the notion of a *free-choice sequence*, the only construction by Brouwer which is formally inconsistent with classical logic. In Martin-Löf theory of types and in G. Sambin's formal topologies, mathematical objects are given together with a *presentation* (e.g., through introduction, elimination, conversion and equality rules): whenever a mathematical object is introduced, we immediately know *what we do* when we speak about it. The temptation to regard the whole of Martin-Löf theory of types as belonging to formal pragmatics is strong: certainly there is no inconsistency in doing so. Martin-Löf does not indulge in the irresistible attitude of those mathematicians, who feel that they have immediate access to mathematical objects *before* knowing how to compute them. But the unrepented mathematician who sticks to the fatal attraction for classical logic is not proved inconsistent by Martin-Löf as he is by Brouwer.

Unlike Martin-Löf and in agreement with Frege, Dalla Pozza and Garola distinguish between "α *is true*" and "$\vdash \alpha$ *is justified*", but extend Frege's framework and enhance the significance of the distinction between the *logic of propositions* and the *logic of judgements*, by giving an intuitionistic interpretation of the pragmatic connectives in the logic of judgements, while retaining classical semantics for the logic of propositions. This approach suggests compatibility between classical and intuitionistic point of views: such a thesis cannot be properly discussed here, but certainly requires further explanations.

It would be clearly a mistake to claim that formal pragmatics settles the philosophical issue of classical realism versus intuitionistic anti-realism: no formal system by itself can settle a philosophical issue! It would also be wrong to claim that it necessarily presupposes a compatibilist solution to the controversy. A realist philosopher, taking advantage of the distinction between semantics and pragmatics, may claim that classical semantics is the ultimate basis of meaning and perhaps that humans have immediate access to mathematical objects independently of their capacity of defining and describing them. An anti-realist philosopher could accept the system of pragmatics and claim that the phenomenon of meaning is only captured by the whole system, reducing the role of classical semantics to that of an *abstract interpretation*, which adequately represents meaning only in particular situations.

As the justification of mathematical judgements relies on proofs, it follows that proof-theory, the typed λ-calculus, categorical logic and their "semantics" are all drafted in the ranks of formal pragmatics, as providers of mathematical models of the processes by which we produce justified acts of judgement. But are we simply giving a new label to these disciplines? Do they need yet another philosophical burden?

In our view, the characterization of the distinction between classical model theory and intuitionistic proof-theory in terms of the distinction between *semantics* and *pragmatics* already achieves two goals. On one hand it makes clear that different criteria of correctness and achievement apply to these different enterprises: traditional logic, based upon classically semantics, is *extended* or *integrated* rather than challenged by intuitionistic pragmatics, the latter having a *different subject matter* than the former. In the same vein, one may also be able to identify the proper subject matter of substructural logics [3] and explain in which sense they are an extension or an integration of traditional logic theory. On the other hand, the problem of the relations between the two becomes more interesting and intriguing than the mere opposition between the classical and the intuitionistic philosophies of logic, and gives additional reason of interest in the mathematical study of the relations between the two.

One direction of the interaction between the intuitionistic pragmatics of assertions and classical semantics is well-known: it is given by Gödel, McKinsey and Tarski's *translation of intuitionistic logic in the modal system* **S4** [11, 18] together with Kripke's semantics for **S4**. In the framework of formal pragmatics, the modal interpretation can be regarded as a *reflection* of formal pragmatics into its own semantic level expanded with the **S4** modality: the reflection of $\vdash \alpha$ is the proposition $\Box\alpha$, which is semantically interpreted in preordered Kripke models. Dalla Pozza and Garola distinguish between *expressive* and *descriptive* uses of the pragmatic operators, where the descriptive use refers to the modal translation; in [9] the framework of formal pragmatics is extended to assertions and obligations and the distinction is made between the *expressive* use of the illocutionary operator of *obligation* and its *descriptive* use in the necessity operator of the deontic system **KD**.

The other direction, the action of classical semantics on intuitionistic pragmatics is less familiar and deserves further investigations. In [8] it is remarked that if the pragmatic expressions are evaluated only with respect to the pragmatic connectives and operators, namely, *negation* ($\sim$), *implication* ($\supset$), *conjunction* ($\cap$) and *disjunction* ($\cup$), and no analysis is made of the propositional contents of judgements according to the classical semantics of the propositional connectives $\neg$, $\rightarrow$, $\wedge$ and $\vee$, then the resulting pragmatic system **ILP** is essentially intuitionistic logic. The consideration of classical reasoning in formal pragmatics emerges in the interaction between classical propositional connectives and pragmatic connectives: e.g., we have that $\vdash(\alpha \rightarrow \beta)$ pragmatically implies $\vdash\alpha \supset \vdash\beta$, but is not implied by it (as we can easily check through the modal translation). A classification of these relations is already in [8, 9].

But if the constructive nature of the illocutionary operators and the notion of semantic reflection is to be taken seriously, then we must ask the following questions: is there is an illocutionary operator which is *dual* to assertion "$\vdash$", i.e., which stands to "$\vdash$" as "$\Diamond$" stands to "$\Box$"? Is the *expressive* use of pragmatic connectives no more restrictive than the *descriptive* use? namely, for every modal expressions α is there a pragmatic expression δ such that the modal reflection δ^M is equivalent in **S4** to α? for instance, shouldn't there be an expression of the pragmatic language whose modal reflection is $\Diamond\Box\alpha$?

The aim of this paper is to give evidence that a positive answer to both questions is possible: this requires performing a mathematical task and making hard philosophical decisions. An act of asserting $\vdash\alpha$ is justified by a proof of the truth of α; its dual must be a the act of *conjecturing* $\mathcal{H}\,\alpha$, which is justified by the absence of a proof of the falsity of α. The "modal reflection" of the pragmatic expressions $\vdash\alpha$ and $\mathcal{H}\,\alpha$ is given by the expressions $\Box\alpha$ and $\Diamond\alpha$, respectively, of *classical* **S4**; in Dalla Pozza's terminology, "$\Box$" and "$\Diamond$" provide the "descriptive" interpretations of the pragmatic operators of assertion and conjecture. In order to extend the expressiveness of formal pragmatics, notice that in addition to Gödel, McKinsey and Tarski's translation of intuitionistic logic into **S4**, there is also a *dual translation*:

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigwedge)^{\Box} & =_{df} & \bot & (\bigwedge)^{\Diamond} & =_{df} & \bot \\
(P)^{\Box} & =_{df} & \Box P & (P)^{\Diamond} & =_{df} & \Diamond P \\
(\delta_1 \supset \delta_2)^{\Box} & =_{df} & \Box(\delta_1^{\Box} \to \delta_2^{\Box}) & (\delta_1 \supset \delta_2)^{\Diamond} & =_{df} & \Diamond(\delta_1^{\Diamond} \to \delta_2^{\Diamond}) \\
(\delta_1 \cap \delta_2)^{\Box} & =_{df} & \delta_1^{\Box} \wedge \delta_2^{\Box} & (\delta_1 \cap \delta_2)^{\Diamond} & =_{df} & \delta_1^{\Diamond} \wedge \delta_2^{\Diamond} \\
(\delta_1 \cup \delta_2)^{\Box} & =_{df} & \delta_1^{\Box} \vee \delta_2^{\Box} & (\delta_1 \cup \delta_2)^{\Diamond} & =_{df} & \delta_1^{\Diamond} \vee \delta_2^{\Diamond}
\end{array}
$$

These translations correspond to the standard topological interpretation of intuitionistic logic, mapping formulas to open sets, and its dual, mapping formulas to closed sets. Now we may extend the system of formal pragmatics in such a way as to have a modal translation $(\ )^M$ and a topological translation extending *both* the standard interpretations and their duals. We let

$$
(\vdash \alpha)^M =_{df} \Box\alpha \qquad\qquad (\mathcal{H}\,\alpha)^M =_{df} \Diamond\alpha
$$

Then we extend the language $\mathcal{L}^P$ with the connectives which naturally arise in the new context: namely, we have a weak negation ($\frown$), a weak implication ($\succ$), conjunction ($\curlywedge$) and disjunction ($\curlyvee$), modally interpreted in **S4** as "*possibly not* ...", "*possibly,* ... *implies* ...", "*possibly* ... *and possibly* ...", "*possibly* ... *or possibly* ...", respectively.

From a technical point of view the first task is to extend the standard topological interpretation of intuitionistic logic and its dual, as well as Gödel's [11], McKinsey and Tarski's [18] interpretation of intuitionistic logic into **S4** by proving the analogue of Kripke's completeness theorem [13] for the extended language. In terms of the topological interpretation, the interest of these connectives lies in the fact that they represent an interaction of the *closed sets* on the *open sets* and vicecersa; we obtain many types of conjunctions and disjunctions and significant relations between the implications and the other connectives. A more abstract mathematical treatment of the whole matter is clearly needed.[1]

The second task is to give an account of classical reasoning in the extended context: here we propose a constructive extension of the intuitionistic system, which is sound and complete with respect to the interpretation in **S4**. This fragment is motivated by the remark that the following rules preserve validity and are semantically invertible in the **S4** translation:

$$
\frac{\sim \mathcal{H}\,\alpha}{\vdash \neg\alpha} \qquad \frac{\vdash \alpha \cap \vdash \beta}{\vdash (\alpha \wedge \beta)} \qquad \frac{\frown \vdash \alpha}{\mathcal{H}\,\neg\alpha} \qquad \frac{\vdash \alpha \succ \mathcal{H}\,\beta}{\mathcal{H}\,(\alpha \to \beta)} \qquad \frac{\mathcal{H}\,\alpha \curlyvee \mathcal{H}\,\beta}{\mathcal{H}\,(\alpha \vee \beta)}
$$

[1] The paper by G. Reyes and H. Zolfaghari [23] may help here.

As a consequence, the sequent calculus for our classical fragment has corresponding *left* and *right* rules which are valid and semantically invertible.

Our proofs of the completeness theorems for the intuitionistic and classical fragments with respect to Kripke semantics over preordered frames follows the well-known *semantic tableaux* procedure for **S4**, which will be quick summarized in Section 2. Hyper-sequents in the style of Pottinger, Avron and Girard [20, 1, 2, 10] provide a convenient technical tool for our proof.

The philosophical task of justifying the introduction of the *conjecture* operator is challenging: we need to explain what counts as a justification of an act of conjecturing, namely, what it means to say that we have no proof of the falsity of α. The epistemic interpretation of such a condition is quite clear, but uninteresting for our purpose: to my statement "I have no proof of that the square root of two is irrational", you may reply: "Think harder, idiot!" or "Poor thing, go to the library!".

More interesting, of course, are the conjectures on which a body of mathematical knowledge relies. A theorem proved supposing the Riemann hypothesis may simply be regarded as a hypothetical judgement. But in formal pragmatics the distinction between *acts of asserting* and *acts of conjecturing* cannot be identified with the distinction between *categorical* and *hypothetical judgements*, as the modal interpretation relates it to the duality between *necessity* and *possibility* in **S4**. Following Martin-Löf's approach, M. Pfening and R. Davies [19] have recently presented constructive notions of necessity and possibility based on the distinction between categorical and hypothetical judgements. The fact that modal notions can be reconstructed with an *epoché* of classical possible-world semantics is very important from the viewpoint of formal pragmatics, but in our framework we also need to explain how such a reconstruction is related to the classical semantics of **S4**.

To say that a act of conjecturing $\mathcal{H}\,\alpha$ is *fully justified* is to say that a proof of $\neg\alpha$ does not exist, independently of the subjective epistemic state of the individual making the assertion. Obviously, for the classical logician, this view is unproblematic. From an orthodox intuitionistic standpoint, the statement that a proof of $\neg\alpha$ does not exist is only understandable if we have a method to derive a contradiction for the hypothesis of the existence of a proof of $\neg\alpha$: but this is already justification for the assertion $\sim \vdash \neg\alpha$. Thus in this view the expression $\mathcal{H}\,\alpha$ is meaningful only as a characterization of an empirical epistemic state and has little mathematical or logical content. The issue at stake here is whether we can say that proofs have a *potential existence*, where "possibility is not understood in the traditional intuitionistic sense as knowledge of a method" to produce such a proof, but as "knowledge independent and tense-less" possibility ([17], pag.83). Professor Prawitz accepts this notion of possibility:

> "That we can prove A is not to be understood as meaning that it is within our practical reach to prove A, but only that it is possible in principle to prove A [...]. Similarly, that there exists a proof of A does not mean that a proof of A will be constructed but only that the possibility is there for constructing a proof of A. [...] I see no objection to conceiving the possibility that there is a specific method for curing cancer, which we may discover one day, but which may also remain undiscovered." ([21], pag. 153-154)

We conclude that our extension of formal pragmatics to conjectures (and to weak connectives) presupposes the philosophical standpoint called *potential intuitionism* by Martino and Uberti, i.e., the belief that proofs have a potential existence, independently of the contingent fact that we have discovered or will ever discover them.

Although the notion of a conjecture may appear of limited interest in the representation of mathematical reasoning, conjectures play an essential role in other forms of reasoning which are good candidates for applications of formal pragmatics. An example is legal reasoning. To charge somebody with a crime is to make a conjecture, while to find somebody guilty is to make an assertion: thus a trial may be regarded as a conventional procedure in which a conjecture may be transformed into an assertion. We would not be able to give a formal account of what a trial is without the notion of a conjecture.

In the presupposition of innocence before the conclusion of a trial and in a guilt verdict at the end of it conjectural and assertive forces play a formal, conventional role which does not depend on the degree of certainty associated with the evidence. Consider the sentence

> "On Sunday, April 26 1998, Monsignor Juan Gerardi Conedera, Auxiliary Bishop of Guatemala City, was killed by a member of a paramilitary death squad".

and the scenarios in which it could have been stated [2]. Such a sentence or its Spanish translation, could have occurred (a) in a communication by the murder himself to his boss immediately after the act, (b) in the statement in which the paramilitary group Jaguar Avengers claimed responsibility for the murder, (c) during the trial, in a statement by the prosecutor (d) during the trial in a statement by one of the defendants, had one of them confessed, (e) in the courtroom, when the guilt sentence was read by one of the judges, (f) in a discussion in the US Senate, e.g., in a statement by Vermont Senator Patrick Leary, when a declassified US Defense Intelligence Agency document revealed that one of the defendants, Col. Byron Disrael Lima Estrada had taken Military Police training at the US Army School of the Americas (SOA).

Had statement (a) been recorded, e.g., from a cellular phone conversation, it would provide direct evidence of the facts, while statement (e) relies on a complex system of information retrieval for its evidence. During the trial, the statement (d) would have had the peculiar pragmatic property of providing legal evidence to its own truth, while statement (c) would have relied on existing evidence. But notice that the process of weighting evidence for and against a case during the trial presupposes that the charge is still a conjecture. On the other hand, if a trial has been conducted according to proper procedure and a guilt sentence has been issued, then a statement of guilt is an assertion no matter what degree of certainty is assigned to the evidence, in particular whether or not evidence came from a confession.

In conclusion, the illutionary forces of statements (a)-(f) would have been very different; in each statement, the speaker might have had several intentions and done several things with his words. Nevertheless, the legal setting provides a simplified arena where assertions and conjectures play a clear and crucial role. For this reason, legal reasoning may provide a fruitful field of application for formal pragmatics and constructive modal logics.

§2. Modalities, pragmatic operators and connectives. The propositional part of the pragmatic language $\mathcal{L}^P$ (which corresponds

[2] The scenarios are fictional, but based on real events; among other sources in the media, see `http://www.peacehost.net/soaw-w/gerardi.html` and `http://leahy.senate.gov/press/199804/980428.html`

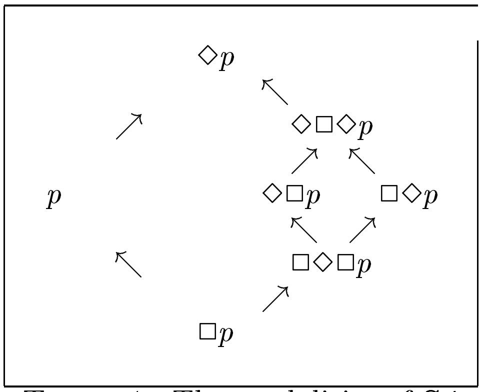

TABLE 1. The modalities of **S4**

to the language of propositional logic) is extended with the modalities "$\Box$" and "$\Diamond$" to a modal language $\mathcal{L}^m$, which is classically interpreted through Kripke's possible-worlds semantics. The modal systems of interest in the framework of formal pragmatics are **K**, **KD** and **S4**. We recall the definitions of the syntax and semantics of these systems in the Appendix 7. We also recall the well-known semantic-tableaux procedure proving the soundess and completeness theorem for these systems: details of this procedure are needed in the proof of the soundness and completeness theorems for the pragmatic sequent calculus in section 5.

It is easy to see, e.g., by using the semantic procedure in the Appendix 7, that there are only seven "modalities" in **S4** (including no modality), the ones in Table 1: indeed, these modalities are idempotent (e.g., $\Box\Box p$ is provably equivalent to $\Box p$ and $\Diamond\Box\Diamond\Diamond\Box\Diamond p$ is provably equivalent to $\Diamond\Box\Diamond p$); moreover applying negation to Table 1 yields a symmetry along the horizontal axis together with a substitution of $\neg p$ for p. Next consider the well-known translation of intuitionistic logic into **S4**, which we now interpret in the framework of the pragmatic language $\mathcal{L}$ as a translation of the sentential formulas *with atomic radical only* and with *assertive illocutionary sign only*:

$$(\vdash \alpha)^{\Box} = \Box\alpha \quad (\delta_1 \supset \delta_2)^{\Box} = \Box(\delta_1^{\Box} \to \delta_2^{\Box}) \quad \sim \delta^{\Box} = \Box(\neg\delta^{\Box})$$
$$(\delta_1 \cup \delta_2)^{\Box} = \delta_1^{\Box} \vee \delta_2^{\Box} \qquad (\delta_1 \cap \delta_2)^{\Box} = \delta_1^{\Box} \wedge \delta_2^{\Box}$$

Notice that of the seven modalities of **S4** only three ever occur in the translation, namely

$$(\vdash p)^{\Box} = \Box p \qquad (\sim\sim \vdash p)^{\Box} = \Box\Diamond\Box p \qquad (\sim \vdash \neg p)^{\Box} = \Box\Diamond p$$

$$
\begin{tikzcd}
 & \mathcal{H}p & & & \\
 & & & \frown\frown \mathcal{H}p \arrow[ull] & \\
p \arrow[uur] & & \frown \mathcal{H}\neg p \arrow[ur] & & \sim\vdash \neg p \arrow[ul] \\
 & & & \sim\sim\vdash p \arrow[ul] \arrow[ur] & \\
 & \vdash p \arrow[uul] \arrow[urr] & & &
\end{tikzcd}
$$

TABLE 2. Asserting and conjecturing

This is not surprising, given that Heyting's interpretation of intuitionistic connectives is in terms of *informal provability* also for the atomic sentences. But in the framework of *potential intuitionism* it makes sense to reason also with the presupposition that an atomic sentence p may be *conjecturally asserted*, in symbols $\mathcal{H}\, p$; *partial justification* for such a conjectural assertion is the belief that p may be *irrefutable*. Similarly, it make sense to *doubt of* an assertion (conjectural or not), in symbols $\frown \delta$; *partial justification* for doubt is the belief that there may be no method to prove that the committments made by δ could be met. If the operator of conjectural assertion and the connective of doubtful negation are acceptable, then we can extend also the above $(\ )^{\Box}$ translation to a translation $(\ )^{M}$ as follows:

$$(\mathcal{H}\alpha)^M = \Diamond\alpha \qquad (\frown \delta)^M = \Diamond(\neg\delta^M)$$

and we give a pragmatic counterpart to three other modalities of **S4**:

$$(\mathcal{H}p)^M = \Diamond p \qquad (\frown\frown \mathcal{H}p)^M = \Diamond\Box\Diamond p \qquad (\frown \mathcal{H}\neg p)^M = \Diamond\Box p$$

§3. The pragmatic language $\mathcal{L}^P$.

DEFINITION 1. (*Syntax*) (i) The language $\mathcal{L}^P$ is built from an infinite set of *propositional letters* p, p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; these expressions are called *radical formulas.* The *elementary formulas* of the pragmatic language are obtained by prefixing a radical formula with a sign of *illocutionary force* "$\vdash$" and "$\mathcal{H}$". There is only one elementary constant for absurdity, namely

$\bigwedge$. Finally, the *sentential formulas* of $\mathcal{L}^P$ are built from the elementary formulas and the constant $\bigwedge$, using the *pragmatic connectives* $\sim$, $\cap$, $\cup$, $\supset$, $\frown$, $\succ$, $\curlywedge$ and $\curlyvee$.

(ii) (*Formation Rules*) The *pragmatic* language $\mathcal{L}^P$ is the union of the sets **Rad** of *radical formulas* and **Sent** of *sentential formulas*. These sets are defined inductively by the following grammar:

$$\alpha \;:=\; p \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\vee\alpha \mid \alpha\rightarrow\alpha \mid$$

$$\delta \;:=\; \vartheta \mid \upsilon \mid$$

$$\vartheta \;:= \vdash\alpha \mid \bigwedge \mid \;\sim\delta \mid \delta\supset\delta \mid \delta\cap\delta \mid \delta\cup\delta \mid$$

$$\upsilon \;:=\; \mathcal{H}\,\alpha \mid \bigwedge \mid \;\frown\delta \mid \delta\succ\delta \mid \delta\curlyvee\delta \mid \delta\curlywedge\delta \mid$$

(iii) The formulas α are called *radical formulas*. The formulas δ are called *pragmatic sentential formulas*; amonge these, $\vdash\alpha$, $\mathcal{H}\,\alpha$, $\bigwedge$ are called *elementary*, the formulas ϑ are *assertive* and the formulas υ are *conjectural*.

We use the letters α, β, α_1, $\ldots$ to denote *radical* formulas, η, η_1, $\ldots\ \ldots$ to denote *elementary sentential* formulas, ϑ, ϑ_1, $\ldots$ to denote *assertive sentences* and υ, υ_1, $\ldots$ to denote *conjectural sentences*.

The *intuitionistic fragment* of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary sentences to those with *atomic radical* only:

$$\bigwedge \quad \vdash p \qquad \mathcal{H}\,p$$

DEFINITION 2. (*Informal Interpretation*) (i) *Radical* formulas are interpreted as propositions, with the Tarskian classical semantics, as usual.

(ii) *Sentential expressions* are interpreted as follows:

1. $\vdash\alpha$ and $\mathcal{H}\,\alpha$ are interpreted as illocutionary acts of *assertion* and *conjectural assertion*, respectively. All such acts are regarded as *impersonal*, i.e., making abstraction from the specific attitudes of the subjects of such acts. Illocutionary acts can be *"justified"* (**J**) or *"unjustified"* (**U**); by extension, so are also the corresponding elementary sentential expressions.
2. $\vdash\alpha$ is *justified* if and only if there is a proof that α is true; it is *unjustified* otherwise. $\vdash\alpha$ is an *assertive* expression.
3. $\mathcal{H}\,\alpha$ is *justified* if there is no refutation of α, i.e., no proof that α is false; it is *unjustified* otherwise. $\mathcal{H}\,\alpha$ is a *conjectural* expression.

4. $\sim \delta$ is *justified* if and only if there is a proof that δ is unjustified; it is *unjustified* otherwise. $\sim \delta$ is an *assertive* expression.
5. $\frown \delta$ is *justified* if and only if there is no proof that δ is justified; it is *unjustified* otherwise. $\frown \delta$ is a *conjectural* expression.
6. $\delta_1 \supset \delta_2$ is *justified* if and only if there is a proof that a justification of δ_1 can be transformed into a justification of δ_2; it is *unjustified*, otherwise. $\delta_1 \supset \delta_2$ is an *assertive* expression.
7. $\delta_1 \succ \delta_2$ is *justified* if and only if there is no proof that δ_1 is justified and δ_2 is unjustified; it is *unjustified*, otherwise. $\delta_1 \succ \delta_2$ is a *conjectural* expression.
8. Let ϑ_1 and ϑ_2 be assertive expressions; then the *assertive expression* $\vartheta_1 \cap \vartheta_2$ is *justified* if and only if *both* ϑ_1 *and* ϑ_2 are justified; it is *unjustified* otherwise. Similarly, $\vartheta_1 \cup \vartheta_2$ is justified if and only if *either* ϑ_1 *or* ϑ_2 is justified.
9. Let υ be a conjectural expression; then the assertive expressions $\upsilon \cap \vartheta$ and $\vartheta \cap \upsilon$ are justified if and only if there is a proof that *both* υ *and* ϑ are justified; they are *unjustified* otherwise. Similarly, $\upsilon \cup \vartheta$ and $\vartheta \cup \upsilon$ are justified if and only if there is a proof that *either* υ *or* ϑ is justified.
10. Let υ_1 and υ_2 be conjectural expressions; then the *conjectural expression* $\upsilon_1 \curlywedge \upsilon_2$ is justified if and only if *both* υ_1 *and* υ_2 are justified; it is *unjustified* otherwise. Similarly, $\upsilon_1 \curlyvee \upsilon_2$ is justified if and only if *either* υ_1 *or* υ_2 is justified.
11. Let ϑ be an assertive expression: then the *conjectual expressions* $\vartheta \curlywedge \delta$ and $\delta \curlywedge \vartheta$ are justified if and only if there is no proof that *either* ϑ *or* δ is unjustified; they are unjustified otherwise. Similarly, $\vartheta \curlyvee \delta$ and $\delta \curlyvee \vartheta$ are justified if and only if there is no proof that *both* ϑ *and* δ are unjustified.

3.1. Topological interpretation. A mathematical model for the system $\mathcal{L}^P$ is obtained through a topological interpretation.

Definition 3. (*topological interpretation*). Let S be a set, let $\cap$, $\cup$ and $(\ )^C$ be the usual operations of intersection, union and complement defined on the powerset $\wp(S)$ of S and let $\mathbf{I} : \wp(S) \to \wp(S)$ and $\mathbf{C} : \wp(S) \to \wp(S)$ be the *interior* and *closure* operators, satisfying

$$\begin{array}{rl} \mathbf{I}(X) \subseteq X & X \subseteq \mathbf{C}(X) \\ \mathbf{I}(X) \subseteq \mathbf{I}(\mathbf{I}(X)) & \mathbf{C}(\mathbf{C}(X)) \subseteq \mathbf{C}(X) \\ X \subseteq Y \Rightarrow \mathbf{I}(X) \subseteq \mathbf{I}(Y) & X \subseteq Y \Rightarrow \mathbf{C}(X) \subseteq \mathbf{C}(Y) \\ \mathbf{C}(X) = (\mathbf{I}(X^C))^C & \text{and } \mathbf{I}(X) = (\mathbf{C}(X^C))^C \end{array}$$

A topological interpretation δ^* of the full language $\mathcal{L}^P$ is given by assigning to each atomic formula P a subset P^* of S and then by proceeding as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigwedge)^* & =_{df} & \emptyset & & & \\
(\vdash \alpha)^* & =_{df} & \mathbf{I}(\alpha^*) & (\mathcal{H}\, \alpha)^* & =_{df} & \mathbf{C}(\alpha^*) \\
(\sim \delta)^* & =_{df} & \mathbf{I}((\delta^*)^C) & (\frown \delta)^* & =_{df} & \mathbf{C}((\delta^*)^C) \\
(\delta_1 \supset \delta_2)^* & =_{df} & \mathbf{I}((\delta_1^*)^C \cup \delta_2^*)) & (\delta_1 \succ \delta_2)^* & =_{df} & \mathbf{C}((\delta_1^*)^C \cup \delta_2^*) \\
(\delta_1 \cap \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cap \mathbf{I}(\delta_2^*) & (\delta_1 \curlywedge \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cap \mathbf{C}(\delta_2^*) \\
(\delta_1 \cup \delta_2)^* & =_{df} & \mathbf{I}(\delta_1^*) \cup \mathbf{I}(\delta_2^*) & (\delta_1 \curlyvee \delta_2)^* & =_{df} & \mathbf{C}(\delta_1^*) \cup \mathbf{C}(\delta_2^*)
\end{array}
$$

Remark. Another translation $(\ )^{**}$ would be possible with the same clauses for the elementary formulas, for negation and implication, but with the following clauses for conjunctions and disjunctions:

$$
\begin{array}{rcl@{\qquad}rcl}
(\delta_1 \cap \delta_2)^{**} & =_{df} & \mathbf{I}(\delta_1^{**} \cap \delta_2^{**}) & (\delta_1 \curlywedge \delta_2)^{**} & =_{df} & \mathbf{C}(\delta_1^{**} \cap \delta_2^{**}) \\
(\delta_1 \cup \delta_2)^{**} & =_{df} & \mathbf{I}(\delta_1^{**} \cup \delta_2^{**}) & (\delta_1 \curlyvee \delta_2)^{**} & =_{df} & \mathbf{C}(\delta_1^{**} \cup \delta_2^{**})
\end{array}
$$

This amounts to defining *new* pragmatic connectives, which diverge from the standard meaning of intuitionistic disjunction and of its dual, conjectural conjunction, in the extended system. A motivation for introducing a new assertive disjunction, corresponding to the translation $(\ )^{**}$, would be the desire to make *"either* $\vdash \alpha$ *or* $\frown \vdash \alpha$*"* a valid principle of **ILP**. Clearly for "potential intuitionism" either there is a proof of α or there isn't one; therefore, it would seem that we would be justified in asserting this fact in general. We follow the translation $(\ )^*$ in the rest of this paper, leaving the issue of new connectives open to further investigation.

3.2. Modal interpretation. Another mathematical interpretation is obtained through an extension of Gödel, McKinsey and Tarski's modal translation $(\ )^{\Box}$ into the logic **S4**, where the notion of *validity* and *valid consequence* in the *Kripke semantics* for **S4** give semantical counterparts of the pragmatic notions of *p-validity* and *p-consequence*. The modal translation δ^M of the full language $\mathcal{L}^P$ is as follows:

Definition 4. (**S4** *translation*)

$$
\begin{array}{rcl@{\qquad}rcl}
(\bigwedge)^M & =_{df} & \bot & & & \\
(\vdash \alpha)^M & =_{df} & \Box\alpha & (\mathcal{H}\,\alpha)^M & =_{df} & \Diamond\alpha \\
(\sim \delta)^M & =_{df} & \Box\neg\delta^M & (\frown \delta)^M & =_{df} & \Diamond\neg\delta^M \\
(\delta_1 \supset \delta_2)^M & =_{df} & \Box(\delta_1^M \to \delta_2^M) & (\delta_1 \succ \delta_2)^M & =_{df} & \Diamond(\delta_1^M \to \delta_2^M) \\
(\vartheta_1 \cap \vartheta_2)^M & =_{df} & \vartheta_1^M \wedge \vartheta_2^M & (\upsilon_1 \curlywedge \upsilon_2)^M & =_{df} & \upsilon_1^M \wedge \upsilon_2^M \\
(\vartheta \cap \upsilon)^M & =_{df} & \vartheta^M \wedge \Box\upsilon^M & (\upsilon \curlywedge \vartheta)^M & =_{df} & \upsilon^M \wedge \Diamond\vartheta^M \\
(\upsilon \cap \vartheta)^M & =_{df} & \Box\upsilon^M \wedge \vartheta^M & (\vartheta \curlywedge \upsilon)^M & =_{df} & \Diamond\vartheta^M \wedge \upsilon^M \\
(\upsilon_1 \cap \upsilon_2)^M & =_{df} & \Box\upsilon_1^M \wedge \Box\upsilon_2^M & (\vartheta_1 \curlywedge \vartheta_2)^M & =_{df} & \Diamond\vartheta^M \wedge \Diamond\vartheta_2^M \\
(\vartheta_1 \cup \vartheta_2)^M & =_{df} & \vartheta_1^M \vee \vartheta_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & =_{df} & \upsilon_1^M \vee \upsilon_2^M \\
(\vartheta \cup \upsilon)^M & =_{df} & \vartheta^M \vee \Box\upsilon^M & (\upsilon \curlyvee \vartheta)^M & =_{df} & \upsilon^M \vee \Diamond\vartheta^M \\
(\upsilon \cup \vartheta)^M & =_{df} & \Box\upsilon^M \vee \vartheta^M & (\vartheta \curlyvee \upsilon)^M & =_{df} & \Diamond\vartheta^M \vee \upsilon^M \\
(\upsilon_1 \cup \upsilon_2)^M & =_{df} & \Box\upsilon_1^M \vee \Box\upsilon_2^M & (\vartheta_1 \curlyvee \vartheta_2)^M & =_{df} & \Diamond\vartheta_1^M \vee \Diamond\vartheta_2^M
\end{array}
$$

§4. Sequent calculus for the pragmatic language $\mathcal{L}^P$. The sequent calculus presented here is *unitary* in the sense that it contains fragments which formalize classical and intuitionistic reasoning, respectively: the classical fragment contains rules which modify the *radical part* of the principal formula in the sequent-conclusion; on the contrary, in the rules of the intuitionistic fragment the *radical parts* are regarded as atomic and remain unchanged throughout the derivation. As a corollary of this fact, the subformula property holds for the intuitionistic fragment, but not for the classical system; for the classical system considered here a weaker property holds, namely, the subformula property *for the radical parts*.

Here it is convenient to discuss the two fragments separately: we shall give the general definitions of a sequent and the identity and structural rules of the calculus first, then the two parts whose union generates the full unitary system for assertive reasoning.

Definition 5. All the sequents S are of the form

$$\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon$$

where

- Θ is a sequence of assertive formulas $\vartheta_1, \ldots, \vartheta_m$;
- Υ is a sequence of conjectural formulas $\upsilon_1, \ldots, \upsilon_n$;
- ϵ is conjectural and ϵ' is assertive and at most one of ϵ, ϵ' occurs in S.

identity rules

S.1: logical axiom: $\vartheta \; ; \; \Rightarrow \; \vartheta \; ;$
S.2: logical axiom: $; \; \upsilon \; \Rightarrow \; ; \; \upsilon$

S.3: falsity axiom: $\Theta, \bigwedge \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon$
S.4: falsity axiom: $\Theta \; ; \; \bigwedge \; \Rightarrow \; ; \; \Upsilon$
S.5: assertion-conjecture: $\vdash \alpha \; ; \; \Rightarrow \; ; \; \mathcal{H}\alpha$

S.6: cut₁:

$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon \qquad \vartheta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

S.7: cut₂:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \qquad \Theta' \; ; \; \upsilon \; \Rightarrow \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

structural rules

S.8: exchange:

$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

S.9: exchange:

$$\frac{\Theta, \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

S.10: contraction:

$$\frac{\vartheta, \vartheta, \Theta \; ; \; \epsilon \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

S.11: contraction:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \upsilon, \upsilon, \Upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \upsilon, \Upsilon}$$

S.12: weakening:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\vartheta, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

S.13: weakening:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon}$$

S.14: weakening:

$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon}$$

S.15: weakening:

$$\frac{\Theta \; ; \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon, \upsilon}$$

TABLE 3. The sequent calculus for $\mathcal{L}^P$, structural rules

§5. Sequent calculi for intuitionistic $\mathcal{L}^P$. It is also convenient to present the sequent calculus for intuitionistic pragmatic reasoning as extensions of the standard system.

1. The *standard system* is the fragment restricted to the *assertive formulas* with atomic radicals. This is just intuitionistic logic formalized in a sequent calculus with focalized hyper-sequents.
2. The *dual* of the standard system is the fragment restricted to the *conjectural formulas* with atomic radicals.
3. The system **ILP** extends the union of the standard system and of its dual with all possible connectives transforming pairs of

logical rules, connective of type $\vartheta \to \vartheta$

A.1: right negation:
$$\frac{\Theta, \vartheta \ ; \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \sim \vartheta \ ; \ \Upsilon}$$

A.2: left negation:
$$\frac{\sim \vartheta, \Theta; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\sim \vartheta, \Theta \ ; \ \Rightarrow \ ; \ \Upsilon}$$

logical rules, connectives of type $\vartheta \times \vartheta \to \vartheta$

A.3: right $\supset_m$:
$$\frac{\Theta, \vartheta_1 \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \supset \vartheta_2 \ ; \ \Upsilon}$$

A.4: left $\supset$:
$$\frac{\vartheta_1 \supset \vartheta_2, \Theta; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

A.5: right $\cap$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \Theta \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \cap \vartheta_2 \ ; \ \Upsilon}$$

A.6: left $\cap_m$:
$$\frac{\Theta, \vartheta_0, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta, \vartheta_0 \cap \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

A.7,8: right $\cup_a^i$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_i \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_0 \cup \vartheta_1 \ ; \ \Upsilon}$$
for $i = 0, 1$.

A.9: left $\cup$:
$$\frac{\Theta, \vartheta_0 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \quad \Theta, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta, \vartheta_0 \cup \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

TABLE 4. The standard (assertive) fragment of **ILP**

assertive or conjectural formulas into assertive or conjectural formulas.

Namely, in **ILP** we have the following connectives:

- all negations $\sim \delta$, $\frown \delta$, for $\delta = \vartheta$ or υ;
- all assertive implications $\delta_1 \supset \delta_2$, assertive conjunctions $\delta_1 \cap \delta_2$ and assertive disjunctions $\delta_1 \cup \delta_2$, for $\delta_i = \vartheta_i$ or υ_i, and $i = 1, 2$;
- all conjectural implications $\delta_1 \succ \delta_2$, conjectural conjunctions $\delta_1 \curlywedge \delta_2$ and conjectural disjunctions $\delta_1 \curlyvee \delta_2$, for $\delta_i = \vartheta_i$ or υ_i, and $i = 1, 2$.

The sequent calculus for intuitionistic pragmatic reasoning **ILP** is the formal system given by the rules in Tables 4, 5, 6 and 7 together with the structural rules of Table 3.

The sequent calculus **ILP** is quite large. There are 15 identity and structural rules (including the falsity rules, a rule relating assertions and conjectures and two cut-rules), and 74 logical rules. Among the logical rules, 9 are in the standard (assertive) fragment and 10

logical rules, connective of type $\upsilon \to \upsilon$

C.1: right doubt:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon, \frown \upsilon}{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon, \frown \upsilon}$$

C.2: left doubt
$$\frac{\Theta;\ \Rightarrow\ ;\ \Upsilon, \upsilon}{\Theta\ ;\ \frown \upsilon\ \Rightarrow\ ;\ \Upsilon}$$

logical rules, connectives of type $\upsilon \times \upsilon \to \upsilon$

C.3: right $\succ_a$:
$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \upsilon_2, \Upsilon, \upsilon_1 \succ \upsilon_2}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.4: right $\succ_m$:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.5: left $\succ$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon_1 \qquad \Theta\ ; \upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \upsilon_1 \succ \upsilon_2\ \Rightarrow\ ;\ \Upsilon}$$

C.6: right $\curlywedge$:
$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_0 \qquad \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_0 \curlywedge \upsilon_1}$$

C.7,8: left $\curlywedge^i_a$:
$$\frac{\Theta\ ;\ \upsilon_i\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \upsilon_0 \curlywedge \upsilon_1\ \Rightarrow\ ;\ \Upsilon}$$
for $i = 0, 1$.

C.9: right $\curlyvee_m$:
$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1, \upsilon_2}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \curlyvee \upsilon_2}$$

C.10: left $\curlyvee$:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon \qquad \Theta\ ;\ \upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \upsilon_1 \curlyvee \upsilon_2\ \Rightarrow\ ;\ \Upsilon}$$

TABLE 5. The dual (conjectural) fragment **ILP**

in its dual (conjectural fragment), and 55 additional (mixed) rules. Rules for binary connectives having one sequent-premise are either *multiplicative* or *additive*: e.g., *A.3: right* $\supset_m$ is multiplicative and the rules *A.7,8: right* $\cup_a$ are additive.

THEOREM 1. *The intuitionistic sequent calculus* **ILP** *without the rules of cut is sound and complete with respect to the modal interpretation in* **S4**.

Given a sequent S of the form

$$\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon$$

logical rules, connective of type $\upsilon \to \vartheta$

CA.1: right negation:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow \sim \upsilon\ ;\ \Upsilon}$$

CA.2: left negation
$$\frac{\sim \upsilon, \Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon}{\sim \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\vartheta \times \upsilon \to \vartheta$

ACA.1: right $\supset_m$:
$$\frac{\Theta, \vartheta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \supset \upsilon\ ;\ \Upsilon}$$

ACA.2: left $\supset$:
$$\frac{\vartheta \supset \upsilon, \Theta;\ \Rightarrow \vartheta\ ;\ \Upsilon \qquad \vartheta \supset \upsilon, \Theta\ ; \upsilon\ \Rightarrow\ ;\ \Upsilon}{\vartheta \supset \upsilon, \Theta\ ;\ \Rightarrow\ ;\ \Upsilon}$$

ACA.3: right $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cap \upsilon\ ;\ \Upsilon}$$

ACA.4: left $\cap_m$:
$$\frac{\Theta, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \vartheta \cap \upsilon\ ;\ \Rightarrow\ ;\ \Upsilon}$$

ACA.5: left $\cap_a$:
$$\frac{\Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \vartheta \cap \upsilon\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

ACA.6: right $\cup_a$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

ACA.7: right $\cup_a$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

ACA.8: left $\cup$:
$$\frac{\Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon \quad \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \vartheta \cup \upsilon\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\upsilon \times \vartheta \to \vartheta$

CAA.1: right $\supset_a$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

CAA.2: right $\supset_a$:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

CAA.3: left $\supset$:
$$\frac{\upsilon \supset \vartheta, \Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon \quad \vartheta, \upsilon \supset \vartheta, \Theta\ ; \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\upsilon \supset \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.4: right $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cap \vartheta\ ;\ \Upsilon}$$

CAA.5: left $\cap_m$:
$$\frac{\Theta, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon \cap \vartheta\ ;\ \Rightarrow\ ;\ \Upsilon}$$

CAA.6: left $\cap_a$:
$$\frac{\Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \upsilon \cap \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.7: right $\cup_a$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

CAA.8: right $\cup_a$:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

CAA.9: left $\cup$:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon \quad \Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \upsilon \cup \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\upsilon \times \upsilon \to \vartheta$

CCA.1: right $\supset_m$:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon, \upsilon_2}{\Theta\ ;\ \Rightarrow\ \upsilon_1 \supset \upsilon_2\ ;\ \Upsilon}$$

CCA.2: left $\supset$:
$$\frac{\upsilon_1 \supset \upsilon_2, \Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \qquad \upsilon_1 \supset \upsilon_2, \Theta\ ; \upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\upsilon_1 \supset \upsilon_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon;\ ;\ \Upsilon}$$

CCA.3: right $\cap$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_0, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon_1, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cap \upsilon_1\ ;\ \Upsilon}$$

CCA.4,5: left $\cap_a^i$:
$$\frac{\Theta\ ;\ \upsilon_i\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon_0 \cap \upsilon_1\ ;\ \Rightarrow\ ;\ \Upsilon}$$
for $i = 0, 1$.

CCA.6,7: right $\cup_a^i$:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_i, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cup \upsilon_1\ ;\ \Upsilon}$$

CCA.8: left $\cup$:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon \quad \Theta\ ;\ \upsilon_2\ \Rightarrow\ \ ;\ \Upsilon}{\Theta, \upsilon_1 \cup \upsilon_2\ ;\ \Rightarrow\ ;\ \Upsilon}$$
for $i = 0, 1$.

TABLE 6. **ILP**, 27 additional mixed-to-assertive rules

logical rules, connective of type $\vartheta \to \upsilon$

AC.1: right doubt:
$$\frac{\Theta,\vartheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon \frown \vartheta}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon, \frown \vartheta}$$

AC.2: left doubt
$$\frac{\Theta; \;\Rightarrow\; \vartheta \;;\; \Upsilon}{\Theta \;;\; \frown \vartheta \;\Rightarrow\; ; \Upsilon}$$

logical rules, connectives of type $\vartheta \times \upsilon \to \upsilon$

ACC.1: right $\succ_m$:
$$\frac{\Theta,\vartheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon,\vartheta \succ \upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\vartheta \succ \upsilon}$$

ACC.2: left $\succ$:
$$\frac{\Theta; \;\Rightarrow \vartheta \;;\; \Upsilon \qquad \Theta;\upsilon \;\Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta \succ \upsilon \;\Rightarrow\; ;\; \Upsilon}$$

ACC.3: right $\curlywedge$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon \qquad \Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\vartheta \curlywedge \upsilon}$$

ACC.4: left $\curlywedge_a$:
$$\frac{\Theta,\vartheta \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta \curlywedge \upsilon \;\Rightarrow\; ;\; \Upsilon}$$

ACC.5: left $\curlywedge_a$:
$$\frac{\Theta, \;;\; \upsilon \;\Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta \curlywedge \upsilon \;\Rightarrow\; ;\; \Upsilon}$$

ACC.6: right $\curlyvee_m$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon,\upsilon}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\vartheta \curlyvee \upsilon}$$

ACC.7: right $\curlyvee_a$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\vartheta \curlyvee \upsilon}$$

ACC.8: left $\curlyvee$:
$$\frac{\Theta,\vartheta \;;\; \Rightarrow\; ;\; \Upsilon \qquad \Theta \;;\; \upsilon \;\Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta \curlyvee \upsilon \;\Rightarrow\; ;\; \Upsilon}$$

logical rules, connectives of type $\upsilon \times \vartheta \to \upsilon$

CAC.1: right $\succ_a$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon,\upsilon \succ \vartheta}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\upsilon \succ \vartheta}$$

CAC.2: right $\succ_a$:
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow\; ;\; \Upsilon,\upsilon \succ \vartheta}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\upsilon \succ \vartheta}$$

CAC.3: left $\succ$:
$$\frac{\Theta; \;\Rightarrow\; ;\; \Upsilon,\upsilon \qquad \vartheta,\Theta \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \upsilon \succ \vartheta \;\Rightarrow\; ;\; \Upsilon}$$

CAC.4: right $\curlywedge$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon \qquad \Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon \curlywedge \vartheta}$$

CAC.5: left $\curlywedge_a$:
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \upsilon \curlywedge \vartheta \;\Rightarrow\; ;\; \Upsilon}$$

CAC.6: left $\curlywedge_a$:
$$\frac{\Theta,\vartheta \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \upsilon \curlywedge \vartheta \;\Rightarrow\; ;\; \Upsilon}$$

CAC.7: right $\curlyvee_m$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta \;;\; \Upsilon,\upsilon}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\upsilon \curlyvee \vartheta}$$

CAC.8: right $\curlyvee_a$:
$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\upsilon \curlyvee \vartheta}$$

CAC.9: left $\curlyvee$:
$$\frac{\Theta \;;\; \upsilon \;\Rightarrow\; ;\; \Upsilon \qquad \Theta,\vartheta \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \upsilon \curlyvee \vartheta \;\Rightarrow\; ;\; \Upsilon}$$

logical rules, connectives of type $\vartheta \times \vartheta \to \upsilon$

AAC.1: right $\succ_m$:
$$\frac{\Theta,\vartheta_1 \;;\; \Rightarrow\; \vartheta_2 \;;\; \Upsilon,\vartheta_1 \succ \vartheta_2}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\vartheta_1 \succ \vartheta_2}$$

AAC.2: right $\succ_a$:
$$\frac{\Theta,\vartheta_1 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\vartheta_1 \succ \vartheta_2}{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon,\vartheta_1 \succ \vartheta_2}$$

AAC.3: left $\succ$:
$$\frac{\Theta; \;\Rightarrow \vartheta_1 \;;\; \Upsilon \qquad \vartheta_2,\Theta \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta_1 \succ \vartheta_2 \;\Rightarrow\; ;\; \Upsilon}$$

AAC.4: right $\curlywedge$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta_1 \;;\; \Upsilon \qquad \Theta \;;\; \Rightarrow\; \vartheta_2;\; \Upsilon,\upsilon}{\Theta \;;\; \Rightarrow\; ;\; \Upsilon,\vartheta_1 \curlywedge \vartheta_2}$$

AAC.5,6: left $\curlywedge_a$:
$$\frac{\Theta,\vartheta_i \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta_0 \curlywedge \vartheta_1 \;\Rightarrow\; ;\; \Upsilon}$$
for $i = 0, 1$.

AAC.7,8: right $\curlyvee_a^i$:
$$\frac{\Theta \;;\; \Rightarrow\; \vartheta_i \;;\; \Upsilon}{\Theta \;;\; \Rightarrow\; ;\; \vartheta_0 \curlyvee \vartheta_1, \Upsilon}$$
for $i = 0, 1$.

AAC.9: left $\curlyvee$:
$$\frac{\Theta,\vartheta_0 \;;\; \Rightarrow\; ;\; \Upsilon \qquad \Theta,\vartheta_1 \;;\; \Rightarrow\; ;\; \Upsilon}{\Theta \;;\; \vartheta_0 \curlyvee \vartheta_1 \;\Rightarrow\; ;\; \Upsilon}$$

TABLE 7. **ILP**, 28 additional mixed-to-conjectural rules

and its modal translation

$$\Theta^M \; ; \; \epsilon^M \;\Rightarrow\; (\epsilon')^M \; ; \; \Upsilon^M$$

we simulate the semantical procedure of section 7.2.3 by inverting the rules of **ILP** and then applying the translation $(\;)^M$ at each step. We claim that this yields either a derivation of S in **ILP** or a countermodel for S^M in **S4**.

We consider the construction of the refutation tree τ^M for S^M as it results through the map $(\;)^M$ from the inversion of the rules in Tables 4, 5, 6 and 7. We classify our 74 logical rules in four groups:

1. *pragmatic rules corresponding to invertible propositional rules only* (6 rules):
   (i) *A.5: right-*$\cap$, *A.6: left-*$\cap_m$, *A.9: left-*$\cup$, and
   (ii) *C.6: right-*$\curlywedge$, *C.9: right-*$\curlyvee_m$, *C.10: left-*$\curlyvee$.
   Inverting one among these six rules corresponds to inverting the propositional rules for disjunction $\vee$ and conjunction $\wedge$, which are valid and semantically invertible.
2. *pragmatic rules corresponding to invertible propositional rules together with* $\Box$*-left or* $\Diamond$*-right and possibly structural rules* (21 rules):
   (i) *A.2* and *CA.2: left negation*;
   (ii) *A.4*, *ACA.2*, *CAA.3* and *CCA.2: left* $\supset$;
   (iii) *ACA.8*, *CAA.9* and *CCA.8: left* $\cup$;
   (iv) *ACA.4* and *CAA.5: left* $\cap_m$;
   (v) *C.1* and *AC.1: right doubt*;
   (vi) *C.4*, *ACC.1* and *AAC.1: right* $\succ_m$;
   (vii) *ACC.3*, *CAC.4* and *AAC.4: right* $\curlywedge$;
   $(viii)$ *ACC.6* and *CAC.7: right* $\curlyvee_m$.

   Inverting one among these twenty-one rules of the sequent calculus **ILP** corresponds to inverting a $\Box$-left rule or $\Diamond$-right rule and a propositional rule in the sequent calculus for **S4**, where an implicit contraction rule may be performed. Since in **S4** the rules $\Box$-left and $\Diamond$-right are valid and semantically invertible, it follows that all of these twenty-one rules are valid and semantically invertible with respect to the modal interpretation.

   Since the semantic-tableaux procedure for **ILP** is an extension of the semantic-tableaux procedure for **S4** summarized in section 7.2.3, notice that if an implicit contraction is performed then the occurrence of the active formula in the upper sequent must be *marked*, and it should not be considered again in the current branches of the refutation tree until a rule the corresponding to a $\Box$-right or $\Diamond$-left inference is performed. (Here we ignore issues of optimization of the semantic-tableaux algorithm).

3. *pragmatic rules corresponding to $\Box$-right or $\Diamond$-left, together with a propositional rule* (17 rules):
   (*i*) *A.1* and *CA.1: right negation*;
   (*ii*) *A.3*, *ACA.1* and *CCA.1: right* $\supset_m$;
   (*iii*) *ACA.3*, *CAA.4* and *CCA.3: right* $\cap$;
   (*iv*) *C.2* and *AC.2: left doubt*;
   (*v*) *C.5*, *ACC.2*, *CAC.3* and *AAC.3: left* $\succ$;
   (*vi*) *ACC.8*, *CAC.9* and *AAC.9: left* $\curlyvee$.

   The rules in (*i*), (*ii*), (*iv*) and (*v*) correspond to a $\Box$-*right* or $\Diamond$-*left* inference $\mathcal{I}_1$ following a propositional rule inference $\mathcal{I}_2$, while the rules in (*iii*) and (*vi*) correspond to an $\Box$-*right* or $\Diamond$-*left* inference $\mathcal{I}_1$ followed by a propositional rule $\mathcal{I}_2$. In both cases the active formula $\mathcal{I}_1$ is the principal formula of $\mathcal{I}_2$.

   The rules $\Box$-*right* and $\Diamond$-*left* in the sequent calculus for **S4** are valid and semantically invertible if the restriction on the passive formulas is satisfied. Therefore all our seventeen rules are valid and invertible with respect to the modal translation if the restrictions on the passive formulas are preserved in the translation. This is guaranteed by the use of focalized hyper-sequents: indeed the translation S^M of an **ILP** sequent has the form

   $$\Box\Theta^M \ , \Diamond(\epsilon^M) \ \Rightarrow \ \Box(\epsilon')^M \ , \Diamond\Upsilon^M$$

   where at most one formula among $\{\Diamond(\epsilon^M),\ \Box(\epsilon')^M\}$ actually occurs in S^M.

4. *pragmatic rules corresponding to propositional and modal rules involving a disjunctive ramification* (30 rules).
   All the *additive* rules of the sequent calculus **ILP** are not semantically invertible in the modal translation. Following the semantic-tableaux procedure for **S4** in 7.2.3, these cases are dealt with (what we call) a *disjunctive ramification*: here the construction of a counterexample requires a countermodel on *both* branches, while only one valid premise suffice to infer the conclusion by an implicit use of *weakening*.

(a) The rules *A.7, 8: right-*$\cup_a$ are similar to those in the first group, but the focalization of the hyper-sequents requires a *disjunctive ramification* of the form

$$\begin{array}{ccc} \Theta\ ;\ \Rightarrow\ \vartheta_0\ ;\ \Upsilon & & \Theta\ ;\ \Rightarrow\ \vartheta_1\ ;\ \Upsilon \\ & \vdots & \\ & \Theta\ ;\ \Rightarrow\ \vartheta_0 \cup \vartheta_1\ ;\ \Upsilon & \end{array}$$

where the left branching inverts the rule *A.7* and the right branching the rule *A.8*. We proceed similarly for the pair of rules *C.7,8: left-*$\curlywedge_a$.

(b) The following rules are similar to those in the third group,
CAA.1: right $\supset_a$ and *CAA.2: right* $\supset_a$
but the semantic-tableaux procedure requires a *disjunctive ramification* of the form

$$\begin{array}{ccc} \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon & & \Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \\ & \vdots & \\ & \Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon & \end{array}$$

The following pairs are handlead similarly:
(*i*) *ACA.6,7 CAA.7,8* and *CCA.6,7: right* $\cup_a$;
(*ii*) *CCA.4,5: left* $\cap^i_a$, for and $i = 0, 1$;
(*iii*) *CAC.1,2: right* $\succ_a$;
(*iv*) *ACC.4,5*, *CAC.5,6* and *AAC.5,6: left* $\curlywedge_a$;
(*v*) *AAC.7,8: right* $\curlyvee^1_a$.

(c) Finally, there are rules that can be treated multiplicatively and are semantically invertible only if the focalized part of the sequent-conclusion is empty. Otherwise, we have two alternatives: either we discard the focalized part and then invert the rule in its *multiplicative* form, or we invert the rule in its *additive* form, thus erasing one of the immediate subformulas of the principal formula. The two alternatives are preserved by the *disjunctive ramification*.

We follow this procedure for the following pairs of rules:
(*i*) *ACA.4: left* $\cap_m$, *ACA.5: left* $\cap_a$;
(*ii*) *CAA.5: left* $\cap_m$, *CAA.6: left* $\cap_a$;
(*iii*) *C.3: right* $\succ_a$, *C.4: right* $\succ_m$;
(*iv*) *ACC.6: right* $\curlyvee_m$, *ACC.7: right* $\curlyvee_a$;
(*v*) *CAC.7: right* $\curlyvee_m$, *CAC.8: right* $\curlyvee_a$;

(*vi*) *AAC.1: right* $\succ_m$, *AAC.2: right* $\succ_a$.

For instance, the first pair may occur in a disjunctive ramification of the form

$$\begin{array}{ccc} \Theta, \vartheta \; ; \; \upsilon \; \Rightarrow \; ; \; \Upsilon & & \Theta, \vartheta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon \\ & \vdots & \\ & \Theta, \vartheta \cap \upsilon \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon & \end{array}$$

In all cases the construction of the refutation tree carries over from the **S4** procedure. Further details are left to the reader.

PROPOSITION 1. *The pragmatic connectives satisfy the following significant equivalences:*

$$\begin{array}{cc}
(\vartheta_1 \supset (\vartheta_2 \supset \vartheta_3)) \equiv ((\vartheta_1 \cap \vartheta_2) \supset \vartheta_3) & (\upsilon_1 \succ (\upsilon_2 \succ \upsilon_3)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \succ \upsilon_3) \\
(\vartheta_1 \supset (\vartheta_2 \supset \upsilon)) \equiv ((\vartheta_1 \cap \vartheta_2) \supset \upsilon) & (\upsilon_1 \succ (\upsilon_2 \succ \vartheta)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \succ \vartheta) \\
(\upsilon_1 \supset (\upsilon_2 \supset \vartheta)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \supset \vartheta) & (\vartheta_1 \succ (\vartheta_2 \succ \upsilon)) \equiv ((\vartheta_1 \cap \vartheta_2) \succ \upsilon) \\
(\upsilon_1 \supset (\upsilon_2 \supset \upsilon_3)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \supset \upsilon_3) & (\vartheta_1 \succ (\vartheta_2 \succ \vartheta)) \equiv ((\vartheta_1 \cap \vartheta_2) \succ \vartheta) \\
 & \\
(\vartheta \supset (\vartheta_1 \cap \vartheta_2)) \equiv (\vartheta \supset \vartheta_1) \cap (\vartheta \supset \vartheta_2) & ((\upsilon \succ \upsilon_1) \curlyvee (\upsilon \succ \upsilon_2)) \equiv (\upsilon \succ (\upsilon_1 \curlyvee \upsilon_2)) \\
(\upsilon \supset (\vartheta_1 \cap \vartheta_2)) \equiv (\upsilon \supset \vartheta_1) \cap (\upsilon \supset \vartheta_2) & ((\vartheta \succ \upsilon_1) \curlyvee (\vartheta \succ \upsilon_2)) \equiv (\vartheta \succ (\upsilon_1 \curlyvee \upsilon_2)) \\
(\vartheta \supset (\upsilon_1 \curlywedge \upsilon_2)) \equiv (\vartheta \supset \upsilon_1) \cap (\vartheta \supset \upsilon_2) & ((\upsilon \succ \vartheta_1) \curlyvee (\upsilon \succ \vartheta_2)) \equiv (\upsilon \succ (\vartheta_1 \curlyvee \vartheta_2)) \\
(\upsilon \supset (\upsilon_1 \curlywedge \upsilon_2)) \equiv (\upsilon \supset \upsilon_1) \cap (\upsilon \supset \upsilon_2) & ((\vartheta \succ \vartheta_1) \curlyvee (\vartheta \succ \vartheta_2)) \equiv (\vartheta \succ (\vartheta_1 \curlyvee \vartheta_2)) \\
 & \\
((\vartheta_1 \cup \vartheta_2) \supset \vartheta) \equiv (\vartheta_1 \supset \vartheta) \cap (\vartheta_2 \supset \vartheta)) & ((\upsilon_1 \succ \upsilon) \curlyvee (\upsilon_2 \succ \upsilon)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \succ \upsilon) \\
((\vartheta_1 \cup \vartheta_2) \supset \upsilon) \equiv (\vartheta_1 \supset \upsilon) \cap (\vartheta_2 \supset \upsilon)) & ((\upsilon_1 \succ \vartheta) \curlyvee (\upsilon_2 \succ \vartheta)) \equiv ((\upsilon_1 \curlywedge \upsilon_2) \succ \vartheta) \\
((\upsilon_1 \curlyvee \upsilon_2) \supset \vartheta) \equiv (\upsilon_1 \supset \vartheta) \cap (\upsilon_2 \supset \vartheta)) & ((\vartheta_1 \succ \upsilon) \curlyvee (\vartheta_2 \succ \upsilon)) \equiv ((\upsilon_1 \cap \upsilon_2) \succ \upsilon) \\
((\upsilon_1 \curlyvee \upsilon_2) \supset \upsilon) \equiv (\upsilon_1 \supset \vartheta) \cap (\upsilon_2 \supset \upsilon)) & ((\vartheta_1 \succ \vartheta) \curlyvee (\vartheta_2 \succ \vartheta)) \equiv ((\vartheta_1 \cap \vartheta_2) \succ \vartheta)
\end{array}$$

§6. Sequent calculus for classical $\mathcal{L}^P$. We briefly consider classical reasoning in the framework of formal pragmatics. Here we take the *principle of compositionality* as an optimal criterion for the definition of pragmatic rules of inference: namely, we are looking for rules of inference which would allow us to infer more complex formulas from simpler ones. Clearly in this way we formalize only a *constructive* fragment of classical reasoning within $\mathcal{L}^P$.

Moreover, we want a sequent calculus where both the *left* and *right* rules *preserve validity* and are *semantically invertible* in the **S4** translation. A set of rules satisfying our requirements is given in the following table 8: this is a fragment of classical reasoning for which the soundness and completeness theorem with respect to the semantic interpretation in **S4** can be easily proved.

right assert-negation:
$$\frac{\Theta\ ;\ \mathcal{H}\alpha\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vdash\neg\alpha\ ;\ \Upsilon}$$

left assert-negation
$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon,\ \mathcal{H}\alpha}{\vdash\neg\alpha, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\Upsilon}$$

right hyp-negation:
$$\frac{\Theta,\ \vdash\alpha\ ;\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon,\ \mathcal{H}\neg\alpha}$$

left hyp-negation
$$\frac{\Theta\ ;\ \Rightarrow\ \vdash\alpha\ ;\ \Upsilon}{\Theta\ ;\ \mathcal{H}\neg\alpha\ \Rightarrow\ ;\ \Upsilon}$$

right hyp-impl:
$$\frac{\Theta,\ \vdash\alpha\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon,\ \mathcal{H}(\alpha\to\beta),\ \mathcal{H}\beta}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon,\ \mathcal{H}(\alpha\to\beta)}$$

left hyp-impl:
$$\frac{\Theta\ ;\ \Rightarrow\vdash\alpha\ ;\ \Upsilon \qquad \Theta\ ;\ \mathcal{H}\beta\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \mathcal{H}(\alpha\to\beta)\ \Rightarrow\ ;\ \Upsilon}$$

right assert-and:
$$\frac{\Theta\ ;\ \Rightarrow\ \vdash\alpha\ ;\ \Upsilon \qquad \Theta\ ;\ \Rightarrow\ \vdash\beta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vdash(\alpha\wedge\beta)\ ;\ \Upsilon}$$

left assert-and:
$$\frac{\Theta,\ \vdash\alpha,\ \vdash\beta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta,\ \vdash(\alpha\wedge\beta)\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

right hyp-or:
$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \mathcal{H}\alpha,\ \mathcal{H}\beta, \Upsilon}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \mathcal{H}(\alpha\vee\beta), \Upsilon}$$

left hyp-or
$$\frac{\Theta\ ;\ \mathcal{H}\alpha\ \Rightarrow\ ;\ \Upsilon \quad \Theta\ ;\ \mathcal{H}\beta\ \Rightarrow\ ;\ \Upsilon}{\Theta,\ \mathcal{H}(\alpha\vee\beta)\ ;\ \Rightarrow\ ;\ \Upsilon}$$

TABLE 8. Classical sequent calculus

DEFINITION 6. (i) Consider the following grammar for *radical formulas*:

$$\begin{array}{lcl|l|l|l|}
\mathbf{P} & := & p & \neg\mathbf{N} & \mathbf{P}\wedge\mathbf{P} & \\
\mathbf{N} & := & p & \neg\mathbf{P} & \mathbf{N}\vee\mathbf{N} & \mathbf{P}\to\mathbf{N}
\end{array}$$

(ii) Consider the sublanguage of $\mathcal{L}^P$ where elementary pragmatic expressions are generated by the following rules:

$$\vartheta\ :=\ \vdash\mathbf{P} \qquad\qquad \upsilon\ :=\ \mathcal{H}\mathbf{N}.$$

Let us call such a language *the basic classical language*.

(iii) The *basic classical sequent calculus* is system of sequent calculus for classical logic where sequents are restricted to elementary formulas in the basic classical language, i.e., sequents have one of the forms

$$\vdash\alpha_1, \dots, \vdash\alpha_m\ ;\ \Rightarrow\ \vdash\alpha\ ;\ \mathcal{H}\beta_1, \dots, \mathcal{H}\beta_n$$
$$\vdash\alpha_1, \dots, \vdash\alpha_m\ ;\ \mathcal{H}\beta\ \Rightarrow\ ;\ \mathcal{H}\beta_1, \dots, \mathcal{H}\beta_n$$

where the α_i, α are of the form $\mathbf{P}$ and the β_j, β are of the form $\mathbf{N}$.

THEOREM 2. *The basic classical sequent calculus is sound and complete with respect to the modal interpretation in* **S4**.

To prove the theorem, notice that in the semantics of **S4** there is a countermodel to the translation of the sequent-conclusion if and only if there is a countermodel to the translation of at least one sequent-premise. For sequents consisting of elementary formulas whose radical is in the basic classical language, there is always a rule in the basic sequent calculus which can be applied, until we reach a sequent where all elementary formulas have atomic radicals. Therefore we can apply the semantical procedure of section 7.2.3 to the translations of the sequents.

Consider the following translation $(\ )^P$:

$$\begin{array}{rcl@{\qquad}rcl}
(\mathbf{P})^P & =_{df} & \vdash p & \text{if } \mathbf{P} & := & p \\
(\mathbf{N})^P & =_{df} & \mathcal{H}\, p & \text{if } \mathbf{N} & := & p \\
(\neg\mathbf{N})^P & =_{df} & \sim(\mathbf{N}^P) & (\mathbf{P}\wedge\mathbf{P})^P & =_{df} & \mathbf{P}^P \cap \mathbf{P}^P \\
(\neg\mathbf{P})^P & =_{df} & \frown(\mathbf{P}^P) & (\mathbf{N}\vee\mathbf{N})^P & =_{df} & \mathbf{N}^P \curlyvee \mathbf{N}^P \\
(\mathbf{P}\to\mathbf{N})^P & =_{df} & \mathbf{P}^P \succ \mathbf{N}^P) & & &
\end{array}$$

where the conditions $\mathbf{P} := p$ and $\mathbf{N} := p$ in the first two rules refer to the productions of the grammar generating the radical formulas.

THEOREM 3. *Let S be a sequent consisting of elementary formulas in the basic classical language. Then S is derivable in the classical sequent calculus if and only if S^P is derivable in the intuitionistic sequent calculus.*

§7. APPENDIX. Syntax of $\mathcal{L}^M$ and semantics for K and S4.

DEFINITION 7. (*Syntax*) (i) The language $\mathcal{L}^m$ is built from an infinite set **Atoms** of *propositional letters* p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\to$; and the modal operators $\Box$ and $\Diamond$.

(ii) (*Formation Rules*) The expressions of the language $\mathcal{L}^m$ are given by the following grammar, where p ranges over **Atoms**:

$$\alpha \ := \ p \mid \bot \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\vee\alpha \mid \alpha\to\alpha \mid \Box\alpha \mid \Diamond\alpha \mid$$

7.1. Frames and Kripke models.

DEFINITION 8. (*Frames and Kripke models*) (i) A *frame* is a pair $\mathcal{F} = (W, \sqsubseteq)$ where

- W is a set (of *"possible worlds"*);
- $\sqsubseteq \subset W \times W$ is a relation (the *"accessibility relation"* between possible worlds).

(ii) A *Kripke model* is a triple $\mathcal{M} = (W, \sqsubseteq, \Vdash)$ where $\mathcal{F} = (W, \sqsubseteq)$ is a frame and $\Vdash \subseteq W \times \mathbf{Atoms}$ is the *forcing relation*, usually written in infix notation: $w \Vdash p$ means *"p is true in the possible world w"* and $w \nVdash p$ means *"p is false in the possible world w"*.

(iii) The relation $\Vdash$ is extended to a relation $\Vdash \subset W \times \mathcal{L}^m$ according to the following rules:

1. $w \nVdash \bot$ for all $w \in W$;
2. $w \Vdash \neg\alpha$ iff $w \nVdash \alpha$;
3. $w \Vdash (\alpha \wedge \beta) = V$ iff $w \Vdash \alpha$ and $w \Vdash \beta$;
4. $w \Vdash (\alpha \vee \beta)$ iff $w \Vdash \alpha$ or $w \Vdash \beta$;
5. $w \Vdash (\alpha \to \beta)$ iff either $w \nVdash \alpha$ or $w \Vdash B$;
6. $w \Vdash \Box\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that $w' \sqsubseteq w$;
7. $w \Vdash \Diamond\alpha$ iff $w' \Vdash \alpha$ for some $w' \in W$ such that $w' \sqsubseteq w$.

If Γ and Δ are sets of formulas in $\mathcal{L}^m$, then the *sequent* $\Gamma \Rightarrow \Delta$ is *true in* $w \in W$ iff $w \Vdash (\bigwedge \Gamma \to \bigvee \Delta)$.

(iv) We say that a formula α is *valid in a model* $\mathcal{M} = (W, \sqsubseteq, \Vdash)$, in symbols $\models_{\mathcal{M}} \alpha$, iff for every $w \in W$ we have $w \Vdash \alpha$. Similarly, given a *sequent* $S = \Gamma \Rightarrow \Delta$ we say that S is *valid in* $\mathcal{M}$ iff for every $w \in W$, S is true in w.

(v) We say that a formula α is *valid in a frame* $\mathcal{F}$ iff for every $\mathcal{M}$ over $\mathcal{F}$ we have $\models_{\mathcal{M}} \alpha$. Similarly, a sequent S is *valid in a frame* $\mathcal{F}$ iff it is valid in every Kripke model over $\mathcal{F}$.

(vi) A formula α [a sequent S] is *valid* in the system **K** iff α [S] it is valid in all Kripke models $\mathcal{M}$.

(viii) A formula α [a sequent S] is *valid* in the system **KD** iff α [S] is valid in all frames without terminal points, i.e., all frames where $\forall w.\exists w'.w' \sqsubseteq w$.

(viii) A formula α [a sequent S] is *valid* in the system **S4** iff α [S] is valid in all preordered frames, i.e., all frames where the accessibility relation $\sqsubseteq$ is reflexive and transitive.

7.2. Sequent calculi G3c, K and S4. Gentzen-Kleene's sequent calculus **G3c** for classical propositional logic (cfr.[25], p. 77) is given by the following sequent-axioms and rules of inference:

identity axioms: $p, \Gamma \Rightarrow \Delta, p$

falsity axioms: $\bot, \Gamma \Rightarrow \Delta$

logical rules

right $\neg$:
$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

left $\neg$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}$$

right $\wedge$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

left $\wedge$:
$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}$$

right $\to$:
$$\frac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \to \beta, \Delta}$$

left $\to$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \to \beta, \Gamma \Rightarrow \Delta}$$

right $\vee$:
$$\frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

left $\vee$:
$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}$$

TABLE 9. Sequent Calculus G3c

DEFINITION 9. (i) Given a notion of semantic validity, a rule of the sequent calculus $\dfrac{S_1, \ldots, S_n}{S}$ *preserves validity* if for every instance of the rule, the sequent conclusion S is valid whenever the sequent-premises $S_1, \ldots, S_n$ are all valid; a rule is *semantically invertible* if for every instance of the rule the sequent-premises are all valid whenever the sequent-conclusion is valid.

PROPOSITION 2. *(i) The rules of the system* **G3c** *preserve validity and are semantically invertible for any modal semantics;*
(ii) the modal rules for the system **K** *preserve validity and are semantically invertible in the semantics of system* **K***;*
(iii) the modal rules for the system **S4** *preserve validity and are semantically invertible in the semantics of the system* **S4***;*
(iv) the rules of weakening *preserve validity but are not semantically invertible:*

$$\frac{\Gamma \Rightarrow \Delta,}{\Gamma \Rightarrow \Delta, \alpha} \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}$$

weakening:
$$\frac{\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta}{\Xi, \Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta, \Lambda}$$

weakening:
$$\frac{\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta}{\Xi, \Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta, \Lambda}$$

modal rules for K

K-$\Box$-*rule*:
$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta}$$

K-$\Diamond$-*rule*:
$$\frac{\Gamma, \alpha \Rightarrow \Delta}{\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta}$$

additional rule for KD

D-*rule*:
$$\frac{\Gamma \Rightarrow \Delta}{\Box\Gamma \Rightarrow \Diamond\Delta}$$

modal rules for S4

$\Box$ *left*:
$$\frac{\alpha, \Box\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}$$

$\Box$ *right*:
$$\frac{\Box\Gamma \Rightarrow \alpha, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box\alpha, \Diamond\Delta}$$

$\Diamond$ *left*:
$$\frac{\Box\Gamma, \alpha \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond\alpha \Rightarrow \Diamond\Delta}$$

$\Diamond$ *right*:
$$\frac{\Gamma \Rightarrow \Delta, \Diamond\alpha, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}$$

TABLE 10. Systems **K**, **KD** and **S4**

7.2.1. *Semantic Tableaux procedure for* **K**. The "semantic tableaux" procedure decides whether a sequent S is valid in the semantics for **K** by building a *refutation tree* labelled with sequents and with S at the root; if S is valid, then it return a derivation of S in the sequent calculus for **K**; if S not valid, it returns a counterexample $\mathcal{M}$ which refutes S.

DEFINITION 10. (o) Start with tree τ_0 consisting of the root S; $(n+1)$ for every leaf S' of the tree τ_n, check whether the sequent S' matches the conclusion of a rule of inference (in some given order, e.g., checking the one-premised rules first). If yes, invert that rule; otherwise, the leaf in question is a sequent of the form

$$p_1, \ldots, p_k, \Box\Gamma, \Diamond\alpha_1, \ldots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \ldots, \Box\beta_n, \Diamond\Delta, q_1, \ldots, q_\ell \qquad (\dagger)$$

and we have four cases:

(a) $p_i = q_j$ for some $i \leq k$, $j \leq \ell$: in this case the sequent (†) is a *logical axiom* and the procedure halts on this branch;
(b) $p_i = \bot$ for some $i \leq k$: in this case the sequent (†) is a *falsity axiom* and the procedure halts on this branch;
(c) otherwise, if (†) is not an axiom and $m = 0 = n$, then the procedure halts on this branch leaving it *open*;
(d) otherwise, (†) is not an axiom and $m + n > 0$: in this case the procedures *branches* considering all possible sub-sequents S_1, ... S_{m+n} of (†) which may be conclusions of a modal inference and from which (†) may be derived by repeated applications of *weakening*, as in the figure below.

$$\dfrac{\Gamma, \alpha_i \Rightarrow \Delta}{\Box\Gamma, \Diamond\alpha_i \Rightarrow \Diamond\Delta}\mathbf{KR} \qquad \cdots \qquad \dfrac{\Gamma \Rightarrow \beta_j, \Delta}{\Box\Gamma \Rightarrow \Box\beta_j, \Diamond\Delta}\mathbf{KR}$$

$$\text{for all } i \leq m \qquad\qquad \text{for all } j \leq n$$

$$\vdots$$

$$p_1, \dots, p_h, \Box\Gamma, \Diamond\alpha_1, \dots, \Diamond\alpha_m \Rightarrow \Box\beta_1, \dots, \Box\beta_n, \Diamond\Delta, q_1, \dots, q_\ell \qquad (\dagger)$$

$$(\bot \neq p_i \neq q_j \text{ for all } i \leq h\ j \leq \ell, \text{ and also } n + m > 0)$$

TABLE 11. "Disjunctive branching"

DEFINITION 11. We define inductively what it means for a refutation tree τ to be *closed* (starting from the *leaves*):

- a *logical axiom* or a *falsity axiom* is closed;
- if τ results from τ_0 by a *one-premise* inference rule, then τ is closed iff τ_0 is closed;
- if τ results from τ_0 and τ_1 by a *two-premises* inference rule, then τ is closed iff τ_0 and τ_1 are both closed;
- if τ results from τ_1, ..., τ_{m+n} by a *disjunctive ramification*, then τ is closed iff at least one τ_i is closed, for $i \leq m + n$.

Fact 1: *The semantic tableax procedure for* **K** *terminates.*

Fact 2: *If a refutation tree τ with conclusion S is closed, then we can obtain a derivation of S in the sequent calculus for* **K** *as follows:*

- for each disjunctive ramification from a sequent of the form (†), first we *prune* τ by selecting a closed subtree τ_i and then obtain the sequent (†) by suitable applications of *weakening.*

Fact 3: *If a refutation tree τ with conclusion S is open, the we can construct a Kripke model $\mathcal{M}$ which refutes S*:

- for every two-premises logical rule, if the sequent-conclusion is open, then we select one of the sequent-premises which is open. In this way we eventually obtain a tree τ' where all branches are open.
- Consider all fragments of branches $\beta_1, \ldots, \beta_z$ obtained from τ' by removing every modal inference and every disjunctive ramification:
  (i) identify β_i with a possible world w_i;
  (ii) put $w_i \sqsubseteq w_j$ if and only if the lowermost sequent of β_i is the premise of a **KR** occurring immediately above a disjunctive ramification from a sequent S^* of the form (†) and S^* is the uppermost sequent of β_j;
  (iii) let $w_i \Vdash p_i$ if and only if p_i occurs in the antecedent of a sequent S^* of the form (†) and S^* is the uppermost sequent of β_i.

From facts 1-3 we obtain the following theorem:

Theorem 4. *The semantic tableaux procedure for* **K** *is sound and complete with respect to the semantics of* **K**.

7.2.2. *Semantic Tableaux procedure for* **KD**. Notice that we may apply a **D**-rule even when $\Box\Gamma$ in the antecedent and $\Diamond\Delta$ in the consequent are empty. Therefore on each branch we eventually have a sequent of the form (†) such that above it only one **D**-rule is inverted of the form

$$\frac{\Rightarrow}{\Rightarrow}\mathbf{D}\text{-}rule$$

which could be iterated forever. Instead, the procedure stops on such a branch, but in correspondence of such a **D**-rule we slightly modify the construction in **Fact 3** by introducing a possible world w_j such that $w_j \sqsubseteq w_j$ and such that $w_j \nVdash p$ for all atom p. This suffices to satisfy the condition for **KD** frames, since in every coutermodel no possible world will be terminal. It follows that

THEOREM 5. *The semantic tableaux procedure for* **KD** *is sound and complete with respect to the semantics of* **KD**.

7.2.3. *Semantic Tableaux procedure for* **S4**. In the case of **S4** two modifications are required to the procedure to deal with the fact that the above procedure inevitably enters infinte loops. The first problem comes form the $\Box$-left and $\Diamond$-right rules, which could be trivially iterated forever. It is enough to mark the modal formulas which such a rule has been applied to and remove the mark only when a $\Box$-right or $\Diamond$-left rule is inverted; in other words we may take modal rules of the forms

$\Box$ *left*:
$$\frac{\alpha, \Gamma, \underline{\Box\alpha}, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}}{\Box\alpha, \Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \underline{\Box\Lambda}}$$

$\Box$ *right*:
$$\frac{\Box\Gamma \Rightarrow \alpha, \Diamond\Delta}{\underline{\Box\Gamma} \Rightarrow \Box\alpha, \underline{\Diamond\Delta}}$$

$\Diamond$ *left*:
$$\frac{\Box\Gamma, \alpha \Rightarrow \Diamond\Delta}{\underline{\Box\Gamma}, \Diamond\alpha \Rightarrow \underline{\Diamond\Delta}}$$

$\Diamond$ *right*:
$$\frac{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \alpha, \underline{\Diamond\alpha}, \underline{\Diamond\Lambda}}{\Gamma, \underline{\Box\Theta} \Rightarrow \Delta, \Diamond\alpha, \underline{\Diamond\Lambda}}$$

The second source of non-termination is, of course, the fact that an inversion of the $\Box$-left and of the $\Diamond$-right rules generally increases rather than reducing the logical complexity of the sequent. However, since the procedure satisfies the *subformula property* and there is only a finite number of modal subformulas in any given sequent, eventually on any branch we come to invert a $\Box$-right or $\Diamond$-left rule with a sequent-conclusion S such that the same rule with the same sequent-conclusion S had already inverted at some point below in the refutation tree. Let $\mathcal{I}$ and let $\mathcal{I}'$ be the lower and the upper applications of the modal rule under consideration. At $\mathcal{I}'$ the procedure stops on that branch.

The construction in **Fact 3** above is modified as follows: let β_j be the fragment of branch whose lowermost sequent is the sequent-premise of $\mathcal{I}$; let w_j be the corresponding possible world (defined as in the case for **K**). If the procedure continued, above the sequent-premise S' of $\mathcal{I}'$ there would be a copy of β_j; therefore we associate S' with the possible world w_j. If the sequent-conclusion of $\mathcal{I}'$ branches from a sequent of the form (†) which belongs to a fragment of branch β_i and if β_i is associated with the possible world w_i, then we put the condition $w_j \sqsubseteq w_i$ on the frame of the countermodel. Details are left to the reader.

This proves the following theorem:

THEOREM 6. *The semantic tableaux procedure for* **S4** *is sound and complete with respect to the semantics of* **S4**.

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