NATURAL DEDUCTION AND TERM ASSIGNMENT FOR CO-HEYTING ALGEBRAS IN POLARIZED BI-INTUITIONISTIC LOGIC.

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Abstract. We reconsider Rauszer's *bi-intuitionistic logic* in the framework of the *logic for pragmatics*: every formula is regarded as expressing an act of *assertion* or *conjecture*, where conjunction and implication are assertive and subtraction and disjunction are conjectural. The resulting system of *polarized bi-intuitionistic logic* (**PBL**) consists of two fragments, positive intuitionistic logic $\mathbf{LJ}^{\supset\cap}$ and its dual $\mathbf{LJ}^{\smallsetminus\curlyvee}$, extended with two negations partially internalizing the duality between $\mathbf{LJ}^{\supset\cap}$ and $\mathbf{LJ}^{\smallsetminus\curlyvee}$. Modal interpretations and Kripke's semantics over *bimodal preordered frames* are considered and a Natural Deduction system **PBN** is sketched for the whole system. A stricter interpretation of the duality and a simpler natural deduction system is obtained when polarized bi-intuitionistic logic is interpreted over **S4** rather than bi-modal **S4** (a logic called *intuitionistic logic for pragmatics of assertions and conjectures* $\mathbf{ILP_{AC}}$). The term assignment for the conjectural fragment $\mathbf{LJ}^{\smallsetminus\curlyvee}$ exhibits several features of calculi for concurrency, such as remote capture of variable and remote substitution. The duality is extended from formulas to proofs and it is shown that every computation in our calculus is isomorphic to a computation in the simply typed λ-calculus.

§1. Preface. We present a *natural deduction system* for propositional *polarized bi-intuitionistic logic* **PBL**, (a variant of) intuitionistic logic extended with a connective of *subtraction* $A \smallsetminus B$, read as "*A but not B*", which is dual to implication.[1] The logic **PBL** is polarized in the sense that its expressions are regarded as expressing acts of *assertion* or of *conjecture*; implications and conjunctions are assertive, subtractions and disjunctions are conjectural. Assertions and conjectures are regarded as dual; moreover there are two negations, transforming assertions into conjectures and viceversa, in some sense internalizing the duality.

Our notion of polarity isn't just a technical device: it is rooted in an analysis of the structure of speech-acts, following the viewpoint of the

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logic for pragmatics. An interesting consequence of polarization is that in **PBL** only intuitionistic principles are provable. The natural deduction system for the conjectural part has *multiple-conclusions* but a *single-premise* and the term assignment associated to it is a purely intuitionistic calculus related to Tristan Crolard's calculus of *coroutines*. The term assignment is related to a simple categorical interpretation of **PBL**; however, we shall not develop the abstract treatment here.

The consideration of Cecylia Rauszer's bi-intuitionistic logic [29, 30] (also called *Heyting-Brouwer* or *subtractive* logic) from the point of view of the logic for pragmatics has been advocated in a previous work [5], where the philosophical background and motivations of a logic of assertions and conjectures are discussed, a general outline of such a logic is presented and a polarized sequent calculus **ILP** complete for Kripke's semantics over preordered frames is given. However, the logic **ILP** considerably departs from Rauszer's tradition, namely, from the works by Lawvere, Makkai, Reyes and Zolfaghari in category theory [21, 31] and the more recent ones by R. Gore and T. Crolard in proof theory [17, 9, 10]. The main difference is precisely in the semantic definition of the *duality* between the ordinary assertive fragment and the conjectural one. This can be seen in the modal translations: **ILP** is translated in the modal system **S4**, while Rauszer's logic has been translated into *temporal* **S4**. It seems that the duality between assertions and conjecture, or between intuitionistic logic and its dual, can be interpreted in many ways; a more general treatment is therefore in order and this is what we begin to do here. In the remainder of the introduction a brief presentation of the *logic for pragmatics* is given first: sections 1.1, 1.3 are a summary of the discussion of the logic of assertions and conjectures in the paper [5]. Next the main features of a system of natural deduction for dual intuitionistic logic are described.

1.1. Logic for pragmatics. The *logic for pragmatics*, as introduced by Dalla Pozza and Garola in [12, 13] and developed in [5, 7, 6], aims at a formal characterization of the logical properties of *illocutionary operators*: it is concerned, e.g., with the operations by which we performs the act of *asserting* a proposition as true, either on the basis of a mathematical proof or by empirical evidence or by the recognition of physical necessity, or the act of taking a proposition as an *obligation*, either on the basis of a moral principle or by inference within a normative system.

The following is a rough account of the viewpoint in Dalla Pozza and Garola [12]. There is a logic of *propositions* and a logic of *judgements*. Propositions are entities which can be *true* of *false*, judgements are acts which can be *justified* or *unjustified*. The logic of propositions is about *truth* according to classical semantics. The logic of judgements gives conditions for the *justification* of acts of judgements. An instance of an elementary act of judgement is the *assertion* of a proposition α, which is justified by the capacity to exhibit a *proof* of it, if α is a mathematical proposition, or some kind of *empirical evidence* if α is about states of affairs. It is then claimed that the justification of complex acts of judgement must be in terms of *Heyting's interpretation of intuitionistic connectives*: for instance, a *conditional judgement* where the assertion of β depends on the assertibility of α is justified by a method that transforms any justification for the assertion of α into a justification for the assertion of β.

In modern logic the distinction between propositions and judgements was established by Frege: a proposition expresses the thought which is the content of a judgement and a judgement is the act of recognizing the truth of its content. The distinction between propositions and judgements has recently been taken up by Martin-Löf: in his formalism "α *prop*" expresses the assertion that α is a well-formed proposition, and "α *true*" expresses the judgement that it is known how to verify α. However, Martin-Löf seems to give propositions a *verificationist* semantics: in order to give meaning to a proposition we must know what counts as a verification of it.

Unlike Martin-Löf and in agreement with Frege, Dalla Pozza and Garola distinguish between the truth of a proposition and the justification of a judgement, but extend Frege's framework by introducing *pragmatic connectives* and by giving them Heyting's interpretation while retaining Tarski's semantics for the logic of propositions. Therefore, Dalla Pozza and Garola seem to embrace a *compatibilist* approach in the controversy between classical and intuitionistic logic: classical logic is *extended* rather than challenged by intuitionistic pragmatics, the latter having a different *subject matter* than the former. Formally, intuitionistic logic may simply be identified with the logic of assertions whose elementary formulas have the form $\vdash p$, for p an atomic proposition, i.e., assertions whose justification is independent of the (classical) propositional structure of their content.

On the crucial issue of the relations between the intuitionistic logic of pragmatic expressions and the underlying level of classical propositions, Dalla Pozza and Garola first extend the classical logic of propositions with *modal operators*, which are interpreted using Kripke's semantics, and then they rely on the **S4** interpretation of intuitionistic logic, due to Gödel, Tarski, McKinsey and Kripke, which they regard as a *reflection* of the pragmatic level on the semantic one. This method has been used also to introduce *extensions* of Dalla Pozza and Garola's approach to logics that exhibit the interactions between the operators for *assertions*, *obligations* and *causal implication* [6] or between *assertions* and *conjectures* [5]. In all these cases, a pragmatics system is defined and a suitable modal extension of classical propositional logic is found which provides a "semantic reflection".

Evidently, the philosophical import of the logic for pragmatics depends on the interpretation of such a reflection. If Kripke's semantics is regarded as faithfully expressing the essential logical content of the pragmatic level, then a reductionist outcome of the logic of pragmatics to classical modal logic seems likely. On the contrary, Kripke's semantics may be regarded as an *abstract interpretation* of intuitionistic pragmatics: the rich content of the latter may be more faithfully expressed in the diverse branches of intuitionistic mathematics, from categorical logic and the typed λ-calculus, to game theory, than in the former.

A possible philosophical interpretation of Dalla Pozza and Garola's logic for pragmatics is in terms of Stewart Shapiro's *epistemic* approach to the philosophy of mathematics [33]. Justification of judgements depends on knowledge; Kripke's possible worlds may be regarded as possible states of knowledge and their preordering may correspond to ways our knowledge could evolve in the future. Having a proof of α *now* rules out the possibility of α being false at any *future* state of knowledge, and the possibility that α may be false at a future state of knowledge propagates the impossibility of having a proof of α backwards to all previous states of knowledge. The epistemic interpretation of Kripke's semantics can be given an ontological significance: in this way Kripke's possible world semantics can be used to reduce intuitionistic mathematics to *classical* epistemic mathematics; presumably, the intensional notion of a proof would be explained away in an ontology of possible states of knowledge.

On the other hand, Dalla Pozza insists that the logic for pragmatics is an *intensional* logic, while Kripke's semantics for modal logics suggests an *extensional* interpretation of intensional notions. In the field of *deontic logic* Dalla Pozza has successfully applied the intensional status of the pragmatic operator of obligation, in opposition to the extensional reading of the **KD** necessity operator, by introducing a distinction between *expressive* and *descriptive* interpretations of norms, which appears to have resolved conceptual confusions [13]. Similarly, Frege's symbol "$\vdash$" may be regarded here as expressing the *intentionality* of an act of judgement, while the **S4** modality "$\Box$" would perhaps describe conditions on the states of knowledge which justify the appropriateness of such an act.

1.2. Assertions and Conjectures. Which criteria shall we follow in extending the logic for pragmatics to a *logic of conjectures*? First of all, such a logic cannot deal with positive justifications of acts of conjecture, e.g., in terms of the *likelihood* of their propositional content being true: such a task would require other tools than those available here. Second, a characterization of the relations between acts of assertion and acts of conjecture may be based upon the similarity between what counts as a justification of the assertion that α is true, on one hand, and what counts as a refutation of the conjecture that α is false, on the other. Certainly in Dalla Pozza and Garola's approach, *proving the truth* of the proposition α is very close to *refuting the truth* of $\neg\alpha$. Thus a formal treatment of the logic of conjectures could have the form of a *calculus of refutations* and the overall system should axiomatize a notion of *duality* between assertions and conjectures.

In [5] the following principles have been identified as plausible starting points for the definition of the extension of the logic for pragmatics with a logic of conjectures: [2]

1. the grounds that justify *asserting* a proposition α certainly suffice also for *conjecturing* it, whatever these grounds may be; in other words $\vdash \alpha \Rightarrow \mathcal{H}\, \alpha$ should be an axiom of our logic of assertions and conjectures;

[2]Notice that in the formula $\mathcal{H}\,\neg\alpha$ of (2), the negation is *classical negation*, not the intuitionistic one: e.g., the conjecture $\mathcal{H}\,\neg\alpha$ may be refuted also by evidence that a certain state of affairs α does not obtain, not necessarily by a proof that there would be a contradiction assuming that α obtains.

2. in any situation, the grounds that justify the assertion $\vdash \alpha$ are also necessary and sufficient to regard $\mathcal{H} \neg\alpha$ as unjustified;
3. pragmatic connectives are operations which express ways of building up complex acts of assertion or of conjecture from elementary acts of assertion and conjecture. The justification of a complex act depends on the justification of the component acts, possibly through intensional operations.

Therefore our extension of the logic of pragmatics deals with *acts of assertion* ϑ and *acts of conjecture* υ. The language $\mathcal{L}^{-}_{\vartheta\upsilon}$ has symbols for *elementary* assertions $\vdash \alpha$ and the constant $\bigvee$, for an assertion which is always justified, and symbols for acts of *composite* type, *conjunctive* $\vartheta_1 \cap \vartheta_2$ and *implicative* $\vartheta_1 \supset \vartheta_2$ ones; similarly, we have symbols for *elementary* conjectures $\mathcal{H}\, \alpha$ and the constant $\bigwedge$, for a conjecture which is always refuted, and symbols for conjectural acts of *composite* type, *disjunctive* $\upsilon_1 \curlyvee \upsilon_2$ and *subtractive* $\upsilon_1 \smallsetminus \upsilon_2$ ones. We define the duality $(\ )^{\perp}$ inductively thus:

$$\begin{array}{rlrl} (\vdash p)^{\perp} & = \mathcal{H} \neg p & (\mathcal{H}\, p)^{\perp} & = \vdash \neg p \\ (\bigvee)^{\perp} & = \bigwedge & (\bigwedge)^{\perp} & = \bigvee \\ (\vartheta_0 \supset \vartheta_1)^{\perp} & = \vartheta_1^{\perp} \smallsetminus \vartheta_0^{\perp} & (\upsilon_0 \smallsetminus \upsilon_1)^{\perp} & = \upsilon_1^{\perp} \supset \upsilon_0^{\perp} \\ (\vartheta_0 \cap \vartheta_1)^{\perp} & = \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp} & (\upsilon_0 \curlyvee \upsilon_1)^{\perp} & = \upsilon_0^{\perp} \cap \upsilon_1^{\perp} \end{array}$$

The methodological principles above indicated support an intuitive interpretation of the duality between acts of assertion and of conjecture. [3] The main contribution of this paper is to define a proof-system for the logic of assertions and conjectures and to show that in this proof-system the above notion of duality can be extended from formulas to proofs.

A further step is to *extend* the language $\mathcal{L}^{-}_{\vartheta\upsilon}$ to a language $\mathcal{L}_{\vartheta\upsilon}$ which has a *strong negation* $\sim \upsilon =_{df} \upsilon \supset \bigwedge$ and a *weak negation* $\frown \vartheta =_{df} \bigvee \smallsetminus \vartheta$. These connectives "internalize" the action of the duality $(\ )^{\perp}$.[4]

However, in order to motivate the specification of a proof-system for our logic, we consider the mathematical interpretation of the language $\mathcal{L}_{\vartheta\upsilon}$ which result from a modal interpretation. Indeed in a logic for pragmatics the acts of *assertions* and *conjecture* must

[3]However, our first principle introduces an asymmetry which is not accounted for here.

[4]It should be clear that strong and weak negations extend the language $\mathcal{L}_{\vartheta\upsilon}$: they turn assertion into conjectures and viceversa, therefore are not definable within $\mathcal{L}^{-}_{\vartheta\upsilon}$.

be related to the epistemic conditions by which they are justified through an *extension* of the Gödel's, McKinsey and Tarski's and Kripke's translation of intuitionistic logic into **S4**.

Thus let $\mathcal{F} = (W, R, S)$ be a bimodal frame, where R and S are arbitrary preorders. The forcing conditions for $\Box\alpha$ and $\boxminus\,\alpha$ are defined thus:

$w \Vdash \Box\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that wRw';
$w \Vdash \boxminus\,\alpha$ iff $w' \Vdash \alpha$ for all $w' \in W$ such that wSw'.

Now define the modal translation $(\mathcal{L}_{\vartheta\upsilon})^M$ inductively thus:

$$\begin{array}{rlrl}
(\vdash\alpha)^M & = \Box\alpha & (\mathcal{H}\,\alpha)^M & = \diamondminus\,\alpha \\
(\bigvee)^M & = \top & (\bigwedge)^M & = \bot \\
(\vartheta_1 \supset \vartheta_2)^M & = \Box(\vartheta_1^M \to \vartheta_2^M) & (\upsilon_1 \setminus \upsilon_2)^M & = \diamondminus\,(\upsilon_2^M \wedge \neg\upsilon_1^M) \\
(\vartheta_1 \cap \vartheta_2)^M & = \vartheta_1^M \wedge \vartheta_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & = \upsilon_1^M \vee \upsilon_2^M
\end{array}$$

from which one easily shows $\vartheta^M \equiv \Box\vartheta^M$ and $\upsilon^M \equiv \diamondminus\,\upsilon^M$. [5]

As R and S are preorders,

$$\Box \boxminus \Box\alpha \to \Box\alpha \qquad \text{and} \qquad \diamondminus\,\alpha \to \diamondminus \Diamond \diamondminus\,\alpha$$

are certainly valid in $\mathcal{F}$. It is easy to see that

$$(1) \qquad \Box\alpha \to \Box \boxminus \Box\alpha \qquad \text{and} \qquad (2) \qquad \diamondminus \Diamond \diamondminus\,\alpha \to \diamondminus\,\alpha$$

are valid in every Kripke model over $\mathcal{F}$ if and only if $R = S$.

Our methodological principles strongly support the identification $R = S$, i.e., defining the modal translation thus:

$$(\mathcal{H}\,\alpha)^M = \Diamond\alpha, \qquad (\upsilon_1 \setminus \upsilon_0)^M = \Diamond(\upsilon_1^M \wedge \neg\upsilon_0^M) \qquad (\S)$$

Indeed, the validity of $\Box\alpha \Rightarrow \Diamond\alpha$ in **S4** satisfies (1) and the equivalence $\Box\alpha \equiv \neg\Diamond\neg\alpha$ satisfies (2). Moreover, by (3) $\upsilon_1 \setminus \upsilon_0$ also expresses an act of conjecture, thus the choices in (§) cannot be separated.

An corollary of this choice is that the modal translations of *strong negation* $(\sim A)^M = \Box\neg A^M$ and of *weak negation* $(\frown A)^M = \Diamond\neg A^M$ support the interpretation of these connectives as inverses: if ϑ is an assertive formula and υ a conjectural one then

$$\sim\frown\vartheta \iff \vartheta \qquad \text{and} \qquad \frown\sim\upsilon \iff \upsilon$$

[5]Notice that the above methodological principles do *not* support the other well-known translation $(\ )^G$ which yields $(p)^G = p$, $(A \supset B)^G = \Box A^G \to B^G$ and $(A \cup B)^G = \Box A^G \vee \Box B^G$: here atomic symbols stand for *propositions*, not for *elementary acts of judgement*.

Moreover, writing $F(\vartheta)$ for $\frown \vartheta$ and $G(\upsilon)$ for $\sim \upsilon$ the following equalities and rules are semantically justified:

$$F(\vdash p) = \mathcal{H} \neg p \qquad\qquad G(\mathcal{H} p) = \vdash \neg p$$

$$F(\vartheta_0 \cap \vartheta_1) = F(\vartheta_0) \curlyvee F(\vartheta_1) \qquad G(\upsilon_0 \curlyvee \upsilon_1) = G(\upsilon_0) \cap G(\upsilon_1)$$

$$F(\bigvee) = \bigwedge \qquad\qquad G(\bigwedge) = \bigvee$$

$$\begin{array}{c} \vartheta \Rightarrow G(\upsilon) \\ \hline\hline F(\vartheta) \Leftarrow \upsilon \end{array} \qquad\qquad \begin{array}{c} G(\upsilon) \Rightarrow \vartheta \\ \hline\hline \upsilon \Leftarrow F(\vartheta) \end{array}$$

1.3. Heyting-Brouwer Logic. Let us give a closer look at the semantics of Rauszer's Heyting-Brouwer logic. A *co-Heyting algebra* is a (distributive) lattice $\mathcal{C}$ such that its opposite $\mathcal{C}^{op}$ is a Heyting algebra. In $\mathcal{C}^{op}$ the operation of Heyting implication $B \rightarrow A$ is defined by the familiar adjunction, thus in the co-Heyting algebra $\mathcal{C}$ *co-implication* (or *subtraction*) $A \smallsetminus B$ is defined dually

$$\frac{C \wedge B \ \leq \ A}{C \ \leq \ B \rightarrow A} \qquad\qquad \frac{A \ \leq \ B \vee C}{A \smallsetminus B \ \leq \ C}$$

In this tradition the crucial move has been to consider *bi-Heyting* algebras, which have both the structure of Heyting and co-Heyting algebras. The topological models of the Heyting-Brouwer logic are bi-topological spaces, but every bi-topological space consists of the final sections of some preorder; the categorical models are bi-Cartesian closed categories, but unfortunately by Joyal's argument bi-CCC's collapse to partial orderings (see [10]). Since in a bi-CCC for every pair of objects A, B, $Hom(A, B)$ has at most one element, in such a categorical model of the Heying-Brouwer logic it is impossible to define a sensible notion of *identity of proofs*.

In the framework of the logic for pragmatics, the objects of an Heyting algebra and of a co-Heyting algebras cannot be identified, but alternative modal translations are possible without such identification. We may define Kripke models for Heyting-Brouwer logic over bimodal preordered frames $\mathcal{F} = (W, R, S)$ where $S = R^{-1}$, namely over *temporal* **S4** rather than **S4**. This shows that our choice (§) is not the only reasonable one, at least from a mathematical viewpoint. Looking at this interpretation in the light of our methodologica criteria, we see that criterion (1) is still satisfied, as $\Box\alpha \Rightarrow \Diamond\alpha$, and so is (3) but not (2): indeed now $\Box\alpha$ and $\Diamond\neg\alpha$ may be both true at some possible world w, i.e., the modal translation into *temporal* **S4** justifies *asserting* α while at the same time *conjecturing* $\neg\alpha$. Nevertheless, even from a philosophical viewpoint it would be premature to rule out alternatives to (§): indeed a naif reading of Kripke's possible worlds as states of knowledge in a temporal sequence may be philosophically questionable.

The pragmatic system for assertions and conjectures whose modal translation is interpreted over bimodal frames $\mathcal{F} = (W, R, S)$ where R and S are *arbitrary* preorders is called *polarized bi-intuitionistic logic* (**PBL**). We reserve the name *intuitionistic pragmatic logic of assertions and conjectures* ($\mathbf{ILP_{AC}}$) for the pragmatic system for assertions and conjectures whose modal translation is interpreted within **S4** (i.e., over bimodal frames $\mathcal{F} = (W, R, S)$ where $R = S$). It follows that **PBL** is more general than our logic $\mathbf{ILP_{AC}}$: indeed if we add the axioms

$$(1) \qquad \vartheta \Rightarrow \sim\frown \vartheta \qquad \text{and} \qquad (2) \qquad \frown\sim \upsilon \Rightarrow \upsilon$$

to an axiomatization of **PBL** then we obtain an axiomatization of $\mathbf{ILP_{AC}}$.

Notice that in **PBL** we only have $\sim\frown \vartheta \Rightarrow \vartheta$ and $\upsilon \Rightarrow \frown\sim \upsilon$. We still have

$$\frac{\vartheta \Rightarrow G(\upsilon)}{F(\vartheta) \Leftarrow \upsilon} \qquad \frac{\upsilon \Leftarrow F(\vartheta)}{G(\upsilon) \Rightarrow \vartheta}$$

but the bottom-up directions no longer hold.

1.4. Natural Deduction. The Curry-Howard correspondence between propositions of intuitionistic logic and types, on one hand, and between proofs in intuitionistic Natural Deduction **NJ** and λ-terms, on the other, and moreover the abstract characterization of the Curry-Howard correspondence in terms of Cartesian closed categories, are remarkable discoveries and powerful motivations for the study of Gentzen systems in the last three decades. The Curry-Howard correspondence can be illustrated in the most elementary way by decorating Natural Deduction derivations with λ-terms, so that the resulting trees may be regarded either as logical deductions or as type derivations; then one shows that in this representation β-reduction actually coincides with the reduction of cuts (*maximal formulas*).

It may be worth remembering that when deductions are regarded as type derivations, they are usually represented as directed trees, whose edges are labelled with *intuitionistic sequents* and vertices are labelled with *deduction rules*, as in the rules for implication[6]:

$$\supset\text{-}E\ (application): \quad \frac{\overline{x} : \Theta \ \rhd \ t : \vartheta_1 \supset \vartheta_2 \qquad \overline{x} : \Theta \ \rhd \ u : \vartheta_1}{\overline{x} : \Theta \ \rhd \ tu : \vartheta_2} \qquad \supset\text{-}I\ (\lambda\text{-}abstraction): \quad \frac{\overline{x} : \Theta, x : \vartheta_1 \ \rhd \ t : \vartheta_2}{\overline{x} : \Theta \ \rhd \ \lambda x.t : \vartheta_1 \supset \vartheta_2}$$

Here all the information needed to verify the correctness of the derivation and to determine which open assumptions an edge depends on (and, in particular, which open assumptions are discharged in an $\supset$-I inference), is exhibited *locally* in each sequent. But Prawitz [27] gives another presentation of deductions as directed trees, where edges are labelled with *formulas* and vertices are labelled with *inference rules*, together with *pointers*, mapping the leaves which are closed assumptions to the inferences by which they are discharged. It should be recalled that such a *proof-graph* does not determine a deduction in a unique way: for instance, if a proof-graph τ represents a deduction d of A from Γ, then it may also represent any deduction of A from Γ', for $\Gamma \subseteq \Gamma'$. Thus additional information must be provided to determine the intended derivation: as in the definition of type derivations, we may regard *deduction rules* as clauses of an inductive definition and we may assume that correct proof-graphs are those which are inductively generated in accordance with the deduction rules[7]. However, an important feature of proof-graphs is that the verification of their correctness as derivations is a *global* affair; in particular, an obvious linear-time algorithm determines for any edge which open assumptions it actually depends on. The more recent representation of non-intuitionistic proofs through *proof-nets* shares this feature with Prawitz's Natural Deduction.

[6]We shall use the symbol "$\Rightarrow$" for the consequence relation in the sequent calculus, "$\vdash$" in the deduction rules of Natural Deduction systems and "$\rhd$" in type derivations.

[7]In Prawitz [27] pp. 19-24, the inference rules of first-order intuitionistic and classical Natural Deduction are listed and then it is pointed out that these rules "do not characterize a system of natural deduction completely, since it is not stated in them how assumptions are discharged" and also how global restrictions on the open assumptions are verified; thus the distinction is introduced between *proper inference rules*, namely, &-I, &-E, $\supset$-E, $\vee$-I, $\forall$-E, $\exists$-I and $\bigwedge_I$, and *improper* ones, namely, $\supset$-I, $\vee$-E, $\forall$-I, $\exists$-E and $\bigwedge_C$, which must be determined by *deduction rules* in order for their correctness to be verified and for the deduction to be fully determined. Since in Prawitz [27] the &-I rule is *multiplicative* but the &-E rules are *additive* (and dually for disjunction), in the normalization process some actual dependencies may become vacuous; thus the issue of the representation of vacuous dependencies affects the definition of normalization of proof-graphs.

Already in the implicational and conjunctive fragment $\mathbf{NJ}^{\supset\cap}$ of $\mathbf{NJ}$ proof-graphs are directed trees with complex additional structure of logical, computational and geometric significance. In a directed proof-tree, the direction from the leaves to the root, may be called *main orientation*. Prawitz's analysis of *branches* in normal deductions for the fragment $\mathbf{NJ}^{\supset\cap}$ ([27] p. 41) identifies an *elimination part*, where vertices are $\supset$-E (*applications*) or $\cap$-E (*projections*), followed by an *introduction part*, where vertices are $\supset$-I (λ-*abstractions*) or $\cap$-I (*pairings*). Branches are connected at an application vertex: the child branch terminates at the minor premise (*argument position*) while the parent branch continues from the major premise (*function position*) to the conclusion. This analysis identifies a "*flow of information*", from the elimination part of a branch to its introduction part, and from a branch to its parent, which may be called the *input-output orientation*[8]. It is a remarkable feature of Natural Deduction for $\mathbf{NJ}^{\supset\cap}$ that the input-output orientation and the main orientation *coincide* in a deduction tree.[9] In the sequent calculus $\mathbf{LJ}$ the input-output orientation is "contravariant" in the antecedent and "covariant" in the succedent, namely, it runs from a formula to its ancestors in the antecedent and conversely in the succedent.

In the full system $\mathbf{NJ}$ the analysis of branches is subsumed in that of *paths* ([27], pp. 52-3). Paths extend branches but in addition they go from the major premise of a $\cup$-elimination $\mathcal{I}$ to any one of the assumptions in the classes discharged by the inference $\mathcal{I}$. Thus in the case of $\cup$-elimination the main orientation diverges from input-output orientation: here the tree structure of the derivation performs a *control function*, namely, the verification that the minor premises of the inference coincide.

[8] The terminology comes from research communities working on process calculi and on linear logic in the early 1990s. From a computational point of view, the analysis is spelt out in [3].

[9] An application of this remark is the explanation of Girard's (or Danos-Regnier's) *correctness conditions* in the theory of proof-nets for *classical multiplicative linear logic* $\mathbf{MLL}^-$ given in [3], section 5.4. Indeed a *long trip* on a proof-net induces the *input-output orientation* of Natural Deduction on a proof-net; moreover such orientations determine translations of formulas and proofs of *classical* $\mathbf{MLL}^-$ into formulas and proofs of *intuitionistic* $\mathbf{MLL}^-$. Abstractly presented, this fact is a special case of Chu's construction: a $*$-autonomous category can be built as $\mathcal{C} \times \mathcal{C}^{op}$ from a free symmetric monoidal closed category with products $\mathcal{C}$ and its opposite $\mathcal{C}^{op}$ (see [4]).

1.5. Natural Deduction for conjectural reasoning. Our Natural Deduction for the *logic of conjectures* is a *single-premise multiple-conclusion* system $\mathbf{NJ}^{\smallsetminus\curlyvee}$ with rules for the conjectural connectives of subtraction ($\smallsetminus$) and disjunction ($\curlyvee$). $\mathbf{NJ}^{\smallsetminus\curlyvee}$ is dual to the familiar system $\mathbf{NJ}^{\supset\cap}$, in a sense to be made precise below.

The deduction rules for subtraction are

$$\smallsetminus\text{-}I: \quad \frac{\epsilon \vdash \Upsilon,\ \upsilon_1}{\epsilon \vdash \Upsilon,\ \upsilon_2,\ \upsilon_1 \smallsetminus \upsilon_2} \qquad \smallsetminus\text{-}E: \quad \frac{\epsilon \vdash \Upsilon, \upsilon_1 \smallsetminus \upsilon_2 \qquad \upsilon_1 \vdash \Upsilon', \upsilon_2}{\epsilon \vdash \Upsilon,\ \Upsilon'}$$

The $\smallsetminus$-introduction rule has following "operational interpretation": if from the conjecture ϵ the alternative conjectures Υ, υ_1 follow, then we may we specify our alternative υ_1 by taking it as "υ_1 *but not* υ_2", on one hand, and by considering also υ_2 as an alternative, on the other hand.

The $\smallsetminus$-elimination rule can be explained as follows. Suppose we have two arguments, one showing that ϵ yields the alternatives Υ or else "υ_1 *but not* υ_2", and another showing that υ_1 yields the alternatives Υ' or υ_2; then after assuming ϵ we are left with the alternatives Υ and Υ', but $\upsilon_1 \setminus \upsilon_2$ is no longer a consistent option.

The *dynamics* of our calculus is illustrated by the following reduction of a cut (*maximal formula*) $\upsilon_1 \smallsetminus \upsilon_2$:

$$\cfrac{\cfrac{\epsilon \vdash \Upsilon, \upsilon_1}{\epsilon \vdash \Upsilon, \upsilon_2, \upsilon_1 \smallsetminus \upsilon_2}\smallsetminus\text{-I} \qquad \upsilon_1 \vdash \Upsilon', \upsilon_2}{\epsilon \vdash \Upsilon, \Upsilon', \upsilon_2}\smallsetminus\text{-E}$$

reduces to

$$\epsilon \vdash \Upsilon,\ \Upsilon', \upsilon_2$$

Thinking in terms of the underlying proof-graph, in order to remove the $\smallsetminus$ introduction-elimination pair we have substituted the branch ending with the premise υ_1 of the $\smallsetminus$-introduction for the open assumption υ_1 in the derivation of the minor premise υ_2 of the $\smallsetminus$-elimination; finally, we have identified this occurrence of υ_2 with the one which was a conclusion of the removed $\smallsetminus$-introduction.

Derivations still have a *main orientation* as directed trees, whose edges are labelled with sequents of the form $\epsilon \vdash \Upsilon$ and vertices with deduction rules. We define paths and the *input-output orientation* as follows. In a $\smallsetminus$-elimination, a path continues from the main premise $\upsilon_1 \smallsetminus \upsilon_2$ to the assumption υ_1 discharged by the inference; any path reaching the minor premise υ_2 ends there. In the $\smallsetminus$-introduction, a path begins at the conclusion υ_2 and another continues from the premise υ_1 to the other conclusion $\upsilon_1 \smallsetminus \upsilon_2$. The relation between these edges $\upsilon_1 \smallsetminus \upsilon_2$ and υ_2 is the same between the major premise and the minor premise of a $\supset$-elimination, i.e., it establishes the relation between the child branch which υ_2 belongs to and its parent branch continuing through $\upsilon_1 \smallsetminus \upsilon_2$. The main difference is that the flow of information here goes from the parent branch to its children branches: if in considering the alternative υ_1 we decide to exclude υ_2, then the new task of exploring the consequences of υ_2 *follows* from our decision.

Natural Deduction in $\mathbf{NJ}^{\smallsetminus\curlyvee}$ may be regarded as a *calculus of refutations*: a deduction, given refutations of the conclusions, yields a refutation of the premise. Thus when a deduction is regarded as a refutation, it would seem that the *flow of information* goes in the opposite direction to the one described above, from the *many conclusions* to the *unique open assumption* of the derivation. However, we have regarded a deduction $\mathbf{NJ}^{\smallsetminus\curlyvee}$ as a process in which the task of refuting the premise is specified in the subtasks of refuting its consequences, as we have just seen in our discussion of the $\smallsetminus$-*introduction* rule; in this sense it seems appropriate to say that here the flow of information goes from the premise to the conclusions.

1.6. Term Assignment for conjectural reasoning. We extend the Curry-Howard correspondence by giving a term assignment to out Natural Deduction system for $\mathbf{NJ}^{\smallsetminus\curlyvee}$[10]. The implicit presence of contraction in the conclusion requires formulas to be labelled with *lists* ℓ of terms. Terms t are defined by the following grammar:

[10]In [11] Tristan Crolard presents a Natural Deduction system and a term assignment for Subtractive Logic, called $\lambda\mu^{\rightarrow+\times-}$-*calculus*, in the tradition of Parigot's $\lambda\mu$-calculus for classical Natural Deduction. Restrictions on the implication-introduction and subtraction-elimination rules are introduced to define a *constructive* system of Subtractive Logic and its term calculus. A crucial difference of the approach presented here is that polarization *prevents* us from expressing full classical logic; there is no obvious way for us to introduce a μ-operator. It is an interesting problem for future research, an important one for the project of logic for pragmatics, to find a more general framework in which *both* the intuitionistic and the classical proof-theory could be fully expressed.

$$\begin{array}{rl} t \; := & x \mid \texttt{false}\;(\ell_1 \ldots \ell_n) \mid \texttt{inl}(\ell) \mid \texttt{inr}(\ell) \mid \texttt{casel}\;(\ell) \mid \texttt{caser}\;(\ell) \mid \\ & \texttt{continue from}(x)\texttt{using}(\ell) \mid \texttt{postpone}(x :: \ell)\;\texttt{until}(\ell') \mid \end{array}$$

and substitution of lists of terms within lists of terms is defined from the usual substitution (avoiding capture of free variables) as follows:

$$\begin{array}{ll} ()[\ell'/x] = () & (t \cdot \ell)[\ell'/x] = t[\ell'/x] \cdot \ell[\ell'/x] \\ t[()/x] = () & t[(u \cdot \ell)/x] = t[u/x] \cdot t[\ell/x] \end{array}$$

All terms are typed with formulas as usual, except for the terms of the form `postpone`: these are *control terms.*[11] The rules for subtraction are labelled as follows:

$\smallsetminus$*-I (continuation):*

$$\frac{y : \epsilon \;\rhd\; \overline{\ell} : \Upsilon,\; \ell_1 : \upsilon_1}{y : \epsilon \;\rhd\; \overline{\ell} : \Upsilon,\; \texttt{z} : \upsilon_2,\; \texttt{continue from}\;(\texttt{z})\;\texttt{using}\;(\ell_1) : \upsilon_1 \smallsetminus \upsilon_2}$$

$\smallsetminus$*-E (postpone):*

$$\frac{y : \epsilon \rhd \overline{\ell} : \Upsilon, \ell : \upsilon_1 \smallsetminus \upsilon_2 \qquad x : \upsilon_1 \;\rhd\; \overline{\ell'} : \Upsilon', \ell_2 : \upsilon_2}{y : \epsilon \rhd\; \overline{\ell} : \Upsilon,\; \overline{\ell'}[\texttt{x}/x] : \Upsilon',\; \texttt{postpone}(x :: \ell_2)\;\texttt{until}(\ell) : \bullet}$$

Notice that in the $\smallsetminus$-elimination rule the term

$$\mathbf{u} = \texttt{postpone}(x :: \ell_2)\;\texttt{until}(\ell) : \bullet$$

binds the occurrences of the variable x which occur in its subterm ℓ_2; but the occurrences of x within $\overline{\ell'}$ also become bound as a side effect of the introduction of the term $\mathbf{u}$. We use the typescript font `x` to indicate that this occurrence of the variable x has been *globally captured* by another term in the context.

A similar remark applies to the $\smallsetminus$-introduction rule: here the conclusions υ_2 and $\upsilon_1 \smallsetminus \upsilon_2$ are introduced; the terms $\texttt{z} : \upsilon_2$ is certainly not a free variable: it is bounded by the term `continue from z using` $\ell_1 : \upsilon_1 \smallsetminus \upsilon_2$.

Le us consider the β-reduction of a redex corresponding to a $\smallsetminus$-reduction. A derivation of the form

$$\dfrac{\dfrac{y : \epsilon \;\rhd\; \overline{\ell} : \Upsilon, \ell_1 : \upsilon_1}{y : \epsilon \;\rhd\; \overline{\ell} : \Upsilon, \texttt{z} : \upsilon_2, \mathbf{t} : \upsilon_1 \smallsetminus \upsilon_2}\,\smallsetminus\text{-I} \qquad x : \upsilon_1 \;\rhd\; \overline{\ell'} : \Upsilon', \ell_2 : \upsilon_2}{y : \epsilon \;\rhd\; \overline{\ell} : \Upsilon, \overline{\ell'}[\texttt{x}/x] : \Upsilon', \texttt{z} : \upsilon_2, \mathbf{u} : \bullet}\,\smallsetminus\text{-E}$$

[11]The meaning of a 'type' $\bullet$ assigned to a `postpone` term resembles that of *absurdity*.

where

$$\mathbf{t} = \texttt{continue from}(\texttt{z})\ \texttt{using}(\ell_1)$$

and

$$\mathbf{u} = \texttt{postpone}(x :: \ell_2)\ \texttt{until}(\texttt{continue from}(\texttt{z})\texttt{using}(\ell_1))$$

is *eventually* reduced to

$$y : \epsilon \ \rhd\ \overline{\ell} : \Upsilon,\ \overline{\ell'}[\ell_1/x] : \Upsilon', \ell_2[\ell_1/x] : \upsilon_2$$

Notice that when the major premise $\upsilon_1 \setminus \upsilon_2$ of a $\setminus$-elimination is the conclusion of a $\setminus$-introduction, then the term $\mathbf{t} : \upsilon_1 \setminus \upsilon_2$ *is not a redex*, but *some control term* $\mathbf{u} : \bullet$ is a redex.

When such a redex $\mathbf{u}$ is removed the control term disappears; we substitute ℓ_1 for x in ℓ_2, which are subterms of $\mathbf{u}$ and thus "locally available". But then the resulting term $\ell_2[\ell_1/x]$ must be substituted for $\texttt{z} : \upsilon_2$ and also ℓ_1 must be substituted for $\texttt{x}$ in $\overline{\ell'}[\texttt{x}/x] : \Upsilon'$; these are *remote substitutions* which take place as the *remote binding* determined by $\mathbf{u}$ is removed. Therefore commands to perform such substitutions must be *broadcast* to the context and the execution of such commands may be performed in parallel. We shall not try to implement remote substitution here: we simply introduce control terms expressing remote substitutions $\{\texttt{x} ::= \ell\}$ and then describe their intended effect. We expect the specification of their action may be achieved in an elegant way using familiar techniques from calculi for concurrency.

We have interpreted Natural Deduction $\mathbf{NJ}^{\setminus \curlyvee}$ as a calculus where the "flow of information" goes from the premise to the conclusions: what we have obtained is a calculus related to Crolard's *coroutines* typed within this intuitionistic system. On the other hand, if the "flow of information" is regarded as going from the conclusions to the open assumption, and therefore variables are assigned to the conclusions and terms to the premises, what we obtain is the familiar simply typed λ-calculus, which is assigned also to the dual fragment $\mathbf{NJ}^{\supset \cap}$. The duality between these points of view can be mathematically expressed as an orthogonality functor $(\ )^{\perp}$ from $\mathbf{NJ}^{\setminus \curlyvee}$ to $\mathbf{NJ}^{\supset \cap}$. The analysis of paths yields a geometric representation of the duality between $\mathbf{NJ}^{\setminus \curlyvee}$ and $\mathbf{NJ}^{\supset \cap}$: the flow of information in a derivation of Υ from ϵ can be represented as the proof-tree of a derivation of $\epsilon^{\perp}$

form $\Upsilon^{\perp}$ "turned upside down" (as in examples given at the end of this paper).

Finally, the systems $\mathbf{NJ}^{\supset\cap}$ and $\mathbf{NJ}^{\smallsetminus\curlyvee}$ can be combined in a larger system; their interaction is handled by rules of *right-*$\sim$, *left-*$\frown$. A proper specification of their behaviour, however, and the interesting issues it evokes concerning the rules of Contraction are left to future research.

§2. The pragmatic language of assertions and conjectures. The language $\mathcal{L}^P$ of the *logic for pragmatics* [5], characterizing the logical properties of the acts of *assertion* and *conjecture*, is based on a countable set of propositional letters $p_1, p_2, \ldots$, from which *radical formulas* α are built using the classical propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. The *elementary formulas* are either the elementary constants $\bigvee$ and $\bigwedge$, or they are obtained by prefixing a radical formula α with a sign of *illocutionary force* "$\vdash$" for *assertion* and "$\mathcal{H}$" for *conjecture*. The *sentential formulas* of $\mathcal{L}^P$ are built from elementary formulas using *pragmatic connectives* of conjunction, disjunction, implication and subtraction; these connectives denote operations building complex acts of assertion or conjecture from the elementary ones. We use ϑ, ϑ_1, $\ldots$ to denote *assertive expressions* and υ, υ_1. $\ldots$ to denote *conjectural expressions.*

In the framework of logic for pragmatics, the treatment of *intuitionistic* logic is obtained by regarding the *radical part* of pragmatic expressions as constant. (For a consideration of some interactions between the radical part and the pragmatic level, see [5].)

Here we are interested only in a fragment of the gigantic language $\mathcal{L}^P$, the language $\mathcal{L}^{-}_{\vartheta\upsilon}$ given as follows.

Definition 1. (i) The language $\mathcal{L}^{-}_{\vartheta\upsilon}$ is generated by the following grammar:

$$\vartheta \ := \ \vdash \alpha \mid \bigvee \mid \vartheta \supset \vartheta \mid \vartheta \cap \vartheta$$

$$\upsilon \ := \ \mathcal{H} \alpha \mid \bigwedge \mid \upsilon \smallsetminus \upsilon \mid \upsilon \curlyvee \upsilon$$

where α is atomic p or the negation $\neg p$ of an atomic proposition.

(ii) We define the duality $(\ )^{\perp}$ on $\mathcal{L}^{-}_{\vartheta\upsilon}$ inductively thus:

$$
\begin{array}{llll}
(\vdash p)^{\perp} & = \mathcal{H}\neg p & (\mathcal{H} p)^{\perp} & = \vdash \neg p \\
(\vdash \neg p)^{\perp} & = \mathcal{H} p & (\mathcal{H} \neg p)^{\perp} & = \vdash p \\
(\bigvee)^{\perp} & = \bigwedge & (\bigwedge)^{\perp} & = \bigvee \\
(\vartheta_0 \supset \vartheta_1)^{\perp} & = \vartheta_1^{\perp} \setminus \vartheta_0^{\perp} & (\upsilon_0 \setminus \upsilon_1)^{\perp} & = \upsilon_1^{\perp} \supset \upsilon_0^{\perp} \\
(\vartheta_0 \cap \vartheta_1)^{\perp} & = \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp} & (\upsilon_0 \curlyvee \upsilon_1)^{\perp} & = \upsilon_0^{\perp} \cap \upsilon_1^{\perp}
\end{array}
$$

(iii) Later, we shall interested in the *extension* $\mathcal{L}_{\vartheta\upsilon}$ of $\mathcal{L}^{-}_{\vartheta\upsilon}$ obtained from the above grammar by adding also the clauses

$$\vartheta \ := \sim \upsilon \qquad \text{and} \qquad \upsilon \ := \frown \vartheta$$

The informal interpretation of the language $\mathcal{L}_{\vartheta\upsilon}$ is as follows.

Definition 2. (*Informal Interpretation*) (i) *Radical* formulas are interpreted as propositions, which can be *true* or *false*.

(ii) *Sentential expressions* ϑ and υ are interpreted as impersonal illocutionary acts of *assertion* and *conjecture*, respectively, which can be *felicitously* or *infelicitously* made. Assertions can be *justified* or *unjustified*, and are *felicitous* or *infelicitous* accordingly. Conjectures can be *refuted* or *unrefuted* and we shall make the (perhaps unintuitive) convention that conjectures are *infelicitous* precisely when they are refuted, and *felicitous* if they are unrefuted.

1. $\bigvee$ is assertive and always *justified*, $\bigwedge$ is conjectural and always *refuted*.
2. $\vdash \alpha$ is *justified* if and only if a proof can be exhibited that α is true. Dually, $\mathcal{H}\, \alpha$ is *refuted* if and only if a proof that α is false can be exhibited.
3. $\vartheta_1 \supset \vartheta_2$ is *justified* if and only if a proof can be exhibited that a justification of ϑ_1 can be transformed into a justification of ϑ_2; it is *unjustified*, otherwise. Dually, $\upsilon_1 \setminus \upsilon_2$ is *refuted* if and only if a proof can be exhibited that a refutation of υ_2 can be transformed into a refutation of υ_2; it is *unrefuted* otherwise.
4. $\vartheta_1 \cap \vartheta_2$ is *justified* if and only if ϑ_1 and ϑ_2 are both *justified*; it is *unjustified*, otherwise. Dually, $\upsilon_1 \curlyvee \upsilon_2$ is *refuted* if and only if υ_1 and υ_2 are both *refuted*; it is *unrefuted* otherwise.
5. $\sim \upsilon$ is *justified* if and only if a proof can be exhibited that to assume υ justified would lead to a contradiction and $\frown \vartheta$ is refuted if and only if a proof can be exhibited that a refutation of ϑ would be inconsistent.

(iii) A fragment of the language $\mathcal{L}^{P}$ is *intuitionistic* if only atomic radicals occur in it.

§3. Bimodal S4. The bimodal language $\mathcal{L}_{\Box,\boxminus}$, an extension of the classical radical part of $\mathcal{L}^P$ is defined as follows.

Definition 3. (i) Let p range over a denumerable set of propositional variables $\mathbf{Var} = \{p_1, p_1, \dots\}$. The language $\mathcal{L}_{\Box,\boxminus}$ is defined by the following grammar.

$$\alpha := p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \Box\alpha \mid \boxminus\alpha$$

Define $\Diamond\alpha =_{df} \neg\Box\neg\alpha$ and $\diamondminus\alpha =_{df} \neg\boxminus\neg\alpha$.

(ii) Let $\mathcal{F} = (W, R, S)$ be a multimodal frame, where W is a set, R and S are *preorders* on W. Given a valuation function $V : \mathbf{Var} \to \wp(W)$, the forcing relations are defined as usual:

- $w \Vdash \Box\alpha$ iff $\forall w'.wRw' \Rightarrow w' \Vdash \alpha$,
- $w \Vdash \boxminus\alpha$ iff $\forall w'.wSw' \Rightarrow w' \Vdash \alpha$.

(iii) We say that a formula A in the language $\mathcal{L}_{\Box,\boxminus}$ is *valid in bimodal* **K** [*valid in bimodal* **S4**] if A is valid in all bimodal frames $\mathcal{F} = (W, R, S)$ [where R adn S are preorders].

Lemma 1. *Let $\mathcal{F} = (W, R, S)$ be a multimodal frame, where R and S are preorders.*
(i) The following are valid in $\mathcal{F}$:

$$\Box \boxminus \Box\alpha \to \Box\alpha \qquad \textit{and} \qquad \diamondminus\alpha \to \diamondminus \Diamond \diamondminus\alpha$$

(ii) The following are equivalent:
1. $R = S$;
2. the following schemes are valid in $\mathcal{F}$

$$(Ax.i) \quad \Box\alpha \to \Box \boxminus \Box\alpha \qquad \textit{and} \qquad (Ax.ii) \quad \boxminus\alpha \to \boxminus \Box \boxminus\alpha$$

3. the following rules is are valid in $\mathcal{F}$:

$$(R.i) \quad \frac{\diamondminus \neg\Box\alpha \Leftarrow \diamondminus\beta}{\Box\alpha \Rightarrow \Box\neg \diamondminus\beta} \qquad \textit{and} \qquad (R.ii) \quad \frac{\Box\neg \diamondminus\beta \Rightarrow \Box\alpha}{\diamondminus\beta \Leftarrow \diamondminus \neg\Box\alpha}$$

Proof of (ii). ($1 \Rightarrow 2$) is obvious. ($2 \Rightarrow 1$): If S is not a subset of R, then given wSv and not wRv define a model on $\mathcal{F}$ where $w' \Vdash p$ for all w' such that wRw' but $v \nVdash p$; thus $\Box p \to \Box \boxminus \Box p$ is false at w. Similarly, using ($Ax.ii$), if R is not a subset of S.

($2 \Rightarrow 3$): If $\diamondminus \neg\Box\alpha \Leftarrow \diamondminus\beta$ is valid in $\mathcal{F}$ then so is $\Box\neg \diamondminus \neg\Box\alpha \Rightarrow \Box\neg \diamondminus\beta$ and the conclusion of ($R.i$) is valid because of ($Ax.i$).

If $\Box\neg \diamondminus\beta \Rightarrow \Box\alpha$ is valid in $\mathcal{F}$, then so is $\diamondminus\neg\Box\neg \diamondminus\beta \Leftarrow \diamondminus\neg\Box\alpha$ and the conclusion of $(R.ii)$ is valid because of $(Ax.ii)$.

$(3 \Rightarrow 2)$: $(Ax.i)$ is obtained by applying $(R.i)$ to $\diamondminus\neg\Box\alpha \Leftarrow \diamondminus\neg\Box\alpha$ and similarly $(Ax.ii)$ is obtained by applying $(R.ii)$ to $\Box\neg \diamondminus\beta \Rightarrow \Box\neg \diamondminus\beta$.

In the Appendix, Section 7, we give sequent calculi for *bimodal* **K** and **S4** and outline a completeness theorem for them, base on semantic tableaux procedure.

3.1. Modal and bimodal interpretations of $\mathcal{L}_{\vartheta\upsilon}$. We give the bimodal interpretation of $\mathcal{L}_{\vartheta\upsilon}$, proper of Polarized Bi-intuitionistic Logic.

Definition 4. (i) The interpretation $(\ )^M$ of $\mathcal{L}_{\vartheta\upsilon}$ into $\mathcal{L}_{\Box,\boxminus}$ is defined inductively thus:

$$
\begin{array}{rll@{\qquad}rll}
(\bigwedge)^M & =_{df} & \bot & (\bigvee)^M & =_{df} & \top \\
(\vdash\alpha)^M & =_{df} & \Box\alpha & (\mathcal{H}\,\alpha)^M & =_{df} & \diamondminus\alpha \\
(\vartheta_1 \supset \vartheta_2)^M & =_{df} & \Box(\vartheta_1^M \to \vartheta_2^M) & (\upsilon_1 \smallsetminus \upsilon_2)^M & =_{df} & \diamondminus(\upsilon_1^M \wedge \neg\upsilon_2^M) \\
(\vartheta_1 \cap \vartheta_2)^M & =_{df} & \vartheta_1^M \wedge \vartheta_2^M & (\upsilon_1 \curlyvee \upsilon_2)^M & =_{df} & \upsilon_1^M \vee \upsilon_2^M \\
(\sim\upsilon)^M & =_{df} & \Box\neg\upsilon^M & (\frown\vartheta)^M & =_{df} & \diamondminus\neg\vartheta^M
\end{array}
$$

It is immediate to prove that $\vartheta^M \iff \Box\vartheta^M$ and $\upsilon^M \iff \diamondminus\upsilon^M$.

(ii) The propositional theory **PBL** is the set of all formulas δ in the language $\mathcal{L}_{\vartheta\upsilon}$ such that δ^M is valid in every preordered bimodal frame (i.e, in any frame (W, R, S) where R and S are arbitrary preorders).

(iii) The propositional theory $\mathbf{ILP_{AC}}$ is the set of all formulas δ in the *intuitionistic fragment* of the language $\mathcal{L}_{\vartheta\upsilon}$ such that δ^M is valid in **S4**[12].

Remark. (i) Let $\mathcal{F} = (W, R, S)$ be a bimodal preordered frame where $S = R^{-1}$. Then by Lemma 1(i) $\sim\frown\vartheta \supset \vartheta$ is valid in $\mathcal{F}$ and $\upsilon\smallsetminus \frown\sim\upsilon$ is contradictory in $\mathcal{F}$. However, if a bimodal frame $\mathcal{F} = (W, R, R^{-1})$ has backwards branching as well as forward branching, then there are models over $\mathcal{F}$ which falsify $\vartheta \supset\sim\frown\vartheta$ and models over $\mathcal{F}$ which satisfy $\frown\sim\upsilon \smallsetminus \upsilon$.

[12]This is the set of all formulas $\delta \in \mathcal{L}_{\vartheta\upsilon}$ such that δ has only *atomic radicals* $\vdash p$ or $\mathcal{H}\,p$ and δ^M is valid in every preordered bimodal frame (W, R, S) where $R = S$.

(ii) Let $\mathcal{F} = (W, R, S)$ be a bimodal frame where R and S are preorders. If the schemes

$$(1) \qquad \vartheta \Rightarrow \sim\frown \vartheta \qquad \text{and} \qquad (2) \qquad \frown\sim \upsilon \Rightarrow \upsilon$$

are valid in $\mathcal{F}$ for all ϑ and for all υ, then by Lemma 1 we have $R = S$. It follows that $\mathbf{ILP_{AC}}$ may be regarded as an axiomatic theory of **PBL**.

(iii) Notice that the duality $(\ )^{\perp}$ cannot be defined in the *intuitionistic fragment* of the language $\mathcal{L}^{-}_{\vartheta\upsilon}$, as it relies on the classical equivalence $p \equiv \neg\neg p$. However, we do not need the full power of classical reasoning here; what is required is a polarization of the atoms as p_0^{+}, p_0^{-}, p_1^{+}, p_1^{-}, $\ldots$, i.e., an involution without fixed points on them.

(iv) In $\mathbf{ILP_{AC}}$, but not in **PBL** $\frown \vdash p$ is equivalent to $\mathcal{H}\neg p$ and also $\sim\mathcal{H}\, p$ is equivalent to $\vdash \neg p$. Therefore the intuitionistic fragment of $\mathcal{L}_{\vartheta\upsilon}$, where only atomic radicals are considered, has the same expressive power than $\mathcal{L}_{\vartheta\upsilon}$.

(v) The *purely assertive intuitionistic* fragments (with formulas in $\mathcal{L}^{-}_{\vartheta}$) of **PBL** and $\mathbf{ILP_{AC}}$ coincide, and so do their *purely conjectural intuitionistic* fragments.

In Polarized Bi-intuitionistic Logic the two connectives "$\sim$" and "$\frown$" are "real negations", not orthogonalities. However, to a certain extent negations internalize the duality $(\ )^{\perp}$ between conjectures and assertions. However, there is a significant difference here between **PBL** and $\mathbf{ILP_{AC}}$, as indicated by the following Lemma, whose proof is immediate from Lemma 1.

LEMMA 2. *Write $F(\vartheta)$ for $\frown \vartheta$ and $G(\upsilon)$ for $\sim \upsilon$.*

(i) The following equalities and rules are valid in $\mathbf{ILP_{AC}}$*:*

(a) $\qquad F(\vdash p) = \mathcal{H}\neg p \qquad\qquad G(\mathcal{H} p) = \vdash \neg p$

(b) $\qquad F(\vartheta_0 \cap \vartheta_1) = F(\vartheta_0) \curlyvee F(\vartheta_1) \qquad G(\upsilon_0 \curlyvee \upsilon_1) = G(\upsilon_0) \cap G(\upsilon_1)$

(c) $\qquad F(\bigvee) = \bigwedge \qquad\qquad G(\bigwedge) = \bigvee$

(d) $$\frac{\vartheta \Rightarrow G(\upsilon)}{\overline{F(\vartheta) \Leftarrow \upsilon}} \qquad\qquad \frac{G(\upsilon) \Rightarrow \vartheta}{\overline{\upsilon \Leftarrow F(\vartheta)}}$$

(ii) In **PBL** *(b) and (c) hold, but (a) doesn't. Moreover, we have*

$$\frac{\vartheta \Rightarrow G(\upsilon)}{F(\vartheta) \Leftarrow \upsilon} \qquad \frac{\upsilon \Leftarrow F(\vartheta)}{G(\upsilon) \Rightarrow \vartheta}$$

but the bottom-up directions do not hold.

§4. Sequent Calculus for PBL. The intuitionistic fragments of $\mathbf{ILP_{AC}}$ and **PBL** are formalized by sequent calculi which contains only rules for the pragmatic connectives. The characteristic feature of **G3** sequent calculi [34] is that the rules of Weakening and Contraction are implicit. The sequent calculus for the logic **PBL** considered here is of this type and so is the sequent calculus system for **ILP** given in [5]. It can be proved [8] that in $\mathbf{ILP_{AC}}$-**G3** the rules of Weakening and Contraction are admissible preserving the depth of the derivation and that the rules Cut can be given the *context-sharing* form, rather than the *multiplicative* form.

Since all formulas of the language $\mathcal{L}^P$ are polarized as *assertive* or *conjectural*, sequents of **PBL-G3** have a restricted form, inspired by Girard's logic **LU** [15].

Definition 5. All the sequents S are of the form

$$\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \Upsilon$$

where

- Θ is a sequence of assertive formulas $\vartheta_1, \ldots, \vartheta_m$;
- Υ is a sequence of conjectural formulas $\upsilon_1, \ldots, \upsilon_n$;
- ϵ is conjectural and ϵ' is assertive and exactly of ϵ, ϵ' occurs.

The rules of **PBL-G3** are given in Table 3.

Remark. The only formal difference between **PBL-G3** and $\mathbf{ILP_{AC}}$-**G3** is the restrictions on the $\supset$-*right*, $\sim$-*right*, $\smallsetminus$-*left* and $\frown$-*left* rules. In **PBL-G3** the rules of $\supset$-*right* and $\smallsetminus$-*left* have the forms

$$\frac{\Theta, \vartheta_1 \;;\; \Rightarrow\; \vartheta_2 \;;}{\Theta \;;\; \Rightarrow\; \vartheta_1 \supset \vartheta_2 \;;\; \Upsilon} \qquad \frac{;\; \upsilon_1 \;\Rightarrow\; ;\; \Upsilon, \upsilon_2}{\Theta \;;\; \upsilon_1 \smallsetminus \upsilon_2 \;\Rightarrow\; ;\; \Upsilon}$$

while in $\mathbf{ILP_{AC}}$-**G3** the formulas in Υ and Θ are allowed in the sequent-premise of $\supset$-*right* and $\smallsetminus$-*left*, respectively; similar remarks apply to $\sim$-*right* and $\frown$-*left*. In Table 3 the restricted rules are marked with (¶). It is obvious that the schemes

$$(1) \qquad \vartheta \Rightarrow \sim\frown \vartheta \qquad \text{and} \qquad (2) \qquad \frown\sim \upsilon \Rightarrow \upsilon$$

(section 3.1) are derivable in **ILP**; conversely, the unrestricted rules of **ILP** become derivable in **PBL** using *cut* with the schemes (1) and (2) taken as axioms.

A proof of the following theorem is sketched in Section 7.

THEOREM 1. *The intuitionistic sequent calculus* **PBL − G3** *without the rules of cut are sound and complete with respect to the interpretation in* bimodal **S4**.

§5. Natural Deducation Systems $\mathbf{INP_{AC}}$ and PBN. We outline two Natural Deduction system: $\mathbf{INP_{AC}}$ for the Intuitionistic Logic for Pragmatics of Assertions and Conjectures and **PBN** for Polarized Bi-intuitionistic Logic. In this paper we will not give full treatments of these systems; the main result here is about *purely conjectural fragment*, which is common to both. A proper treatment of the (most interesting) negation rules is left to future work.

Notation. (i) $\mathbf{INP_{AC}}$ is the Natural Deduction system for the Intuitionistic Logic for Pragmatics of Assertions and Conjectures on the language $\mathcal{L}_{\vartheta\upsilon}$, with rules of inference and rules of deduction for assertive connectives, conjectural connectives and negations. There are β-reductions for all connectives except $\bigvee$ and $\bigwedge$ and commutations for $\bigvee$-introduction and $\bigwedge$-elimination.

(ii) **PBN** is the Natural Deduction system for Polarized Bi- intuitionistic Logic. It is like $\mathbf{INP_{AC}}$, except that the rules $\supset$-introduction, $\smallsetminus$-elimination, $\sim$-introduction and $\frown$-elimination have restrictions corresponding to those for the corresponding rules of the sequent calculus (which are marked (¶) in Table 3). There are β-reductions for all connectives except $\bigvee$ and $\bigwedge$ and commutations for all connectives except $\cap$ and $\curlyvee$.

(iii) Leaving out the rules of negation and working with the language $\mathcal{L}^{-}_{\vartheta\upsilon}$, our Natural Deduction systems split in two parts:

• $\mathbf{NJ}^{\supset\cap}$, the familiar *many-premises single-conclusion* Natural Deduction system with rules for intuitionistic implication ($\supset$), conjunction ($\cap$) and validity ($\bigvee$);

• $\mathbf{NJ}^{\smallsetminus\curlyvee}$, a *single-premise multiple-conclusion* Natural Deduction system with rules for subtraction ($\smallsetminus$), weak disjuncion ($\curlyvee$) and absurdity ($\bigwedge$).

Definition 6. (*Proof-graphs and Rules of Inference*) (i) A *proof-graph* is a directed acyclic and connected labelled graph $\mathcal{G} = (V, E, L, C, D)$ where

- the labelling function L maps edges in E to formulas of the language $\mathcal{L}_{\vartheta\upsilon}$ and vertices in V to *rules of inference*; vertices with no incoming edge are called *assumptions* and vertices with no outgoing edge are called *conclusions*;
- C is an equivalence relation on edges such that e_1Ce_2 implies $L(e_1) = L(e_2)$. Equivalence classes of assumption [conclusions] are called *assumption classes* [*conclusion classes*] .
- the partial *discharge function* D maps assumptions classes to vertices (namely, an assumption class to the inference in virtue of which all formulas in the assumption class are discharged).

(ii) The rules of inference for $\mathbf{INP_{AC}}$ must have the form indicated below in Table 4.

(iii) The rules of inference proper of **PBN** are the *restricted rules* ($\supset$-introduction, $\smallsetminus$-elimination, $\sim$-introduction and $\frown$-elimination) given in Table 8; all other rules are unrestricted and have the same form as those for $\mathbf{INP_{AC}}$ given in Table 4.

Remarks on notation. (i) The absence of explicit *structural rules* is a distinctive feature of Prawitz's Natural Deduction for $\mathbf{NJ}^{\supset\cap}$. Implicit left Contraction is implemented through the equivalence relation C which collects assumptions into assumption classes, and becomes active in the discharging operations associated with $\supset$-I. Notice that the latter implicitly involves also the structural rule Exchange left. Following the convention of Prawitz [27], we write $[A]$ for a class of assumptions of the form A in a proof-graph which are discharged by an inference $\supset$-I. In **PBN** this rule is restricted and has *extra premises* $\vartheta'_1, \ldots, \vartheta'_m$; every extra premise ϑ'_i discharges an assumption class $[\vartheta'_i]$.

(ii) Following a common convention in the literature, the notation

$$\begin{array}{c} d_1 \\ \vdots \\ [\vartheta] \\ \vdots \\ d_2 \end{array}$$

in a reduction (or commutation) rule indicates that a copy of d_1 has been susbtituted for each open assumption in the assumption class $[\vartheta]$ of d_2 (identifying each assumption vertex in $[\vartheta]$ with the conclusion vertex of the corresponding copy of d_1) and that open assumption classes $[\vartheta_i]'$ and $[\vartheta_i]''$ in different copies d_1' and d_1'' of d_1 have been merged.

(iii) Symmetrically, an inference $\smallsetminus$-E also "discharges" some conclusions and we write (C) for a conclusion class which is discharged in virtue of such an inference. Here again an implicit use is made of Exchange right. However, unlike assumption-classes, this information affects the specification of the form of inference rule, i.e., the number of *minor premises* of the inference in question, thus in some sense the implicit use of Contraction right is manifest in the form of the $\smallsetminus$-E rule[13]. In **PBN** this rule is restricted and has *extra premises* $(\upsilon_1'), \ldots, (\upsilon_m')$ and *extra conclusions* $\upsilon_1', \ldots, \upsilon_m'$, again with an implicit use of Exchange right and Contraction right. A symmetric convention to that in (ii) applies here with respect to the notation

$$\begin{array}{c} d_2 \\ \vdots \\ (\upsilon) \\ \vdots \\ d_1 \end{array}$$

in a reduction or commutation rule. Namely, we substitute a copy of the deduction d_1 for each conclusion of d_2 in (υ) (identifying each conclusion vertex in (υ) with the unique assumption υ of the corresponding copy of d_1) and merging the conclusion classes $(\upsilon_j)'$ and $(\upsilon_j)''$ occurring in different copies d_1' and d_1'' of d_1.

(iii) Right Contraction appears in $\mathbf{INP_{AC}}$ also in the $\sim$-I rule, a rule introducing a single assertive formula from a variable number of conjectural premises of the form $\bigwedge$.

Definition 7. (*Rules of deduction*) (i) The *deduction rules* that characterize the systems $\mathbf{INP_{AC}}$ are given in Table 5.

(ii) The deduction rules characterizing the system **PBN** are the same as those for $\mathbf{INP_{AC}}$ except for those corresponding to *restricted rules of inference* given in Table 9.

(iii) Let $\mathcal{G}$ be a proof-graph for $\mathbf{INP_{AC}}$ [or for **PBN**]. We say that $\mathcal{G}$ represents a *Natural Deduction derivation* in $\mathbf{INP_{AC}}$ [in **PBN**] if

[13]This "anomaly" is removed in an alternative formulation of the Rule of Inference for $\smallsetminus$-E, mentioned in the Remark in section 6.2.

$\mathcal{G}$ can be built inductively, using the deduction rules for $\mathbf{INP_{AC}}$ [for **PBN**] as inductive clauses.

(iv) *Symmetric reductions* (*β-reductions*) and *commutations* for the system $\mathbf{INP_{AC}}$ are given in Table 6 and 7.

(v) *Symmetric reductions* involving the restricted rules of system **PBN** are given in Table 10; the other symmetric reductions (for $\cap$ and for $\curlyvee$) are the same as for $\mathbf{INP_{AC}}$. The *commutations* for **PBN** are those for $\mathbf{INP_{AC}}$ (Table 7) and in addition those in Table 11.

Remark. In **PBN** (as in the corresponding **PBL-G3** sequent calculus rules) applications of $\supset$-I, $\sim$-I, $\smallsetminus$-E and $\frown$-E require a global control of the context of the inference. For an application of $\supset$-I to be correct, the premise must be derivable with a subderivation having no conjectural assumption and no conjectural conclusions. As a consequence, if $\supset$-I had the same familiar form as in $\mathbf{INP_{AC}}$, then an application of it may become invalid as a consequence of a β-reduction. A solution is provided by giving it the form of a *promotion rule*: such a rule has *extra premises*, as many as the open assumption classes in the derivation of the major premise, and all these open assumptions are discharged in virtue of the rule application. The problem and this solution are well-known from work on Natural Deduction for linear logic, but were already familiar in the literature about modal logic (see [27], pp. 74-80 and [2]). A similar problem arises with applications of the $\smallsetminus$-E, where the minor premise must be derivable with a subderivation having no assertive assumptions and no assertive conclusion. It occurs also in the rules $\sim$-I and $\frown$-E, which are superficially similar to $\supset$-I and $\smallsetminus$-E, respectively.

The definitions of a *path*, a *segment* in a path and of *maximal* and *minimal* segments in **PBN** follow from the analogue definition of Prawitz [27] and are not given here. We shall not give a normalization theorem for $\mathbf{INP_{AC}}$ nor for **PBN**. A strong normalization theorem for $\mathbf{NJ}^{\smallsetminus\curlyvee}$ is a corollary of the same result for $\mathbf{NJ}^{\supset\cap}$ and the isomorphism theorem below.

§6. Term assignment for the assertive and conjectural fragments. For the purely assertive fragment $\mathbf{NJ}^{\supset\cap}$ we take the familiar term assignment with terms of the simply typed λ-calculus. Namely, we have an infinite list of variables, denoted by x, and terms are given by the following grammar:[14]

$$t \ := x \mid \texttt{true}\ (t_1 \ldots t_n) \mid \ < t_0, t_1 > \ \mid \pi_0 t \mid \pi_1 t \mid \lambda x.t \mid t_0 t_1$$

Usually we will assign terms to natural deductions explicitly written as type derivations with deduction rules and sequents of the form

$$\overline{x} : \Theta \rhd t : \vartheta.$$

However, we shall sometimes use the more concise notation and decorate *proof-graphs* with terms.

6.1. Term assignment for PBN, conjectural fragment. For the purely conjectural fragment $\mathbf{NJ}^{\setminus\curlyvee}$ we will give type derivations with sequents of the form

$$\overline{x} : \upsilon \rhd \overline{\ell} : \Upsilon$$

but here the term-assignment presents some novelties. One is that we must label formulas with *lists* ℓ of terms, rather than just with terms, in order to account for contraction of conclusions. Another is the presence of *global binding*: as explained in the Preface (Section 1.6), variables become bound as a consequence of the introduction of another term in the context (*control term*) and "frozen", until the computation of the latter makes them available again for substitution. For each symbol of variable x there will correspond exactly one symbol $\mathtt{x}$ for a *globally bound* variable (the presence in the same context of a free variable x and of its counterpart $\mathtt{x}$ is ruled out by suitable conventions).

Since the process of computation may take place in control terms, substitution becomes a *global process*, which has to be broadcast and performed in remote terms. We shall express a *command of remote substitution* by control expressions of the form $\{\mathtt{x} ::= \ell\}$. Remote substitution shall coexist with *ordinary substitution* (avoiding capture of free variables), denoted by $t[u/x]$ as usual. A parameter $\mathtt{x}$ in a term can be substituted by a term through a command for remote substitution; a free variable cannot occur in the left-hand side of a command for remote substitution. However, we shall not attempt to indicate an implementation of *remote substitution*: we will simply say that in presence of a control term of the form $\{\mathtt{x} ::= u\}$ a term of the form $\ell[\mathtt{x}/x]$ will *eventually* become $\ell[u/x]$.

[14]In the global system **PBN** the restriction on the $\supset$-I rule also prevents unrestricted substitutions in the term calculus. Therefore we need terms of the form $\texttt{promote}\ \overline{t}\ \texttt{for}\ \overline{x}\ \texttt{in}\ \lambda x.t$, instead of just $\lambda x.t$.

DEFINITION 8. We are given a countable set of *free variables* (denoted by x) and a contable set of *globally bound variables* (denoted by $\mathtt{x}$) together with a bijection between them.
(i) Terms and lists of terms are defined simulaneously by the following grammar:[15]

$$\begin{array}{rcl} t & := & x \mid \mathtt{x} \mid \mathtt{false}\,(\ell_1 \ldots \ell_n) \mid \mathtt{inl}(\ell) \mid \mathtt{inr}(\ell) \mid \mathtt{casel}\,(\ell) \mid \mathtt{caser}\,(\ell) \mid \\ & & \mathtt{continue\ from(x)using}(\ell) \mid \mathtt{postpone}\,(x :: \ell)\,\mathtt{until}(\ell') \\ \ell & := & () \mid t \cdot \ell \end{array}$$

with the usual associative operation of *append*:

$$() * \ell' = \ell' \qquad\qquad (t \cdot \ell) * \ell' = t \cdot (\ell * \ell').$$

If $\overline{\ell}$ and $\overline{\ell'}$ are vectors of lists of the same length n, then $\overline{\ell} * \overline{\ell'} = (\ell_1 * \ell'_1, \ldots, \ell_n * \ell'_n)$.
(ii) *Term expansion:* Let $\mathtt{op}$ ( ) be one of
$\mathtt{false}\,(\ell_1 \ldots (\ ) \ldots \ell_n)$, $\mathtt{inl}$ ( ), $\mathtt{inr}$ ( ) $\mathtt{casel}$ ( ), $\mathtt{caser}$( ),
$\mathtt{continue\ from\ x\ using}$ ( ) or $\mathtt{postpone}\,(x :: \ell')$ $\mathtt{using}$ ( ).
Then the *expansion* of $\mathtt{op}\ (\ell)$ is the list defined inductively thus:

$$\mathtt{op}\ () = () \qquad\qquad \mathtt{op}\ (t \cdot \ell) = \mathtt{op}\ (t) \cdot \mathtt{op}\ (\ell)$$

Remark. By term expansion, a term consisting of an operator applied to a list of terms can always be turned into a list of terms; thus terms may always be trasformed into an expanded form where operators are applied only to terms, *except for expressions* $(x :: \ell')$ occurring in terms of the form $\mathtt{postpone}\ (x :: \ell')\ \mathtt{using}\ (t)$.

DEFINITION 9. The *free variables* $FV(\ell)$ in a list of terms ℓ are defined as follows:

[15] In the full system **PBN** we may need to consider $\mathtt{postpone}$ terms of a more elaborate form, such as $\mathtt{postpone}\ (x :: \ell)\ \mathtt{until}(\ell')\ \mathtt{with}\ \overline{\ell}[\mathtt{x}/x]\ \mathtt{for}\ \overline{\mathtt{y}}$. We also need a *closure requirement*, i.e., the condition that in a term $\mathtt{postpone}\ (x :: \ell)$ $\mathtt{until}\ (\ell')\ \mathtt{with}\ \overline{\ell}[\mathtt{x}/x]\ \mathtt{for}\ (\overline{\mathtt{y}})$ we must have

$$FV(\ell) \setminus \{x\} \ \cup FV(\overline{\ell}[\mathtt{x}/x]) = \emptyset.$$

This corresponds the condition that in an $\smallsetminus$-E inference υ_1 must be the only open assumption on which the minor premises depends, in particular they cannot depend on an *assertive* assumption.

$$\begin{array}{rl} FV(()) = & \emptyset \\ FV(t \cdot \ell) = & FV(t) \cup FV(\ell) \\ FV(x) = & \{x\} \\ FV(\mathtt{x}) = & \emptyset \\ FV(\mathtt{false}\ (\ell_1 \dots \ell_n)) = & \bigcup_{i \le n} FV(\ell_i) \\ FV(\mathtt{inl}\ (\ell)) = FV(\mathtt{inr}\ (\ell)) = & FV(\ell) \\ FV(\mathtt{casel}\ (\ell)) = FV(\mathtt{caser}\ (\ell)) = & FV(\ell) \\ FV(\mathtt{continue\ from}\ (\mathtt{x})\ \mathtt{using}\ (\ell)) = & FV(\ell) \\ FV(\mathtt{postpone}\ (x :: \ell)\ \mathtt{until}\ (\ell')) = & FV(\ell') \cup FV(\ell) \setminus \{x\}. \end{array}$$

DEFINITION 10. Substitution of *lists of terms* within lists of terms is defined from the usual substitution (avoiding capture of free variables) as follows:

$$\begin{array}{cc} ()[\ell'/x] = () & t \cdot \ell[\ell'/x] = t[\ell'/x] \cdot \ell[\ell'/x] \\ t[()/x] = () & t[u \cdot \ell/x] = t[u/x] \cdot t[\ell/x] \end{array}$$

If $\overline{\ell}$ is a vector $(\ell_1, \dots, \ell_m)$, then $\overline{\ell}[\ell'/x] = (\ell_1[\ell'/x], \dots, \ell_m[\ell'/x])$.

DEFINITION 11. β-reduction $\ell \rightsquigarrow_\beta \ell'$ for lists of terms in the purely conjectural fragment is defined as follows:

$$\begin{array}{rl} \mathtt{casel}\ (\mathtt{inl}\ \ell) \rightsquigarrow_\beta \ \ell; & \mathtt{caser}\ (\mathtt{inr}\ \ell) \rightsquigarrow_\beta \ \ell; \\ \mathtt{casel}\ (\mathtt{inr}\ \ell) \rightsquigarrow_\beta (); & \mathtt{caser}\ (\mathtt{inl}\ \ell) \rightsquigarrow_\beta (); \\ \mathtt{postpone}\ (y :: \ell')\ \mathtt{until} & \\ (\mathtt{continue\ from}\ (\mathtt{x})\ \mathtt{using}(\ell)) \rightsquigarrow_\beta & \{\mathtt{x} ::= \ell'[\ell/y]\}, \{\mathtt{y} ::= \ell\} \end{array}$$

6.2. Typing judgement for the term calculus. The typing judgements for the purely conjectural fragment $\mathbf{NJ}^{\setminus \curlyvee}$ are in the following table.

We shall make the following *pure variable requirement*:

Different axioms are labelled with a different free variable.

Remark. (i) The deduction rule for $\curlyvee$-E in the form given in Table 5 results from the one given above using substitution.

(ii) There is an alternative *one premise* definition to the *rule of inference* $\setminus$ which takes the *discharged assumption* $x : \upsilon_1$ in our official definition (Table 4) as *conclusion* of the inference:

Typing judgement for $\mathbf{NJ}^{\diagdown\curlyvee}$

$$
\begin{array}{ccc}
\textit{exchange:} & \textit{contraction:} & \textit{weakening:} \\
\dfrac{\epsilon \ \rhd \ \Upsilon, x : \upsilon, y : \upsilon', \Upsilon'}{\epsilon \ \rhd \ \Upsilon, y : \upsilon', x : \upsilon, \Upsilon'} & \dfrac{\Theta \ \rhd \ \Upsilon, \ell : \upsilon, \ell' : \upsilon}{\Theta \ \rhd \ \Upsilon, \ell * \ell' : \upsilon} & \dfrac{\Theta \ \rhd \ \Upsilon}{\Theta \ \rhd \ \Upsilon, () : \upsilon}
\end{array}
$$

$$
\begin{array}{cc}
\textit{assumption}: & \\
x : \upsilon \rhd x : \upsilon & \dfrac{\epsilon \rhd \Upsilon, \ell : \upsilon \qquad x : \upsilon \rhd \overline{\ell} : \Upsilon'}{\epsilon \rhd \Upsilon, \overline{\ell}[\ell/x] : \Upsilon'} \textit{-substitution}
\end{array}
$$

$$
\dfrac{\epsilon \rhd \ell : \bigwedge}{\epsilon \rhd \texttt{false}\ (\ell) : \upsilon_1 \ \ldots \ \texttt{false}\ (\ell) : \upsilon_n} \bigwedge\text{-E}
$$

$$
\dfrac{\epsilon \rhd \ \ell : \upsilon_1, \Upsilon}{\epsilon \rhd \ \texttt{y} : \upsilon_2, \ \texttt{continue from (y) using}\ (\ell) : \upsilon_1 \diagdown \upsilon_2, \Upsilon} \diagdown\text{-I}
$$

$$
\dfrac{\epsilon \ \rhd \ \Upsilon, \ell' : \upsilon_1 \diagdown \upsilon_2 \qquad\qquad x : \upsilon_1 \rhd \ell : (\upsilon_2), \ \overline{\ell} : \Upsilon'}{\epsilon \rhd \ \Upsilon, \ \overline{\ell}[\texttt{x}/x] : \Upsilon, \ \texttt{postpone}\ (x :: \ell)\ \texttt{until}\ \ell' : \bullet} \diagdown\text{-E}
$$

$$
\dfrac{\epsilon \rhd \ell : \upsilon_0, \Upsilon}{\epsilon \rhd \texttt{inl}\ (\ell) : \upsilon_0 \curlyvee \upsilon_1, \Upsilon} \curlyvee_0\text{-I} \qquad \dfrac{\epsilon \rhd \ell : \upsilon_1, \Upsilon}{\epsilon \rhd \texttt{inr}\ (\ell) : \upsilon_0 \curlyvee \upsilon_1, \Upsilon} \curlyvee_1\text{-I}
$$

$$
\dfrac{\epsilon \rhd \ \Upsilon, \ell : \upsilon_0 \curlyvee \upsilon_1}{\epsilon \rhd \ \Upsilon, \texttt{casel}\ (\ell) : \upsilon_0, \texttt{caser}\ (\ell) : \upsilon_1} \curlyvee\text{-E}
$$

TABLE 1. Typing judgements for $\mathbf{NJ}^{\diagdown\curlyvee}$

$$
\begin{array}{c}
[\epsilon] \\
\vdots \\
\dfrac{\ell' : \upsilon_1 \setminus \upsilon_0}{\texttt{x} : \upsilon_1} \diagdown\text{-E} \\
\vdots \qquad \vdots \\
\ell : \upsilon_0 \quad \overline{\ell} : \Upsilon \\
\texttt{postpone}\ (x :: \ell)\ \texttt{until}\ (\ell') : \bullet
\end{array}
$$

This alternative definition gives *proof-graphs* for $\mathbf{NJ}^{\diagdown\curlyvee}$ deductions a suggestive geometric form, i.e., a $\mathbf{NJ}^{\supset\cap}$ proof-tree turned upside down; it shall be used in the examples in Section 6.5.

6.3. β-reductions. In $\diagdown\curlyvee$ the β-reductions for $\diagdown$ and for $\curlyvee$ correspond to the following transformations of derivations.

$\smallsetminus$-REDUCTION:

$$\dfrac{\dfrac{\epsilon \rhd\ \Upsilon, \ell_1 : \upsilon_1}{\epsilon \rhd\ \Upsilon,\ \mathtt{z} : \upsilon_2,\ \mathbf{t} : \upsilon_1 \smallsetminus \upsilon_2} \qquad x : \upsilon_1\ \rhd\ \ell_2 : \upsilon_2,\ \overline{\ell'} : \Upsilon'}{\epsilon\ \rhd\ \Upsilon,\ \mathtt{z} : \upsilon_2,\ \mathbf{u} : \bullet,\ \overline{\ell'}[\mathrm{x}/x] : \Upsilon'}$$

where $\mathbf{t}$ is $\texttt{continue from (z) using}$ (ℓ_1) and
$\mathbf{u}$ is $\texttt{postpone}$ $(x :: \ell_2)$ $\texttt{until (continue from (z) using}$ $(\ell_1))$,

reduces to

$$\epsilon\ \rhd\ \Upsilon,\ \mathtt{z} : \upsilon_2,\ \{\mathtt{z} ::= \ell_2[\ell_1/x]\},\ \overline{\ell'}[\mathrm{x}/x] : \Upsilon',\ \{\mathtt{x} ::= \ell_1\}$$

After remote substitutions are performed, the sequent becomes
$\epsilon\ \rhd\ \Upsilon,\ \ell'[\ell/x] : \upsilon_2,\ \overline{\ell'}[\ell/x] : \Upsilon'$

$\curlyvee$-REDUCTIONS:

$$\dfrac{\dfrac{\epsilon\ \rhd\ \Upsilon, \ell_0 : \upsilon_0}{\epsilon\ \rhd\ \Upsilon, \mathtt{inl}(\ell_0) : \upsilon_0 \curlyvee \upsilon_1}}{\epsilon\ \rhd\ \Upsilon, \mathtt{casel\ inl}(\ell_0) : \upsilon_0, \mathtt{caser\ inl}(\ell_0) : \upsilon_1}$$

reduces to

$$\epsilon\ \rhd\ \Upsilon, \ell_0 : \upsilon_0$$

and

$$\dfrac{\dfrac{\epsilon\ \rhd\ \Upsilon, \ell_1 : \upsilon_1}{\epsilon\ \rhd\ \Upsilon, \mathtt{inr}(\ell_1) : \upsilon_0 \curlyvee \upsilon_1}}{\epsilon\ \rhd\ \Upsilon, \mathtt{casel\ inr}(\ell_1) : \upsilon_0, \mathtt{caser\ inr}(\ell_1) : \upsilon_1}$$

reduces to

$$\epsilon\ \rhd\ \Upsilon, \ell_1 : \upsilon_1$$

Remark. Notice that the assignment of *lists of terms* (instead of just terms) to deductions in $\mathbf{NJ}^{\smallsetminus\curlyvee}$ implements the implicit use of Contraction right in minor premise of a $\smallsetminus$-E and in the $\smallsetminus$-reductions, as described in the *Remark on notation*, section 5. Given a redex

$$\texttt{postpone}\ (x :: \ell_2)\ \texttt{until continue from (z) using}\ (\ell_1) : \bullet$$

after the command $\{\mathtt{z} ::= \ell_2[\ell_1/x]\}$ is executed, a *list of terms* $\ell_2[\ell_1/x] = (r_1, \ldots, r_k)$ has been substituted for $\mathtt{z} : \upsilon_2$. If υ_j occurs below υ_2 and its term assignment was $\mathtt{s(z)} : \upsilon_j$, then it is

$\mathbf{s}(r_1, \ldots, r_k) : \upsilon_j$ after the β-reduction. Now *term expansion* transforms

$$\mathbf{s}(r_1, \ldots, r_k) : \upsilon_j \qquad \text{into} \qquad (\mathbf{s}(r_1), \ldots, \mathbf{s}(r_k)) : \upsilon_j$$

This implements precisely the implicit action of Contraction right, merging the conclusion classes (υ_j) after a possible action of *copying* required by the β-reduction.

6.3.1. *Commutations.* In the assertive fragment the commutation

$$\frac{\overline{y} : \Theta \ \rhd \ t : \vartheta \qquad x : \vartheta, \overline{x} : \Theta' \ \rhd \ \mathtt{true}\ (x, \overline{x}) : \bigvee}{\overline{y} : \Theta, \overline{x} : \Theta' \rhd \ \mathtt{true}\ (t, \overline{x}) : \bigvee}$$
$$\rightsquigarrow_c$$
$$\overline{y} : \Theta, \overline{x} : \Theta' \rhd \ \mathtt{true}\ (\overline{y}, \overline{x}) : \bigvee$$

can be stipulated as a single-step rewriting rule as well as the result of several rewritings, corresponding to the one-step commutations indicated in Table 7.

In the conjectural fragment the commutation

$$\dfrac{\begin{array}{c}\vdots \\ \epsilon \ \rhd \ \ell : \bigwedge\end{array}}{\epsilon \ \rhd \ \mathtt{false}\ (\ell) : \upsilon_1, \ldots, \mathtt{false}\ (\ell) : \upsilon_n}\ \bigwedge\text{-E}$$
$$\begin{array}{ccc} \vdots & & \vdots \\ \epsilon \ \rhd \ \overline{\ell_1} : \Upsilon_1, & \ldots, & \overline{\ell_n} : \Upsilon_n \end{array}$$

commutes to $(\rightsquigarrow_c)$

$$\dfrac{\begin{array}{c}\vdots \\ \epsilon \ \rhd \ \ell : \bigwedge\end{array}}{\epsilon \ \rhd \ \overline{\mathtt{false}\ (\ell)} : \Upsilon_1, \ldots, \overline{\mathtt{false}\ (\ell)} : \Upsilon_n}\ \bigwedge\text{-E}$$

may be obtained through several one-step rewritings, defined in correspondence to the one-step commutations indicated in Table 7.

6.4. Isomorphism theorem. Consider the map $(\ )^\perp$ of $\mathcal{L}_{\vartheta\upsilon}$:

DEFINITION 12. (*Duality*)

$$\begin{array}{rcl@{\qquad}rcl}
(\vdash p)^\perp & =_{df} & \mathcal{H}\neg p & (\mathcal{H} p)^\perp & =_{df} & \vdash \neg p \\
(\bigvee)^\perp & =_{df} & \bigwedge & (\bigwedge)^\perp & =_{df} & \bigvee \\
(\vartheta_1 \supset \vartheta_2)^\perp & =_{df} & \vartheta_2^\perp \smallsetminus \vartheta_1^\perp & (\upsilon_1 \smallsetminus \upsilon_2)^\perp & =_{df} & \upsilon_2^\perp \supset \upsilon_1^\perp \\
(\vartheta_0 \cap \vartheta_1)^\perp & =_{df} & \vartheta_0^\perp \curlyvee \vartheta_1^\perp & (\upsilon_0 \curlyvee \upsilon_1)^M & =_{df} & \upsilon_0^\perp \cap \upsilon_1^\perp
\end{array}$$

THEOREM 2. (isomorphism) *Modulo α-equivalence, there exists a bijection $(\ )^\perp$ between proof-terms for the* assertive *and the* conjectural *fragments of* **PBL** *such that*

(i) *if* $\overline{x} : \Theta \;\triangleright\; t : \vartheta$ *then* $(t)^{\perp}$ *has the form* $\overline{\ell}$ *and* $x : \vartheta^{\perp} \;\triangleright\; \overline{\ell} : \Theta^{\perp}$ *and conversely, if* $x : \epsilon \;\triangleright\; \overline{\ell} : \Upsilon$ *then* $(\overline{\ell})^{\perp}$ *has the form* t *and* $\overline{x} : \Upsilon^{\perp} \;\triangleright\; t : \epsilon^{\perp}$*; moreover,* $(t)^{\perp\perp} = t$ *and* $(\overline{\ell})^{\perp\perp} = \overline{\ell}$*.*

(ii) *if* t_0 *β-reduces to* t_1 *then* $(t_0)^{\perp}$ *β-reduces to* $(t_1)^{\perp}$*; conversely, if* $\overline{\ell}_0$ *β-reduces to* $\overline{\ell}_1$*, then* $(\overline{\ell}_0)^{\perp}$ *β-reduces to* $(\overline{\ell}_1)^{\perp}$*.*

Proof. We write $\overline{x} : \Upsilon^{\perp}$ for $x_1 : \upsilon_1^{\perp}, \ldots x_n : \upsilon_1^{\perp}$ and $\overline{\mathtt{false}}\ (x) : \Upsilon$ for $\mathtt{false}_{\upsilon_1}(x) : \upsilon_1, \ldots\ \mathtt{false}_{\upsilon_n}(x) : \upsilon_n$, and so on. The duality $(\ )^{\perp}$ on proof-terms is defined as follows. Setting $x^{\perp} = x$, the judgement $x : \vartheta \;\triangleright\; x : \vartheta$ is mapped to $x^{\perp} : \vartheta^{\perp} \;\triangleright\; x^{\perp} : \vartheta^{\perp}$ and conversely.

$$(1.1) \qquad (x : \bigwedge \;\triangleright\; \overline{\mathtt{false}}\ (x) : \Upsilon)^{\perp} \;=\; \overline{x} : \Upsilon^{\perp} \;\triangleright\; \mathtt{true}\ (\overline{x}) : \bigvee$$

$$(1.2) \qquad (\overline{x} : \Theta \triangleright \mathtt{true}\ (\overline{x}) : \bigvee)^{\perp} \;=\; x : \bigwedge \triangleright \overline{\mathtt{false}}\ (x) : \Theta^{\perp}$$

$$(2.1) \qquad (z : \vartheta_1 \supset \vartheta_2, y : \vartheta_1 \triangleright y : \vartheta_2)^{\perp} \;=\; x : \vartheta_2^{\perp} \triangleright \mathbf{y} : \vartheta_1^{\perp}, r : \vartheta_2^{\perp} \setminus \vartheta_1^{\perp}$$

$$(2.2) \qquad (x : \upsilon_1 \triangleright \mathbf{y} : \upsilon_2, r : \upsilon_1 \setminus \upsilon_2)^{\perp} \;=\; y : \upsilon_2^{\perp},\ z : \upsilon_2^{\perp} \supset \upsilon_1^{\perp},\ \triangleright\ zy : \upsilon_1^{\perp}$$

where $r = \mathtt{continue\ from}\ (\mathtt{y})\ \mathtt{using}\ (x)$.

$$(3.1) \qquad (y : \vartheta_0 \cap \vartheta_1 \triangleright \pi_0(y) : \vartheta_0)^{\perp} \;=\; x : \vartheta_0^{\perp} \triangleright \mathtt{inl}\ (x) : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp}$$

$$(3.2) \qquad (x : \upsilon_0 \triangleright \mathtt{inl}\ (x) : \upsilon_0 \curlyvee \upsilon_1)^{\perp} \;=\; y : \upsilon_0^{\perp} \cap \upsilon_1^{\perp} \triangleright \pi_0(y) : y : \upsilon_0^{\perp}$$

$$(4.1) \qquad (y : \vartheta_1 \cap \vartheta_1 \triangleright \pi_1(y) : \vartheta_1)^{\perp} \;=\; x : \vartheta_1^{\perp} \triangleright \mathtt{inr}\ (x) : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp}$$

$$(4.2) \qquad (x : \upsilon_1 \triangleright \mathtt{inr}\ (x) : \upsilon_0 \curlyvee \upsilon_1)^{\perp} \;=\; y : \upsilon_0^{\perp} \cap \upsilon_1^{\perp} \triangleright \pi_1(y) : y : \upsilon_1^{\perp}$$

Now suppose

$$(\overline{x} : \Theta \triangleright t_i : \vartheta_i)^{\perp} = y_i : \vartheta_i^{\perp} \triangleright \overline{\ell_i} : \Theta^{\perp}$$

for $i = 0$ and 1. We set

$$(5.1) \qquad (\overline{x} : \Theta \triangleright < t_0, t_1 >: \vartheta_0 \cap \vartheta_1)^{\perp} \;=\; z : \vartheta_0^{\perp} \curlyvee \vartheta_1^{\perp} \;\triangleright\; \mathbf{r}_0 * \mathbf{r}_1 : \Theta^{\perp}$$

where $\mathbf{r}_0 = \overline{\ell_0}[\mathtt{casel}\ (z)/y_0]$ and $\mathbf{r}_1 = \overline{\ell_1}[\mathtt{caser}\ (z)/y_1$.
Next suppose

$$(y_i : \upsilon_i \triangleright \overline{\ell_i} : \Upsilon)^{\perp} = \overline{x} : \Upsilon^{\perp} \triangleright t_i : \upsilon_i^{\perp}$$

for $i = 0$ and 1. We set

$$(5.2) \qquad (z : \upsilon_0 \curlyvee \upsilon_1 \;\triangleright\; \mathbf{r}_0 : \Theta_0^{\perp}, \mathbf{r}_1 : \vartheta_1^{\perp})^{\perp} \;=\; \overline{x} : \Upsilon^{\perp} \;\triangleright < t_0, t_1 >: \upsilon_0^{\perp} \cap \upsilon_1^{\perp}$$

where again $\mathbf{r}_0 = \overline{\ell_0}[\texttt{casel}\ (z)/y_0]$ and $\mathbf{r}_1 = \overline{\ell_1}[\texttt{caser}\ (z)/y_1$.

Now suppose

$$(x : \vartheta_1,\ \overline{x} : \Theta\ \ \rhd\ \ t : \vartheta_2)^\perp = y : \vartheta_2^\perp \rhd \ell_1 : \vartheta_1^\perp, \dots , \ell_m : \vartheta_1^\perp, \overline{\ell} : \Theta^\perp$$

We set

$$(6.1)\quad (\overline{x} : \Theta \rhd \lambda x.t : \vartheta_1 \supset \vartheta_2)^\perp = z : \vartheta_2^\perp \smallsetminus \vartheta_1^\perp\ \ \rhd \overline{\ell}[\mathtt{y}/y] : \Theta^\perp,\ \mathbf{u} : \bullet$$

where $\mathbf{u} = \texttt{postpone}\ (y :: \ell_1 * \dots * \ell_m)\ \texttt{until}\ (z)$. Finally suppose

$$(x : \upsilon_1 \rhd \ell_1 : \upsilon_2, \dots , \ell_m : \upsilon_2, \overline{\ell} : \Upsilon)^\perp = y : \upsilon_2^\perp,\ \overline{x} : \Upsilon^\perp\ \ \rhd\ \ t : \upsilon_1^\perp$$

We set

$$(6.2)\quad (z : \upsilon_2 \smallsetminus \upsilon_1\ \ \rhd\ \ \overline{\ell}[\mathtt{y}/y] : \Upsilon,\ \mathbf{u} : \bullet)^\perp = \overline{x} : \Upsilon^\perp\ \ \rhd\ \ \lambda y.t : \upsilon_2^\perp \supset \upsilon_1^\perp$$

where $\mathbf{u} = \texttt{postpone}\ (y :: \ell_1 * \dots * \ell_m)\ \texttt{until}\ (z)$.

We need to show the following fact:

LEMMA 3. *If*

$$(i)\qquad (\Theta_2 \rhd u : \vartheta)^\perp = a : \vartheta^\perp \rhd \overline{\ell}_2 : \Theta_2^\perp$$

and

$$(ii)\qquad (x : \vartheta, \Theta_1 \rhd t : \vartheta_0)^\perp = b : \vartheta_0^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell : \vartheta^\perp$$

then

$$(\Theta_2, \Theta_1 \rhd t[u/x] : \vartheta_0)^\perp = b : \vartheta_0^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell_2[\ell/a] : \Theta_2^\perp$$

and simmetrically for substitutions in the conjectural part.

We prove the lemma by induction on t. Let us consider the case of $t = \lambda y.s$. Given (i) and

$$(ii)\quad (x : \vartheta, \Theta_1 \rhd \lambda y.s : \vartheta_1 \supset \vartheta_2)^\perp\ =\ b : \vartheta_2^\perp \smallsetminus \vartheta_1^\perp\ \ \rhd\ \ \overline{\ell}_1[\mathtt{c}/c] : \Theta_1^\perp, \ell[\mathtt{c}/c] : \vartheta^\perp,\ \mathbf{r} : \bullet$$

where $\mathbf{r} = \texttt{postpone}\ (c :: \ell_1)\ \texttt{until}\ (b)$, we need to show that

$$((\lambda y.s)[u/x])^\perp\ =\ \overline{\ell}_1[\mathtt{c}/c], \overline{\ell}_2[\ell[\mathtt{c}/c]/a],\ r : \bullet.$$

We may assume that (ii) results by an application of (6.1) and thus that we have

$$(iii)\qquad (x : \vartheta, y : \vartheta_1, \Theta_1\ \ \rhd\ \ s : \vartheta_2)^\perp = c : \vartheta_2^\perp \rhd \overline{\ell}_1 : \Theta_1^\perp, \ell_1 : \vartheta_1^\perp, \ell : \vartheta^\perp$$

The induction hypothesis is that

$$(iii)\quad (y : \vartheta_1, \Theta_1, \Theta_2\ \ \rhd\ \ s[u/x] : \vartheta_2)^\perp\ =\ c : \vartheta_2^\perp\ \ \rhd\ \ \overline{\ell}_1 : \Theta_1^\perp, \ell_1 : \vartheta_1^\perp, \overline{\ell}_2[\ell/a] : \Theta_2^\perp.$$

By applying (6.1) to (iii) we obtain

$$(\Theta_1, \Theta_2 \rhd \lambda x.s[u/y] : \vartheta_1 \supset \vartheta_2)^\perp = \\ b : \vartheta_2^\perp \setminus \vartheta_1^\perp \ \rhd \ \overline{\ell}_1[\mathsf{c}/c] : \Theta_1^\perp, \overline{\ell}_2[\ell/a][\mathsf{c}/c] : \Theta_2^\perp, \mathbf{r} : \bullet$$

Since we may assume that $(\lambda x.s)[u/y] = \lambda x.s[u/y]$ and since the variable c does not occur in $\overline{\ell}_2$, the desired result follows.

Part (i) of Theorem 2 is proved by a straightforward induction on t or $\overline{\ell}$. To prove part (ii) of Theorem 2 there are four cases to check; we consider only that of a $\setminus$-reduction. Let

$$\overline{\ell} = (\overline{\ell}_1, \mathsf{c}, \overline{\ell}_2[\mathsf{a}/a], \mathbf{s})$$

where $\mathbf{s} = \texttt{postpone}\ (a :: \ell_2)\ \texttt{using}\ (\texttt{continue from}\ (\mathsf{c})\ \texttt{using}\ (\ell_1))$ and suppose

$$\overline{\ell} \quad \rightsquigarrow_\beta \quad \overline{\ell_1}, \mathsf{c}, \overline{\ell}_2[\mathsf{a}/a]\{\mathsf{a} ::= \ell_1\}, \{\mathsf{c} ::= \ell_2[\ell_1/a]\}$$

A typing derivation of $\overline{\ell}$ is obtained as follows: we have a derivation d_1 ending with the inference

$$(o) \qquad \frac{\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \ell_1 : \upsilon_1}{\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \mathsf{c} : \upsilon_2, \mathbf{r} : \upsilon_1 \setminus \upsilon_2}$$

where $\mathbf{r} = \texttt{continue from}\ (\mathsf{c})\ \texttt{using}\ (\ell_1)$ and also a derivation d_2 of

$$(i) \qquad a : \upsilon_1 \ \rhd \ \ell_2 : \upsilon_2, \overline{\ell}_2 : \Upsilon_2$$

and we apply the inference $\setminus$-E to the conclusions of d_1 and d_2, yielding

$$\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \mathsf{c} : \upsilon_2, \overline{\ell}_2[\mathsf{a}/a] : \Upsilon_2, \mathbf{s} : \bullet$$

But the same typing of $\overline{\ell}$ may also be obtained by first deriving

$$(ii) \qquad b : \upsilon_1 \setminus \upsilon_2 \ \rhd \ \overline{\ell}_2[\mathsf{a}/a] : \Upsilon_2, \mathbf{s}(b) : \bullet$$

from (i), where $\mathbf{s}(b) = \texttt{postpone}\ (a :: \ell_2)\ \texttt{using}\ (b)$ and then substituting $\mathbf{r}$ for b in the terms in (ii) which contain it free, i.e., in the "control term" $\mathbf{s}(b)$. Moreover, we have

$$(iii) \qquad (\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \ell_1 : \upsilon_1)^\perp = x : \upsilon_1^\perp, \overline{y} : \Upsilon_1^\perp \rhd u : \epsilon^\perp$$

and

$$(iv) \qquad (a : \upsilon_1 \ \rhd \ \ell_2 : \upsilon_2,\ \overline{\ell}_2 : \Upsilon_2)^\perp = y : \upsilon_2^\perp, \overline{x} : \Upsilon_2^\perp \ \rhd \ t : \upsilon_1^\perp$$

By applying Lemma 3 to (iii) and (2.2) we have

$$(v) \qquad (\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \mathsf{c} : \upsilon_2, \mathbf{r} : \upsilon_1 \setminus \upsilon_2)^\perp = \\ y : \upsilon_2^\perp, z : \upsilon_2^\perp \supset \upsilon_1^\perp, \overline{y} : \Upsilon_1^\perp \ \rhd u[zy/x] : \epsilon^\perp$$

By applying Lemma 3 to (iv) and (6.2) we have

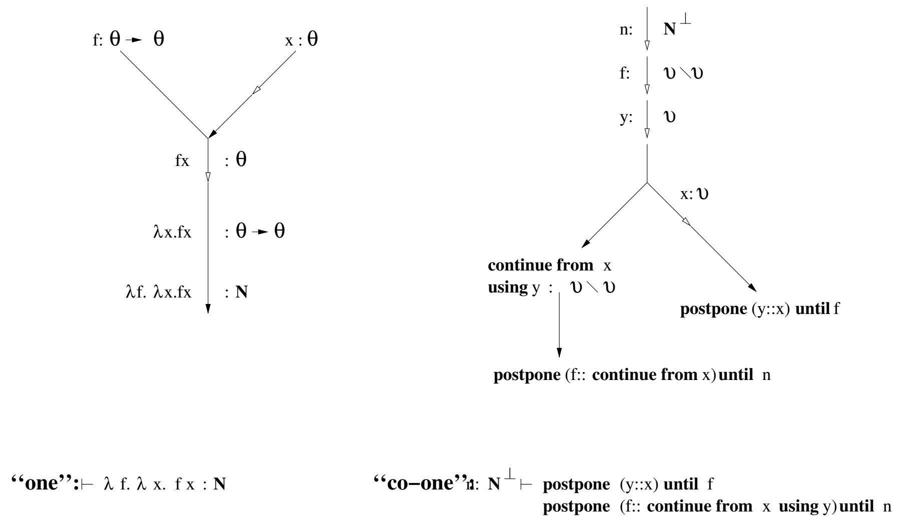

FIGURE 1. Church's one.

$$(vi) \qquad (b : \upsilon_1 \setminus \upsilon_2 \ \rhd \ \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \mathbf{s}(b) : \bullet)^\perp \ = \\ \overline{x} : \Upsilon_2^\perp \ \rhd \ \lambda y.t : \upsilon_2^\perp \supset \upsilon_1^\perp$$

Again by Lemma 3 applied to (v) and (vi) we conclude

$$(vii) \qquad (\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \mathtt{c} : \upsilon_2, \overline{\ell}_2[\mathtt{a}/a] : \Upsilon_2, \ \ \mathbf{s} : \bullet = \\ = y : \upsilon_2^\perp, \overline{y} : \Upsilon_1^\perp, \overline{x} : \Upsilon_2^\perp \ \ \rhd u[(\lambda y.t)y/x] : \epsilon^\perp$$

Now the right-hand side of (vii) reduces to $u[t/x]$. But also by Lemma 3 applied to (iii) and (iv) we obtain

$$(\epsilon \ \rhd \ \overline{\ell}_1 : \Upsilon_1, \ell_2[\ell_1/a] : \upsilon_2, \overline{\ell}_2[\ell_1/a])^\perp = y : \upsilon_2^\perp, \overline{y} : \Upsilon_1^\perp, \overline{x} : \Upsilon_2^\perp \rhd u[t/x] : \epsilon^\perp$$

and the argument of the left-hand side is exactly what $\overline{\ell}$ reduces to when the global substitutions are eventually performed. This concludes the proof.

6.5. Examples. In the case of deductions in the purely implicative and subtractive fragments, it is possible to give a suggestive graphic representation of the isomorphism $(\ )^\perp$.
In Fig. 1 we have drawn a refutation

$$n : \mathbf{N}^\perp \rhd \quad \mathtt{postpone}\ (y :: \mathtt{x})\ \mathtt{until}\ (f) : \bullet \\ \mathtt{postpone}\ (f :: \mathtt{continue\ from}\ (\mathtt{x})\ \mathtt{using}\ (\mathtt{y}))\ \mathtt{until}\ (n) : \bullet$$

which is formally given in $\mathbf{NJ}^{\setminus}$ as follows:

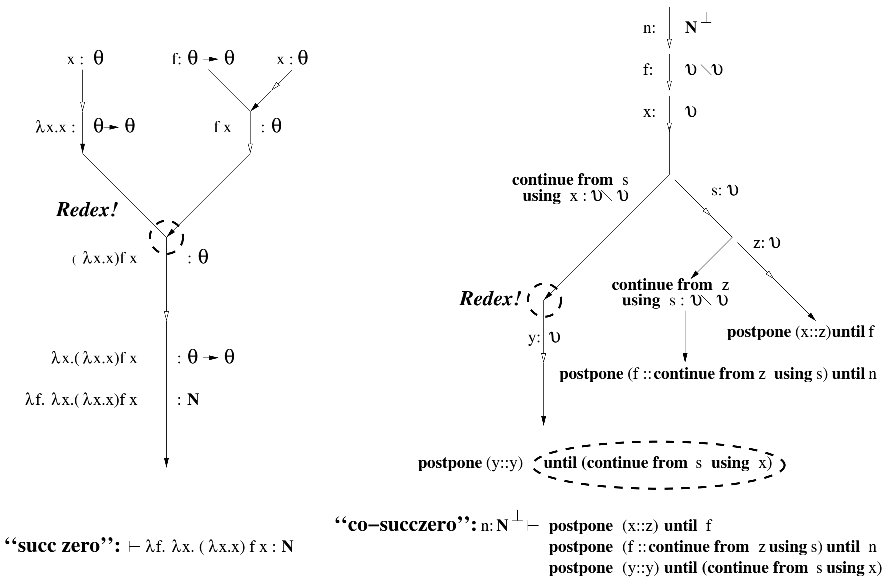

FIGURE 2. "succ zero".

$$\dfrac{n:\mathbf{N}^{\perp} \rhd n:\mathbf{N}^{\perp} \qquad \dfrac{f:\upsilon\setminus\upsilon \rhd f:\upsilon\setminus\upsilon \qquad \dfrac{y:\upsilon \rhd y:\upsilon}{y:\upsilon \rhd \mathtt{x}:\upsilon, \mathbf{s}:\upsilon\setminus\upsilon}}{f:\upsilon\setminus\upsilon \rhd \mathbf{t}:\bullet,\ \mathbf{s}[\mathtt{y}/y]:\upsilon\setminus\upsilon}}{n:\mathbf{N}^{\perp} \rhd \mathbf{t}:\bullet, \mathbf{u}:\bullet}$$

where $\mathbf{s} = \mathtt{continue\ from}\ (\mathtt{x})\ \mathtt{using}\ (y)$,
$\mathbf{t} = \mathtt{postpone}\ (y :: \mathtt{x})\ \mathtt{until}\ (f)$ and
$\mathbf{u} = \mathtt{postpone}\ (f :: \mathtt{continue\ from}\ (\mathtt{x})\ \mathtt{using}\ (\mathtt{y}))\ \mathtt{until}\ (n)$.

In Fig. 2 we draw a part of the computation that **succ**(**zero**) = **one** and its dual.

We leave it as an exercise to the reader to write the formal proof corresponding to the drawing.

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§7. APPENDIX I. Completeness theorem for bimodal K and S4. We outline a *semantic tableaux* procedure for *bimodal* **K** and **S4**. Since these are classical systems, for simplicity we consider the fragment of the language $\mathcal{L}_{\Box,\boxminus}$ consisting of formulas in *negation normal form* given by the grammar

$$\alpha \ := \ p \mid \neg p \mid \top \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \ \boxminus\alpha \mid \ \diamondminus\alpha$$

Our calculi use *succedent only* sequents of the form $\Rightarrow \Gamma$ and are based on (a variant of) Gentzen-Kleene's sequent calculus **G3c** for classical propositional logic (cfr.[34], p. 77), where the rules of *weakening* and *contraction* are implicit. The axioms and the rules are given in Table 2. Given a notion of semantic validity, a rule of the sequent calculus $\dfrac{S_1, \dots , S_n}{S}$ *preserves validity* if for every instance of the rule, the sequent conclusion S is valid whenever the sequent-premises S_1, ..., S_n are all valid; a rule is *semantically invertible* if for every instance of the rule the sequent-premises are all valid whenever the sequent-conclusion is valid.

Proposition 1. *The propositional rules of the classical sequent calculus* **G3c** *preserve validity and are semantically invertible. The modal rules for the systems* bimodal **K** *and* **S4** *preserve validity and are semantically*

SEQUENT CALCULUS G3c FOR CLASSICAL LOGIC

axioms: $\Rightarrow \Delta, p, \neg p$

truth axioms: $\Rightarrow \Delta, \top$

right exchange:
$$\frac{\Rightarrow \Delta, \alpha, \beta, \Delta'}{\Rightarrow \ \Delta, \beta, \alpha, \Delta'}$$

right $\wedge$*:*
$$\frac{\Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Rightarrow \ \Delta, \alpha \wedge \beta}$$

right $\vee$*:*
$$\frac{\Rightarrow \Delta, \alpha, \beta}{\Rightarrow \ \Delta, \alpha \vee \beta}$$

EXTENSION TO MODAL SYSTEMS

weakenings

$$\frac{\Rightarrow \ \Box\alpha, \Diamond\Delta}{\Rightarrow \ \Box\alpha, \Diamond\Delta, \Box\Gamma, \boxminus\Gamma' \ \diamondminus\Delta', \Pi} \qquad \frac{\Rightarrow \boxminus\alpha, \diamondminus\Delta'}{\Rightarrow \boxminus\alpha, \diamondminus\Delta', \boxminus\Gamma', \Box\Gamma, \Diamond\Delta, \Pi}$$

where Π is a sequence of atoms and negations of atoms.

rules for bimodal K

K-$\Box$-*rule*:
$$\frac{\Rightarrow \alpha, \Delta}{\Rightarrow \ \Box\alpha, \Diamond\Delta}$$

K-$\boxminus$-*rule*:
$$\frac{\Rightarrow \alpha, \Delta}{\Rightarrow \boxminus\alpha, \diamondminus\Delta}$$

rules for bimodal S4

$\Diamond$ *right*:
$$\frac{\Rightarrow \Delta, \alpha, \Diamond\alpha}{\Rightarrow \ \Delta, \Diamond\alpha}$$

$\Box$ *right*:
$$\frac{\Rightarrow \ \alpha, \Diamond\Delta}{\Rightarrow \ \Box\alpha, \Diamond\Delta}$$

$\diamondminus$ *right*:
$$\frac{\Rightarrow \ \Delta, \diamondminus\alpha, \alpha}{\Rightarrow \ \Delta, \diamondminus\alpha}$$

$\boxminus$ *right*:
$$\frac{\Rightarrow \ \alpha, \diamondminus\Delta}{\Rightarrow \boxminus\alpha, \diamondminus\Delta}$$

TABLE 2. Sequent calculi for bimodal **K** and **S4**

invertible with respect to their semantics. The rules of weakening *preserve validity but are not semantically invertible.*

7.0.1. *Semantic Tableaux procedure for* **K**. The "semantic tableaux" procedure decides whether a sequent S is valid in the semantics for *bimodal* **K** by building a *refutation tree* labelled with sequents and with S at the root; if S is valid, then it return a derivation of S in the sequent calculus for *bimodal* **K**; if S not valid, it returns a counterexample $\mathcal{M}$ which refutes S.

Starting with sequent S at the root, the procedure builds the tree by inverting the propositional rules in some order on all branches, whenever possible. A propositional rule cannot be inverted on a leaf of the form

$$\Rightarrow \boxminus\, \alpha_1, \ldots, \boxminus\, \alpha_m, \diamondminus \Gamma, \Box\beta_1, \ldots, \Box\beta_n, \Diamond\Delta, \Pi \tag{$\dagger$}$$

where Π is a sequence of atoms and negations of atoms. Rewrite the sequent ($\dagger$) as a *hypersequent* as follows:

$$\Rightarrow [\Rightarrow \Pi] \ldots [\Rightarrow \boxminus\, \alpha_i, \diamondminus \Gamma] \ldots [\Rightarrow \Box\beta_j, \Diamond\Delta] \ldots \tag{$\ddagger$}$$

We call this step a *disjunctive ramification.* Now there are three cases:

(a) an atom $\top$ or a pair p_i, $\neg p_i$ occurs in Π: in this case the sequent ($\dagger$) is a *logical axiom* or a *truth axiom* and the procedure halts on this branch, which is *closed.*
(b) otherwise, if ($\dagger$) is not an axiom and $m = 0 = n$, then the procedure halts on this branch leaving it *open*;
(c) otherwise, ($\dagger$) is not an axiom and $m + n > 0$: in this case the procedures *branches* by inverting the $\boxminus$-R or $\Box$-R rules in the remaining $m + n$ sequents of the hypersequent.

We define inductively what it means for a refutation tree τ to be *closed*: a *logical axiom* or a *truth axiom* is closed; the conclusion of a *one-premise* [*two-premises*] inference rule is closed if and only the subtree[s] ending with the premise[s] is [are] closed; a *hypersequent* resulting from an $m+n$ *disjunctive ramification* branching with $\tau_1, \ldots, \tau_{m+n}$ subtrees is closed if and only if at least one τ_i is closed, for $i \leq m + n$.

Fact 1: *The semantic tableax procedure for* **K** *terminates.* Indeed at each inversion step the complexity of the sequents is reduced.

Fact 2: *If a refutation tree τ with conclusion S is closed, then we can obtain a derivation of S in the sequent calculus for* bimodal **K**. At each disjunctive ramification branching with subtrees $\tau_1, \ldots, \tau_{m+n}$, first we select a *closed* subtree τ_k and remove the others and the hypersequent notation; then to the endsequent of τ_k has the form we apply *weakening* to obtain the required sequent ($\dagger$).

Fact 3: *If a refutation tree τ with conclusion S is open, the we can construct a Kripke model $\mathcal{M}$ over a frame (W, R_1, R_2) which refutes S.* For every two-premises logical rule, if the sequent-conclusion is open, then we select one of the sequent-premises which is open. In this way we eventually obtain a tree τ' where all branches are open. Consider all fragments of branches $\beta_1, \ldots, \beta_z$ obtained from τ' by removing every modal inference:

(i) identify β_i with a possible world w_i;

(ii) put $w_i R_1 w_j$ if and only if the lowermost sequent of β_j is the sequent-premise of a **K**-$\Box$-*rule* and the sequent-conclusion of such a rule occurs in the hypersequent which is the uppermost sequent of β_i;

(ii) put $w_i R_2 w_j$ if and only if the lowermost sequent of β_j is the sequent-premise of a **K**-$\boxminus$-*rule* and the sequent-conclusion of such a rule occurs in the hypersequent which is the uppermost sequent of β_i;

(iv) let $w_i \Vdash p_i$ if and only if $\neg p_i$ occurs in the uppermost sequent of β_i.

From facts 1-3 we obtain the following theorem:

THEOREM 3. *The semantic tableaux procedure for* bimodal **K** *is sound and complete with respect to Kripke's semantics of bimodal frames* $\mathcal{F} = (W, R_1, R_2)$. *The system* bimodal **K** *has the finite model property.*

The procedure sketched here can be extended to *bimodal* **S4**, by standard techniques to guarantee termination: these require restricting the inversion of the rules $\diamondminus$-R and $\Diamond$-R, on one hand, and loop-detection, on the other hand.

Remark. The restriction on the rules $\diamondminus$-R and $\Diamond$-R, and also on $\boxminus$-L and $\Box$-L in two sided-sequents, can be obtained by marking the "contracted" active formula in the sequent-premise and by removing the mark whenever the corresponding symmetric rule is applied. For instance, in the case of the $\Box$-rules in two sided calculus we have:

$$\begin{array}{ccc} \Box \ \textit{left}: & \qquad & \Box \ \textit{right}: \\ \dfrac{\underline{\Box\alpha}, \alpha, \Gamma \ \Rightarrow \ \Delta,}{\Box\alpha, \Gamma \ \Rightarrow \ \Delta} & \qquad & \dfrac{\Box\Gamma \Rightarrow \alpha}{\underline{\Box\Gamma} \ \Rightarrow \ \Box\alpha} \end{array}$$

Thus the system *bimodal* **S4** is sound and complete with respect to Kripke's semantics over *preordered* bimodal frames $\mathcal{F}$.

Moreover it is an easy exercise to extend the above procedure to the whole bimodal language using *two-sided* sequent calculi of type **G3c** for **K** and **S4**.

7.1. Proof of Theorem 1. *The intuitionistic sequent calculus* **PBL-G3** *without the rules of cut are sound and complete with respect to the interpretation in* bimodal **S4**.

Given a sequent $S = \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon$, we construct a "refutation tree" τ for the sequent S^M according to the semantics tableaux procedure for *bimodal* **S4** in the formulation using *two-sided* sequents. If the refutation tree τ is not closed, then from an open tree τ' obtained by pruning τ we obtain a countermodel for S^M over a bimodal frame $(\mathcal{F}, R, R^{-1})$ which may be regarded as a countermodel for S itself. If the refutation tree τ is closed, then by the completeness for *bimodal* **S4** we have a derivation d of S^M in the sequent calculus for bimodal **S4** and we need to produce a derivation of S in **PBL-G3**.

Given a derivation d of S^M in bimodal **S4**, the only difficulty to transform it into a derivation of S in **PBL-G3** lies in the rules $\supset$-L [and $\setminus$-R][16]. Indeed to a $\setminus$-*right* rule

$$\frac{\Theta \; ; \; \epsilon \;\Rightarrow\; ; \; \Upsilon, \upsilon_1 \qquad \Theta \; ; \; \upsilon_2 \;\Rightarrow\; ; \; \Upsilon, \upsilon_1 \setminus \upsilon_2}{\Theta \; ; \; \epsilon \;\Rightarrow\; ; \; \Upsilon, \upsilon_1 \setminus \upsilon_2}$$

there corresponds in d a triple of inferences, i.e., a $\neg$-*right*, a $\wedge$-*right* and a $\diamond\!\!\!-$-*right* application:

$$\dfrac{\dfrac{\Theta^M, \epsilon^M \;\Rightarrow\; \Upsilon^M, \upsilon_1^M, \underline{(\upsilon_1 \setminus \upsilon_2)}^M \qquad \dfrac{\Theta^M, \epsilon^{M\,*}, \upsilon_2^{M\,*} \;\Rightarrow\; \Upsilon^M, (\upsilon_1 \setminus \upsilon_2)^M}{\Theta^M, \epsilon^M, \Rightarrow\; \Upsilon^M, (\upsilon_1 \setminus \upsilon_2)^M, \neg\upsilon_2^M}\,\neg\text{-R}}{\Theta^M, \epsilon^M \;\Rightarrow\; \Upsilon^M, (\upsilon_1 \setminus \upsilon_2)^M, \upsilon_1^M \wedge \neg\upsilon_2^M}\,\wedge\text{-R}}{\Theta^M, \epsilon^M \;\Rightarrow\; \Upsilon^M, \diamond\!\!\!-(\upsilon_1^M \wedge \neg\upsilon_2^M)}\,\diamond\!\!\!-\text{R}$$

It is easy to see that the underlined formula $(\upsilon_1 \setminus \upsilon_2)^M$ (and its ancestors) can be removed preserving provability. But in order for the sequent-premise of the $\neg$-R inference to be the translation of a **PBL-G3**-sequent, one of the two starred formulas $\epsilon^{M\,*}$, $\upsilon_2^{M\,*}$ has to be removed, and it is not obvious how to do it.

Definition 13. Let us call the inferences $\Box$-R, $\Diamond$-L $\boxminus$-R and $\diamond\!\!\!-$-L *promotions* and *promoted formula* their principal formula. We may assume that only $\Box$-R or $\diamond\!\!\!-$-L inferences occur in a *cut-free* **PBL-G3** derivation d of the sequent S^M.

Let $\overline{\tau}$ be a fragment (a "truncated subtree") of the derivation tree d such that

- the root of $\overline{\tau}$ is the conclusion of d or the sequent-premise of a promotion inference;
- the leaves of $\overline{\tau}$ are either axioms or conclusions of a promotion inference in d;
- no promotion inference occurs in $\overline{\tau}$.

We say that a formula-occurrence A in a sequent of $\overline{\tau}$ is *traceable to axioms or promotion* if an ancestor of A in a leaf of $\overline{\tau}$ is a principal formula of an axiom or was a promoted formula in d.

Remark. Three obvious remarks are essential here.
(i) If a starred formula $\epsilon^{M\,*}$ or $\upsilon_2^{M\,*}$ is not traceable to an axiom or promotion, then it can be removed from the proof-tree d together with all its ancestors: indeed, all its ancestors in the leaves of $\overline{\tau}$ are formulas introduced by weakening.

[16]The difficulty is related to the well-known fact $\supset$-L is invertible only in the right subderivation but not in the right one; similarly, $\setminus$-R is invertible in the left subderivation but not in the right one.

(ii) If the formula $\upsilon_2^{M\,*}$ is removed, then the right-uppermost sequent in the figure coincides with the lowemost one, so all the inferences in the figure can be removed, together with the left branch.
(iii) An inference $\mathcal{I}$ of classical propositional logic or $\diamond\!\!-$R or $\Box$-L can always be permuted above any inference whose principal formula is not active in $\mathcal{I}$.

We show how to transform the fragment $\overline{\tau}$ of the given derivation d into another where only one occurrence of the two the starred formulas $\epsilon^{M\,*}$ and $\upsilon_2^{M\,*}$ occurs in any sequent. This actually proves the theorem, because by iterating the given procedure we eventually obtain a derivation d^+ of S^M where in every sequent there is exactly one occurrence of either a formula $\diamond A$ in the antecedent or a formula $\Box A$ in the consequent.

Suppose $\epsilon^{M\,*}$ is traceable to an axiom or to a promotion: then on each branch leading to such an axiom or a modal inference we permute the indicated block of three inferences $\neg$-R, $\wedge$-R and $\diamond\!\!-$R upwards; this is possible by (iii). Eventually, on each such branch β the resulting subderivation will have sequents where $\upsilon_2^{M\,*}$ is no longer traceable to an axiom or promotion, hence it can be removed together with the block of three inferences located in β, according to (ii).

We may therefore assume that $\overline{\tau}$ has been transformed into a fragment $\overline{\tau'}$ where $\epsilon^{M\,*}$ is no longer traceable to an axiom or promotion; then $\epsilon^{M\,*}$ can be removed together with its ancestors, in accordance with (i).

In conclusion, a block of three inferences indicated in the figure where ϵ^M no longer occurs in the right branch and the underlined formula $(\underline{\upsilon_1 \setminus \upsilon_2})^M$ does not occur in the left branch is the exact counterpart in *bimodal* **S4** of a $\setminus$-L inference in **PBL**, as required.

§8. APPENDIX II: Rules for the Sequent Calculus PBL and the Natural Deduction systems INP$_{\mathbf{AC}}$ and PBN.

In Tables 3 we give the rules of the Sequent Calculus **PBL**. In Tables 4 to 7 we give the Rules of Inference, Deduction, Reduction and Commutation of the Natural Deduction System **INP$_{\mathbf{AC}}$**. In Tables 8 to 11 we give the rules that are specific of the system **PBN**.

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Sequent Calculus PBL-G3: axioms and rules

identity rules

logical axiom:
$$\vartheta, \Theta \ ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon$$

logical axiom:
$$\Theta \ ; \ \upsilon \ \Rightarrow \ ; \ \Upsilon, \upsilon$$

cut_1:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta \ ; \ \Upsilon \qquad \vartheta, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon'}{\Theta, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \Upsilon'}$$

cut_2:
$$\frac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon \qquad \Theta' \ ; \ \upsilon \ \Rightarrow \ \Upsilon'}{\Theta, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \Upsilon'}$$

structural rules

exchange:
$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

exchange:
$$\frac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

ASSERTIVE LOGICAL RULES

validity axiom:
$$\Theta \ ; \ \Rightarrow \ \bigvee \ ; \ \Upsilon$$

(¶) *right $\supset$:*
$$\frac{\Theta, \vartheta_1 \ ; \ \Rightarrow \ \vartheta_2 \ ;}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \supset \vartheta_2 \ ; \ \Upsilon}$$

left $\supset$:
$$\frac{\vartheta_1 \supset \vartheta_2, \Theta ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

right $\cap$:
$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \Theta \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \cap \vartheta_2 \ ; \ \Upsilon}$$

left $\cap$:
$$\frac{\vartheta_0, \vartheta_1, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vartheta_0 \cap \vartheta_1, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

CONJECTURAL RULES

absurdity axiom:
$$; \ \bigwedge \ \Rightarrow \ ; \ \Upsilon$$

right $\smallsetminus$:
$$\frac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_1 \qquad \Theta \ ; \ \upsilon_2 \ \Rightarrow \ ; \ \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

(¶) *left $\smallsetminus$:*
$$\frac{; \ \upsilon_1 \ \Rightarrow \ ; \ \Upsilon, \upsilon_2}{\Theta \ ; \ \upsilon_1 \smallsetminus \upsilon_2 \ \Rightarrow \ ; \ \Upsilon}$$

right $\curlyvee$:
$$\frac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_0, \upsilon_1}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

left $\curlyvee$:
$$\frac{\Theta \ ; \ \upsilon_1 \ \Rightarrow \ ; \ \Upsilon \qquad \Theta \ ; \ \upsilon_2 \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \upsilon_1 \curlyvee \upsilon_2 \ \Rightarrow \ ; \ \Upsilon}$$

DUALITIES (MIXED-TYPE NEGATIONS):

(¶) *right* $\sim$:
$$\frac{\Theta \ ; \ \upsilon \ \Rightarrow \ ;}{\Theta \ ; \ \Rightarrow \sim \upsilon \ ; \ \Upsilon}$$

left $\sim$:
$$\frac{\sim \upsilon, \Theta ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \upsilon}{\sim \upsilon, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

right $\frown$:
$$\frac{\Theta, \vartheta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \frown \vartheta}{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \frown \vartheta}$$

(¶)*left* $\frown$:
$$\frac{; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\Theta \ ; \ \frown \vartheta \ \Rightarrow \ ; \ \Upsilon}$$

TABLE 3. The sequent calculus **PBL-G3**

Natural Deduction $\mathbf{INP_{AC}}$ - Rules of Inference

ASSERTIVE RULES

$$\begin{array}{c}\vdots \qquad \vdots\\ \dfrac{\vartheta_1 \;\dots\; \vartheta_m}{\upsilon_1 \;\dots \upsilon_n \;\bigvee}\ \bigvee\text{-I}\\ \vdots \qquad \vdots \quad \vdots\end{array} \qquad \begin{array}{c}[\vartheta_1]\\ \vdots\\ \dfrac{\vartheta_2}{\vartheta_1 \supset \vartheta_2}\ \supset\text{-I}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots \qquad \vdots\\ \dfrac{\vartheta_1 \supset \vartheta_2 \qquad \vartheta_1}{\vartheta_2}\ \supset\text{-E}\\ \vdots\end{array}$$

$$\begin{array}{c}\vdots \qquad \vdots\\ \dfrac{\vartheta_0 \qquad \vartheta_1}{\vartheta_0 \cap \vartheta_1}\ \cap\text{-I}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\vartheta_0 \cap \vartheta_1}{\vartheta_0}\ \cap_0\text{-E}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\vartheta_0 \cap \vartheta_1}{\vartheta_1}\ \cap_1\text{-E}\\ \vdots\end{array}$$

CONJECTURAL RULES

$$\begin{array}{c}\vdots \quad \vdots \qquad \vdots\\ \dfrac{\bigwedge \;\vartheta_1 \;\dots\; \vartheta_m}{\upsilon_1 \;\dots\; \upsilon_n}\ \bigwedge\text{-E}\\ \vdots \qquad \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\upsilon_1}{\upsilon_2 \qquad \upsilon_1 \smallsetminus \upsilon_2}\ \smallsetminus\text{-I}\\ \vdots \qquad \vdots\end{array} \qquad \begin{array}{c}\qquad\qquad [\upsilon_1]\\ \vdots \qquad\qquad \vdots\\ \dfrac{\upsilon_1 \smallsetminus \upsilon_2 \qquad (\upsilon_2)}{}\ \smallsetminus\text{-E}\end{array}$$

$$\begin{array}{c}\vdots\\ \dfrac{\upsilon_0}{\upsilon_0 \curlyvee \upsilon_1}\ \curlyvee_0\text{-I}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\upsilon_1}{\upsilon_0 \curlyvee \upsilon_1}\ \curlyvee_1\text{-I}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\upsilon_0 \curlyvee \upsilon_1}{\upsilon_0 \qquad \upsilon_1}\ \curlyvee\text{-E}\\ \vdots \qquad \vdots\end{array}$$

NEGATIONS

$$\begin{array}{c}[\upsilon]\\ \vdots\\ \dfrac{(\bigwedge)}{\sim \upsilon}\ \sim\text{-I}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots \qquad \vdots\\ \dfrac{\sim \upsilon \qquad \upsilon}{\bigwedge}\ \sim\text{-E}\\ \vdots\end{array}$$

$$\begin{array}{c}\vdots\\ \dfrac{\bigvee}{\vartheta \qquad \frown \vartheta}\ \frown\text{-I}\\ \vdots \qquad \vdots\end{array} \qquad \begin{array}{c}\qquad [\bigvee]\\ \vdots \qquad \vdots\\ \dfrac{\frown \vartheta \qquad \vartheta}{}\ \frown\text{-E}\end{array}$$

TABLE 4. Natural Deduction $\mathbf{INP_{AC}}$ - Rules of Inference

Natural Deduction $\mathbf{INP_{AC}}$: Rules of Deduction

STRUCTURAL RULES

exchange:
$$\frac{\Theta, \vartheta, \vartheta', \Theta' \vdash \Upsilon}{\Theta, \vartheta', \vartheta, \Theta' \vdash \Upsilon}$$

contraction:
$$\frac{\vartheta, \vartheta, \Theta \vdash \Upsilon}{\vartheta, \Theta \vdash \Upsilon}$$

weakening:
$$\frac{\Theta \vdash \Upsilon}{\vartheta, \Theta \vdash \Upsilon}$$

exchange:
$$\frac{\Theta \vdash \Upsilon, \upsilon, \upsilon', \Upsilon'}{\Theta \vdash \Upsilon, \upsilon', \upsilon, \Upsilon'}$$

contraction:
$$\frac{\Theta \vdash \Upsilon, \upsilon, \upsilon}{\Theta \vdash \Upsilon, \upsilon}$$

weakening:
$$\frac{\Theta \vdash \Upsilon}{\Theta \vdash \Upsilon, \upsilon}$$

ASSERTIVE RULES

assumption:
$$\vartheta \vdash \vartheta$$

substitution:
$$\frac{\Theta \vdash \vartheta, \Upsilon \qquad \vartheta, \Theta', \epsilon \vdash \epsilon', \Upsilon'}{\Theta, \Theta' \vdash \Upsilon, \Upsilon'}$$

validity axiom:
$$\frac{}{\Theta \vdash \bigvee, \Upsilon}$$

$\supset$*-I:*
$$\frac{\Theta, \vartheta_1 \vdash \vartheta_2, \Upsilon}{\Theta \vdash \vartheta_1 \supset \vartheta_2, \Upsilon}$$

$\supset$*-E:*
$$\frac{\Theta_1 \vdash \vartheta_1 \supset \vartheta_2, \Upsilon_1 \qquad \Theta_2 \vdash \vartheta_1, \Upsilon_2}{\Theta_1, \Theta_2 \vdash \vartheta_2, \Upsilon_1, \Upsilon_2}$$

$\cap$*-I:*
$$\frac{\Theta \vdash \vartheta_1, \Upsilon \qquad \Theta \vdash \vartheta_2, \Upsilon}{\Theta \vdash \vartheta_1 \cap \vartheta_2, \Upsilon}$$

$\cap_0$*-E:*
$$\frac{\Theta \vdash \vartheta_0 \cap \vartheta_1, \Upsilon}{\Theta \vdash \vartheta_0, \Upsilon}$$

$\cap_1$*-E:*
$$\frac{\Theta \vdash \vartheta_0 \cap \vartheta_1, \Upsilon}{\Theta \vdash \vartheta_1, \Upsilon}$$

CONJECTURAL RULES

assumption:
$$\upsilon \vdash \upsilon$$

substitution:
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon \qquad \Theta', \upsilon \vdash \Upsilon'}{\Theta, \Theta' \vdash \Upsilon, \Upsilon'}$$

absurdity axiom:
$$\frac{}{\Theta, \bigwedge \vdash \Upsilon}$$

$\smallsetminus$*-I:*
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_1}{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_2, \upsilon_1 \smallsetminus \upsilon_2}$$

$\smallsetminus$*-E:*
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_1 \smallsetminus \upsilon_2 \qquad \Theta', \upsilon_1 \vdash \upsilon_2, \Upsilon'}{\Theta, \Theta', \epsilon \vdash \epsilon', \Upsilon, \Upsilon'_1}$$

$\curlyvee_0$*-I:*
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_0}{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

$\curlyvee_1$*-I:*
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_1}{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_0 \curlyvee \upsilon_1}$$

$\curlyvee$*-E:*
$$\frac{\Theta', \epsilon \vdash \epsilon', \Upsilon', \upsilon_0 \curlyvee \upsilon_1 \qquad \Theta, \upsilon_0 \vdash \Upsilon \qquad \Theta, \upsilon_1 \vdash \Upsilon}{\Theta', \Theta, \epsilon \vdash \epsilon', \Upsilon, \Upsilon'}$$

NEGATION RULES

$\sim$-I:
$$\frac{\Theta, \upsilon \vdash \bigwedge, \Upsilon}{\Theta \vdash \sim \upsilon, \Upsilon}$$

$\sim$*-E*
$$\frac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon}{\sim \upsilon, \Theta, \epsilon \vdash \epsilon', \Upsilon}$$

$\frown$*-I:*
$$\frac{\vartheta, \Theta, \epsilon \vdash \epsilon', \Upsilon}{\Theta, \epsilon \vdash \epsilon', \Upsilon, \frown \vartheta}$$

$\frown$*-E:*
$$\frac{\Theta, \bigvee \vdash \vartheta, \Upsilon}{\Theta, \frown \vartheta \vdash \Upsilon}$$

where the assumption ϵ is conjectural, the conclusion ϵ' is assertive and exacty one among ϵ and ϵ' is in the consequence relation.

TABLE 5. Natural Deduction $\mathbf{INP_{AC}}$ - Rules of Deduction

Natural Deduction $\mathbf{INP_{AC}}$ - Reduction Rules

$\cap$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ \vartheta_0 & & \vartheta_1\end{matrix}}{\vartheta_0 \cap \vartheta_1}\,\cap\text{-I}}{\begin{matrix}\vartheta_i\\ \vdots\end{matrix}}\,\cap_i\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \vartheta_i\\ \vdots\end{matrix}$$

$\supset$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}[\vartheta_1]\\ \vdots\\ \vartheta_2\end{matrix}}{\vartheta_1 \supset \vartheta_2}\,\supset\text{-I} \qquad \begin{matrix}\vdots\\ \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_2\\ \vdots\end{matrix}}\,\supset\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ [\vartheta_1]\\ \vdots\\ \vartheta_2\\ \vdots\end{matrix}$$

$\curlyvee$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \upsilon_i\end{matrix}}{\upsilon_0 \curlyvee \upsilon_1}\,\curlyvee_i\text{-I}}{\begin{matrix}\upsilon_0 & & \upsilon_1\\ \vdots & & \vdots\end{matrix}}\,\curlyvee\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \upsilon_i\\ \vdots\end{matrix}$$

$\setminus$-REDUCTION

$$\setminus\text{-I}\,\dfrac{\begin{matrix}\vdots\\ \upsilon_1\end{matrix}}{\begin{matrix}\upsilon_2 & & \upsilon_1 \setminus \upsilon_2\\ \vdots & & \end{matrix}} \qquad \begin{matrix}[\upsilon_1]\\ \vdots\\ (\upsilon_2) & \vdots\end{matrix} \qquad \underline{\qquad\qquad\qquad}\,\setminus\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ \upsilon_1\\ \vdots\\ (\upsilon_2) & \vdots\\ \vdots\end{matrix}$$

$\sim$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}[\upsilon]\\ \vdots\\ (\bigwedge)\end{matrix}}{\sim \upsilon}\,\supset\text{-I} \qquad \begin{matrix}\vdots\\ \upsilon\end{matrix}}{\begin{matrix}\bigwedge\\ \vdots\end{matrix}}\,\supset\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ [\upsilon]\\ \vdots\\ (\bigwedge)\\ \vdots\end{matrix}$$

$\frown$-REDUCTION

$$\setminus\text{-I}\,\dfrac{\begin{matrix}\vdots\\ \bigvee\end{matrix}}{\begin{matrix}\vartheta & & \frown \vartheta\\ \vdots & & \end{matrix}} \qquad \begin{matrix}[\bigvee]\\ \vdots\\ \vartheta & \vdots\end{matrix} \qquad \underline{\qquad\qquad\qquad}\,\setminus\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots\\ [\bigvee]\\ \vdots\\ \vartheta & \vdots\\ \vdots\end{matrix}$$

TABLE 6. Natural Deduction $\mathbf{INP_{AC}}$ - Reduction Rules

Natural Deduction $\mathbf{INP_{AC}}$ - Commutation rules

$\bigvee$-COMMUTATIONS

$$\dfrac{\begin{array}{c}\vdots\\ \vartheta_1\end{array}\ \dots\ \dfrac{\begin{array}{cc}\vdots & \vdots\\ \vartheta'_1\ \dots & \vartheta'_k\end{array}}{\vartheta_i}\ \dots\ \begin{array}{c}\vdots\\ \vartheta_n\end{array}}{\bigvee\quad \upsilon'_1\dots\upsilon'_m}\ \bigvee\text{-I} \atop \vdots\quad\vdots\quad\vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{ccccc}\vdots & & \vdots\ \ \vdots & & \vdots\\ \vartheta_1\ \dots & & \vartheta'_1\ \dots\ \vartheta'_k & & \dots\ \vartheta_n\end{array}}{\bigvee\quad \upsilon'_1\dots\upsilon'_m}\ \bigvee\text{-I} \atop \vdots\quad\vdots\quad\vdots$$

and

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \vartheta'_1\ \dots & \vartheta'_n\end{array}}{\bigvee\quad \upsilon_1\ \dots\ \upsilon_i\ \dots\ \upsilon_m}\ \bigvee\text{-I} \atop \vdots\quad\vdots\quad \overline{\upsilon'_1\dots\upsilon'_k}\quad\vdots \atop \vdots\quad\vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \vartheta'_1\ \dots & \vartheta'_n\end{array}}{\bigvee\quad \upsilon_1\ \dots\ \upsilon'_1\dots\upsilon'_k\ \dots\ \upsilon_m}\ \bigvee\text{-I} \atop \vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots$$

$\bigwedge$-COMMUTATIONS

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \bigwedge & \vartheta_1\end{array}\ \dots\ \dfrac{\begin{array}{cc}\vdots & \vdots\\ \vartheta'_1\ \dots & \vartheta'_k\end{array}}{\vartheta_i}\ \dots\ \begin{array}{c}\vdots\\ \vartheta_n\end{array}}{\upsilon'_1\dots\upsilon'_m}\ \bigwedge\text{-E} \atop \vdots\quad\vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{cccccc}\vdots & \vdots & & \vdots\ \ \vdots & & \vdots\\ \bigwedge & \vartheta_1\ \dots & & \vartheta'_1\ \dots\ \vartheta'_k & & \dots\ \vartheta_n\end{array}}{\upsilon'_1\ \dots\ \upsilon'_m}\ \bigwedge\text{-E} \atop \vdots\quad\vdots$$

and

$$\dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ \bigwedge & \vartheta'_1\dots & \vartheta'_n\end{array}}{\upsilon_1\dots\ \ \upsilon_i\ \ \dots\upsilon_m}\ \bigwedge\text{-E} \atop \vdots\quad \overline{\upsilon'_1\dots\upsilon'_k}\quad\vdots \atop \vdots\quad\vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{ccc}\vdots & \vdots & \vdots\\ \bigwedge & \vartheta'_1\dots & \vartheta'_n\end{array}}{\upsilon_1\ \dots\ \upsilon'_1\dots\upsilon'_k\ \dots\ \upsilon_m}\ \bigwedge\text{-E} \atop \vdots\quad\vdots\quad\vdots\quad\vdots$$

$\bigwedge$ / $\sim$-COMMUTATION

$$\dfrac{\begin{array}{c}\vdots\\ \sim\upsilon\end{array}\qquad \dfrac{\begin{array}{c}\vdots\\ \bigwedge\end{array}}{\upsilon}\ \bigwedge\text{-E}}{\bigwedge}\ \sim\text{-E} \atop \vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \bigwedge & \sim\upsilon\end{array}}{\bigwedge}\ \bigwedge\text{-E} \atop \vdots$$

$\frown$ / $\bigvee$-COMMUTATION

$$\frown\text{-I}\ \dfrac{\begin{array}{c}\vdots\\ \bigvee\end{array}}{\bigvee\text{-I}\ \dfrac{\vartheta}{\bigvee}\qquad \frown\vartheta} \atop \vdots\qquad\vdots \qquad \text{commutes to} \qquad \dfrac{\begin{array}{c}\vdots\\ \bigvee\end{array}}{\bigvee\quad \frown\vartheta}\ \bigvee\text{-I} \atop \vdots\quad\vdots$$

TABLE 7. $\mathbf{INP_{AC}}$ - Commutation Rules

Natural Deduction PBN - Restricted Rules of Inference

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vartheta'_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ \vartheta'_m\end{array}\qquad \begin{array}{c}[\vartheta_1]\ [\vartheta'_1]\ \ldots\ [\vartheta'_m]\\ \vdots\\ \vartheta_2\end{array}}{\vartheta_1 \supset \vartheta_2}\supset\text{-I}\\ \vdots\end{array}
\qquad
\dfrac{\begin{array}{c}\vdots\\ \upsilon_1 \diagdown \upsilon_2\end{array}\qquad \begin{array}{c}[\upsilon_1]\\ \vdots\\ (\upsilon_2)\end{array}\quad (\upsilon'_1)\ \ldots\ (\upsilon'_m)}{\begin{array}{ccc}\upsilon'_1 & \ldots & \upsilon'_m\end{array}}\diagdown\text{-E}
\quad\begin{array}{ccc}\vdots & & \vdots\end{array}$$

NEGATIONS

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \vartheta'_1\end{array}\ \ldots\ \begin{array}{c}\vdots\\ \vartheta'_m\end{array}\qquad \begin{array}{c}[\upsilon]\ [\vartheta'_1]\ \ldots\ [\vartheta'_m]\\ \vdots\\ (\bigwedge)\end{array}}{\sim \upsilon}\sim\text{-I}\\ \vdots\end{array}
\qquad
\dfrac{\begin{array}{c}\vdots\\ \frown \vartheta\end{array}\qquad \begin{array}{c}[\bigvee]\\ \vdots\\ \vartheta\end{array}\ (\upsilon'_1)\ \ldots\ (\upsilon'_m)}{\begin{array}{ccc}\upsilon'_1 & \ldots & \upsilon'_m\end{array}}\frown\text{-E}
\quad\begin{array}{ccc}\vdots & & \vdots\end{array}$$

TABLE 8. **PBN** - Restricted Rules of Inference

Natural Deduction PBN: Restricted Rules of Deduction

$\supset$-*I:*

$$\dfrac{\Theta_1 \vdash \vartheta'_1, \Upsilon_1 \quad \ldots \quad \Theta_n \vdash \vartheta'_n, \Upsilon_n \qquad \vartheta'_1 \ldots \vartheta'_n, \vartheta_1 \vdash \vartheta_2}{\Theta_1, \ldots, \Theta_n \vdash \vartheta_1 \supset \vartheta_2, \Upsilon_1, \ldots, \Upsilon_n}$$

$\diagdown$-*E:*

$$\dfrac{\Theta, \epsilon \vdash \epsilon', \Upsilon, \upsilon_1 \diagdown \upsilon_2 \qquad \upsilon_1 \vdash \upsilon'_1, \ldots, \upsilon'_n, \upsilon_2 \qquad \Theta_1, \upsilon'_1 \vdash \Upsilon_1 \quad \ldots \quad \Theta_n, \upsilon'_n \vdash \Upsilon_n}{\Theta, \Theta_1, \ldots, \Theta_n, \epsilon \vdash \epsilon', \Upsilon, \Upsilon_1, \ldots, \Upsilon_n}$$

$\sim$-I:

$$\dfrac{\Theta_1 \vdash \vartheta'_1, \Upsilon_1 \quad \ldots \quad \Theta_n \vdash \vartheta'_n, \Upsilon_n \qquad \vartheta'_1, \ldots, \vartheta'_n, \upsilon \vdash \bigwedge}{\Theta_1, \ldots, \Theta_n \vdash \sim \upsilon, \Upsilon_1, \ldots, \Upsilon_n}$$

$\frown$-*E:*

$$\dfrac{\bigvee \vdash \vartheta, \upsilon'_1, \ldots, \upsilon'_n \qquad \Theta_1, \upsilon'_1 \vdash \Upsilon_1 \quad \ldots \quad \Theta_n, \upsilon'_n \vdash \Upsilon_n}{\Theta_1, \ldots, \Theta_n, \frown \vartheta \vdash \Upsilon_1, \ldots, \Upsilon_n}$$

TABLE 9. **PBN** - Restricted Rules of Deduction

Natural Deduction PBN - Restricted Reduction Rules

$\supset$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ \vartheta'_1 & \dots & \vartheta'_m\end{matrix} \qquad \begin{matrix}[\vartheta_1]\ [\vartheta'_1]\ \dots\ [\vartheta'_m]\\ \vdots \\ \vartheta_2\end{matrix}}{\vartheta_1 \supset \vartheta_2}\supset\text{-I} \qquad \begin{matrix}\vdots\\ \vartheta_1\end{matrix}}{\begin{matrix}\vartheta_2\\ \vdots\end{matrix}}\supset\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots & \vdots & & \vdots\\ [\vartheta_1] & [\vartheta'_1] & \dots & [\vartheta'_m]\\ & \vdots & & \\ & \vartheta_2 & & \\ & \vdots & & \end{matrix}$$

$\setminus$-REDUCTION

$$\dfrac{\setminus\text{-I}\ \dfrac{\begin{matrix}\vdots\\ \upsilon_1\end{matrix}}{\begin{matrix}\upsilon_2 \\ \vdots\end{matrix} \qquad \upsilon_1 \setminus \upsilon_2} \qquad \begin{matrix} & [\upsilon_1] & & \\ & \vdots & & \\ (\upsilon_2) & (\upsilon'_1) & \dots & (\upsilon'_m)\end{matrix}}{\begin{matrix}\upsilon'_1 & \dots & \upsilon'_m\\ \vdots & & \vdots\end{matrix}}\setminus\text{-E} \qquad \text{reduces to} \qquad \begin{matrix} & \vdots & & \\ & \upsilon_1 & & \\ & \vdots & & \\ (\upsilon_2) & (\upsilon'_1) & \dots & (\upsilon'_m)\\ \vdots & \vdots & & \vdots\end{matrix}$$

$\sim$-REDUCTION

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ \vartheta'_1 & \dots & \vartheta'_m\end{matrix} \qquad \begin{matrix}[\upsilon]\ [\vartheta'_1]\ \dots\ [\vartheta'_m]\\ \vdots \\ (\bigwedge)\end{matrix}}{\sim \upsilon}\sim\text{-I} \qquad \begin{matrix}\vdots\\ \upsilon\end{matrix}}{\begin{matrix}\bigwedge\\ \vdots\end{matrix}}\sim\text{-E} \qquad \text{reduces to} \qquad \begin{matrix}\vdots & \vdots & & \vdots\\ [\vartheta_1] & [\vartheta'_1] & \dots & [\vartheta'_m]\\ & \vdots & & \\ & (\bigwedge) & & \\ & \vdots & & \end{matrix}$$

$\frown$-REDUCTION

$$\dfrac{\frown\text{-I}\ \dfrac{\begin{matrix}\vdots\\ \bigvee\end{matrix}}{\begin{matrix}\vartheta \\ \vdots\end{matrix} \qquad \frown \vartheta} \qquad \begin{matrix} & [\bigvee] & & \\ & \vdots & & \\ \vartheta & (\upsilon'_1) & \dots & (\upsilon'_m)\end{matrix}}{\begin{matrix}\upsilon'_1 & \dots & \upsilon'_m\\ \vdots & & \vdots\end{matrix}}\setminus\text{-E} \qquad \text{reduces to} \qquad \begin{matrix} & \vdots & & \\ & (\bigvee) & & \\ & \vdots & & \\ \vartheta & (\upsilon'_1) & \dots & (\upsilon'_m)\\ \vdots & \vdots & & \vdots\end{matrix}$$

TABLE 10. **PBN** - Restricted Reduction Rules

Natural Deduction PBN - $\supset$- and $\smallsetminus$-Commutations

$$\dfrac{\begin{array}{c}\vdots\\ \vartheta'_1\ldots\end{array}\quad \dfrac{\begin{array}{c}\vdots\quad\vdots\\ \vartheta''_1\ldots\vartheta''_k\end{array}}{\vartheta'_i}\quad \begin{array}{c}\vdots\\ \ldots\vartheta'_m\end{array}\quad \begin{array}{c}[\vartheta_1]\ [\vartheta'_1]\ldots[\vartheta_i]\ldots[\vartheta'_m]\\ \vdots\\ \vartheta_2\end{array}}{\vartheta_1\supset\vartheta_2}\supset\text{-I}$$
$$\vdots$$

commutes to

$$\dfrac{\begin{array}{c}\vdots\quad\vdots\quad\vdots\quad\vdots\\ \vartheta'_1\ldots\vartheta''_1\ldots\vartheta''_k\ldots\vartheta'_m\end{array}\quad \begin{array}{c}[\vartheta_1]\ [\vartheta'_1]\quad \dfrac{[\vartheta''_1]\ldots[\vartheta''_k]}{\ldots\ \vartheta'_i\ \ldots}\quad [\vartheta'_m]\\ \vdots\\ \vartheta_2\end{array}}{\vartheta_1\supset\vartheta_2}\supset\text{-I}$$
$$\vdots$$

and

$$\dfrac{\begin{array}{c}\vdots\\ \upsilon_1\smallsetminus\upsilon_2\end{array}\quad \begin{array}{c}[\upsilon_1]\\ \vdots\\ (\upsilon_2)\quad(\upsilon'_1)\ \ldots\ (\upsilon'_m)\end{array}}{\upsilon'_1\ \ldots\ \upsilon'_i\ \ldots\ \upsilon'_m}\smallsetminus\text{-E}$$
$$\vdots\qquad\overline{\upsilon''_1\ldots\upsilon''_k}\qquad\vdots$$
$$\vdots\quad\vdots$$

commutes to

$$\dfrac{\begin{array}{c}\vdots\\ \upsilon_1\smallsetminus\upsilon_2\end{array}\quad (\upsilon_2)\quad(\upsilon'_1)\ \ldots\begin{array}{c}[\upsilon_1]\\ \vdots\\ (\upsilon'_i)\\ \overline{\upsilon''_1\ldots\upsilon''_k}\end{array}\ldots(\upsilon'_m)}{\upsilon'_1\ldots\upsilon''_1\ldots\upsilon''_k\ldots\upsilon'_m}\smallsetminus\text{-E}$$
$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

TABLE 11. **PBN** - Commutation Rules