Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic

Gianluigi Bellin   Valeria de Paiva   Eike Ritter

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1 Motivation

The benefits of the extended Curry-Howard Correspondence relating the simply typed lambda-calculus to proofs of intuitionistic propositional logic and to appropriate classes of categories are widely known. In this note we pursue an analogous correspondence for a basic intuitionistic (or constructive) modal logic **IK** with both the necessity ($\Box$) and the possibility ($\Diamond$) operator. Similar work for intuitionistic S4 (**IS4**) has appeared in [BdP00, AMPR98, GdPR98].

This is a natural logic to consider, if one is determined to push the frontiers of the Curry-Howard Isomorphism, as far as they will go. Modal system **K** is less symmetric than S4 as far as the modality rules are concerned. Modal S4 has introduction and elimination rules for both necessity and possibility, whereas modal **K** has only one rule for necessity and one rule for possibility. But modal **K** (to be precise, a multimodal modal **K**) is also a solution to a question first posed by McCarthy [McC93] and expounded on by Buvac, Buvac and Mason in [BBM95], when discussing future work:

> We also plan to define non-Hilbert style formal systems for context. Probably the most relevant is a natural deduction system, which would be in line with McCarthy's original proposal of treating contextual reasoning as a strong version of natural deduction. In such a system, entering a context would correspond to making an assumption in natural deduction, while exiting a context corresponds to discharging an assumption.

The application of $\mathbf{IK}_\Box$ (the fragment of **IK** with only the necessity operator) to "logics of context" in AI will be discussed in a companion paper. This note describes the syntax (in some equivalent versions, see below) and semantics of the simple, unimodal system.

Our goal is a propositional system that, like classical or intuitionistic logic, can be presented either as an axiomatic system, or a sequent calculus or a Natural Deduction system and such that these different presentations are proved equivalent; that is all presentations prove the same theorems.

The sequent calculus and the axiomatic presentations are relatively well-known and references are given in the next section. The natural deduction version of this basic modal logic is more controversial: we present one version and discuss two other (equivalent) presentations in the appendix. First we discuss the problem with a Gentzen-style natural deduction for modal system **K** and then we explore the solution implicit in the first author's 1985 paper *A system of natural deduction for GL*. Then we develop a type theory with usual syntactic properties (strong normalization, subject reduction, confluence) for a Gentzen-like (as opposed to Fitch-like) Natural Deduction version of modal system **K**. After that we extend our work on natural deduction and term assignment systems, with the further correspondence between typed λ-calculus and category theory, that is usually referred as the "extended Curry-Howard isomorphism".

Other approaches (Basin *et al*[BMV98], Martini and Masini[MM96], etc) that tie in the semantics of modal logics (in terms of possible worlds) with their syntactic presentation have been devised. We won't have much to say about this line of work here.

2 The Logical System

2.1 Sequent Calculus and Axiomatic System

We take the sequent calculus described below as defining the logic we are interested in. Add to the sequent calculus rules for intuitionistic propositional logic (IPL), a single rule for necessity:

$$\frac{\Gamma \vdash B}{\Box\Gamma \vdash \Box B}$$

(This rule does double-duty as both a left and a right introduction rules for the $\Box$ operator.) We call this system $\mathbf{IK}_\Box$, for the $\Box$-only fragment of **IK** . Add to it a single rule for possibility:

$$\frac{\Gamma, B \vdash B}{\Box\Gamma, \Diamond B \vdash \Diamond C}$$

(this again is considered both a right and a left rule). The calculus thus obtained, call it **IK** , for the intuitionistic system **K**, is our main interest in this paper.

This calculus satisfies cut-elimination, as for instance shown (in a more complicated form) in [Wij90]. This calculus is different from what is usually taken as the basic constructive system **K**, as we do not assume the distribution of possibility ($\Diamond$) over disjunctions neither in its binary form

$$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$$

nor in its nullary form

$$\Diamond\bot \to \bot$$

The sequent calculus above corresponds to an axiomatic formulation given by axioms for intuitionistic logic, plus axioms:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

$$\Box(A \to B) \to (\Diamond A \to \Diamond B)$$

$$\Box A \times \Diamond B \to \Diamond(A \times B)$$

together with rules for Modus Ponens and Necessitation:

$$\frac{\vdash A \to B \qquad \vdash A}{\vdash B}\text{MP} \qquad \frac{\vdash A}{\vdash \Box A}\mathit{Nec}$$

Wijesekera proves a Craig interpolation theorem, one of the usual consequences of syntactic cut-elimination and produces Kripke, algebraic and topological semantics for a calculus very similar to the one above. The only difference is that he does assume $\Diamond\bot \to \bot$. From our "wish list" for logical systems only a natural deduction formulation and a categorical semantics are missing. These we proceed to discuss.

2.2 Natural Deduction for Modal Logic: the problem

Prawitz in his classic monograph on Natural Deduction only discusses natural deduction formulations for the basic modal logics S4 and S5. As Bull and Segerberg note in their survey "Basic Modal Logic" in page 27

> [..]It has proved difficult to extend this sort of analysis [the Natural Deduction one] to the great multitude of other systems of modal logic.

The problem with a natural deduction formulation of **IK** is easy to see: if one wants to think of the sequent calculus rule above for $\Box$ as applying to derivation-trees we have that a tree

$$\begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\,\pi \\ B \end{array}$$

has to be transformed into a tree

$$\begin{array}{c} \Box A_1, \Box A_2, \ldots, \Box A_k \\ \hline A_1, A_2, \ldots, A_k \\ \vdots\,\pi \\ B \\ \hline \Box B \end{array}$$

but while transforming the conclusion of a natural deduction derivation tree is perfectly acceptable, modifying its premisses is not allowed. Moreover, if modifying of premisses was allowed, we would still have a problem as the inference

$$\frac{\Box A}{A}$$

is valid in S4, but not in **K**.

One solution to the problem above for (certain) modal logics can be seen as an easy corollary of work of the first author done in the 80's. We recap this work here and extend it, both with a term assignment system (for which we can prove the usual syntactic properties such as subject reduction, strong normalization and Church-Rosser property) and a categorical model.

But two other solutions exist: The first was proposed by Fitch [F52] in the early 50's. His modification of Natural Deduction, for classical or intuitionistic logic, is simply a notational variant of Gentzen's natural deduction: one writes linear derivations instead of tree-like derivations. But when modal operators are considered, Fitch-style Natural Deduction and Gentzen-style Natural Deduction seem further apart than one might expect. Fitting's monograph "Proof Methods for Modal and Intuitionistic Logics"[F83] considers several Fitch-style natural deduction systems for different modal logics built from system **K** plus axioms. Borghuis, in his doctoral thesis developed Fitch-style ND systems to obtain a Curry-Howard or "proofs-as-types"[1] interpretation for several systems of modal logic. These systems do not seem easily transformable into Gentzen-style natural deduction ones. And at present we cannot see how to provide categorical semantics for them.

Finally a third solution, inspired by work on systems where the context is divided in zones, such as Girard's LU or Plotkin and Barber's DILL is presented. This solution works at the moment for the $\Box$-only fragment. We do not know how to extend it to the $\Diamond$-fragment. Relating this seemingly different solutions is also left to future work.

2.3 Gentzen-style Modal Natural Deduction

We present the Gentzen-style natural deduction calculus for intuitionistic modal system **IK** . This calculus can be seen as revisiting the first author's ideas in [Bel85], except that we simplify our basis to a *constructive* one and we add terms to the natural deduction system, to obtain a Curry-Howard

[1]Borghuis system seems strange in that, despite the fact that his basic systems are all constructive, the modal logics on top are classical. This choice seems dictated more by tradition (Barendregt's cube is constructive, while most work on modal logic is classical) than by application suitability.

isomorphism[2]. We have judgements of the form

$$\Gamma \vdash M : A$$

meaning that logically M is a (term coding up a) proof of the Natural Deduction sequent $\Gamma \vdash A$ (see next section). There are usual rules for conjunctions and implications.

The main point is that the tree

$$\begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\, \pi \\ B \end{array}$$

is transformed to a tree like

$$\dfrac{\Box A_1, \Box A_2, \ldots, \Box A_k \qquad \begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\, \pi \\ B \end{array}}{\Box B}$$

where $A_1, \ldots, A_k$ have all been closed for substitutions. The only place in this tree where substitutions can occur, is on top of the boxed A_i, ie $\Box A_i$. This shape of rule, which was fairly sensible in the S4-case (as in S4 $\Box A_i$ does imply A_i) is less obvious here, but still reasonable.

Thus the rule in Martin Löf-style Natural Deduction looks like:

$$\frac{\Delta \vdash \Box\vec{A} \qquad A_1, A_2, \ldots, A_k \vdash B}{\Delta \vdash \Box B}$$

where $\Box\vec{A} = \Box A_1, \Box A_2, \ldots, \Box A_k$, so $\Delta \vdash \Box\vec{A}$ really means a collections of derivations $\Delta \vdash \Box A_1, \ldots, \Delta \vdash \Box A_k$.

Note that this rule for $\Box$ is mixed (i.e. it is both an introduction and an elimination rule), as the assumption already mentions the $\Box A_i$ that we are defining how to introduce. This is a "bad" characteristic of modal Natural Deduction, which also happens in the S4 case.

To provide the Natural Deduction rule for the possibility operator, we use the same trick of "cutting up" the tree. Then the tree

$$\begin{array}{c} A_1, A_2, \ldots, A_k, B \\ \vdots\, \pi \\ C \end{array}$$

gets transformed into

$$\dfrac{\Box A_1, \Box A_2, \ldots, \Box A_k, \Diamond B \qquad \begin{array}{c} A_1, A_2, \ldots, A_k, B \\ \vdots\, \pi \\ C \end{array}}{\Diamond C}$$

[2]Note that the main goal of the paper[Bel85] was to deal with the provability logic GL, the work on **K** was only necessary background to it. Here we concentrate on the base case, ie system **K**, and leave GL to possible future work.

The Natural Deduction presentation obtained is equivalent to the sequent and axiomatic formulations. The obvious term calculus associated to it, discussed in the next section, satisfies normalization (an easy corollary of work in [Bel85]). This calculus requires some commuting conversions, but it is basically well-behaved.

3 Term Calculus for IK

We have judgements of the form $\Gamma \vdash M\colon A$, meaning that M is a proof of the sequent $\Gamma \vdash A$. We have the following raw terms:

$$M ::= x \mid \lambda x\colon A.M \mid MM \mid \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{x} \mid \texttt{Dia}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{x}$$

The typing rules are the usual typing rules for the simply typed λ-calculus plus the following rules for the modalities:

$$\frac{\Gamma \vdash \vec{N_i}\colon \Box\vec{A} \qquad x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k \vdash M : B}{\Gamma \vdash \texttt{Box}\ M\ \texttt{with}\ N_i\ \texttt{for}\ x_i\colon \Box B}$$

$$\frac{\Gamma \vdash \vec{N_i}\colon \Box\vec{A} \qquad \Gamma \vdash N\colon \Diamond\vec{B} \qquad x_1 : A_1, x_2 : A_2, \ldots, x_k : A_k, x\colon B \vdash M : C}{\Gamma \vdash \texttt{Dia}\ M\ \texttt{with}\ N_i, N\ \texttt{for}\ x_i, x\colon \Diamond C}$$

where by $\vec{a}\colon\vec{A}$ we mean $a_1\colon A_1, a_2\colon A_2, \ldots a_n\colon A_n$.

For the reduction version of the rules we consider only β-rules and commuting conversions, no η-rules. We have the following commuting conversions:

$$\begin{aligned}
\texttt{Box}\ (\texttt{Box}\ M\ \texttt{with}\ N_i\ \texttt{for}\ x_i)\ \texttt{with}\ P_j\ \texttt{for}\ y_j &\rightsquigarrow \texttt{Box}\ \texttt{Box}\ M\ \texttt{with}\ x_i\ \texttt{for}\ x_i \\
&\qquad \texttt{with}\ \texttt{Box}\ N_i\ \texttt{with}\ P_j\ \texttt{for}\ y_j\ \texttt{for}\ x_i \\[1em]
(\texttt{Box}\ M\ \texttt{with}\ N_i\ \texttt{for}\ x_i)R &\rightsquigarrow \texttt{Box}\ (MR)\ \texttt{with}\ N_i\ \texttt{for}\ x_i \\[1em]
(\texttt{Dia}\ M\ \texttt{with}\ N_i\ \texttt{for}\ x_i)R &\rightsquigarrow \texttt{Dia}\ (MR)\ \texttt{with}\ N_i\ \texttt{for}\ x_i \\[1em]
\texttt{Dia}\ (\texttt{Dia}\ M\ \texttt{with}\ N_i\ \texttt{for}\ x_i)\ \texttt{with}\ P_j\ \texttt{for}\ y_j &\rightsquigarrow \texttt{Dia}\ \texttt{Dia}\ M\ \texttt{with}\ x_i\ \texttt{for}\ x_i \\
&\qquad \texttt{with}\ \texttt{Dia}\ N_i\ \texttt{with}\ P_j\ \texttt{for}\ y_j\ \texttt{for}\ x_i
\end{aligned}$$

Traditional syntactic results like subject reduction, strong normalisation and confluence hold for $\mathbf{IK}_\Box$.

Theorem 1 (Properties of $\mathbf{IK}_\Box$). *Subject Reduction, Strong Normalization (SN) and the Church-Rosser property hold for* $\mathbf{IK}_\Box$.

4 Models

A categorical model for **IK** is easier to describe into two steps: first we describe the model for $\mathbf{IK}_\Box$ and then we describe the modelling of possibility ($\Diamond$).

Recall that categorical models distinguish between different proofs of the same formula. A category consists of objects, which model the propositional variables, and for every two objects A and B each morphism in the category from A to B, corresponds to a proof of B using A as hypothesis. *Cartesian closed categories* are the categorical models for (disjunction-free) intuitionistic propositional logic. For a proper explanation the reader should consult Lambek and Scott[LS85].

To model the $\Box$ modality we need a *monoidal* endofunctor. The monoidicity of the functor corresponds to the modelling of the **K** characteristic axiom $\Box(A \to B) \to \Box A \to \Box B$. In previous work [BdP00] it was shown that to model the S4 necessity $\Box$ operator one needs a *monoidal comonad.* Here we have less structure and hence only the monoidal endofunctor remains.

Definition 2. *A* $\mathbf{IK}_\Box$ *-category consists of a cartesian closed category* $\mathcal{C}$, *together with a monoidal endofunctor* $\Box\colon \mathcal{C} \longrightarrow \mathcal{C}$. *The functor being monoidal means the existence of a natural transformation with components:* $m_{A,B}\colon \Box A \times \Box B \to \Box(A \times B)$ *and of a morphism* $m_1\colon 1 \to \Box 1$ *satisfying the well-known commuting diagrams below.*

The soundness theorem simply describes the required constructs to model the modality rule:

Theorem 3 (Soundness). *Let* $\mathcal{C}$ *be any* $\mathbf{IK}_\Box$ *-category. Then there is a canonical interpretation* $[\![\_]\!]$ *of* $\mathbf{IK}_\Box$ *in* $\mathcal{C}$ *such that*

- *a formula* A *is mapped to an object* $[\![A]\!]$ *of* $\mathcal{C}$*;*
- *a natural deduction proof* ψ *of* B *using formulae* $A_1, \ldots, A_n$ *as hypotheses is mapped to a morphism* $[\![\psi]\!]$ *from* $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ *to* $[\![B]\!]$*;*
- *each two natural deduction proofs* ψ *and* ψ *of* B *using formulae* $A_1, \ldots, A_n$ *as hypotheses which are equal (modulo normalisation of proofs) are mapped to the same morphism, in other words* $[\![\phi]\!] = [\![\psi]\!]$.

Proof. We use an induction over the structure of natural deduction proofs. Describing the modality rule, consider a proof ψ

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\,\phi_1\\ \Box A_1\end{array} \quad \cdots \quad \begin{array}{c}\Gamma_n\\ \vdots\,\phi_n\\ \Box A_n\end{array} \qquad \begin{array}{c}[A_1 \cdots A_n]\\ \vdots\,\phi\\ B\end{array}}{\Box B}\ \Box_{\mathcal{I}}$$

By induction hypothesis, let $f_1, \ldots, f_n, f$ be the interpretation of $\phi_1, \ldots, \phi_n, \phi$ respectively. Then the interpretation of ψ is

$$(\Box f) \circ m_{A_1,\ldots,A_n} \circ (f_1 \times \cdots \times f_n)$$

where $m_{A_1,\ldots,A_n}$ is inductively defined by

$$m_{A_1,\ldots,A_{m-1},A_m} = m_{A_1 \times \cdots \times A_{m-1},A_m} \circ (m_{A_1,\ldots,A_{m-1}}) \times id_{A_m}$$

$\Box$

A trivial degenerate example of an $\mathbf{IK}_\Box$ -category consists of taking any ccc, say set for example and considering the identity functor (as a monoidal functor) on it. Less trivial models should be investigated. To prove categorical completeness we use a term model construction.

Theorem 4. *(i) There exists an $\mathbf{IK}_\Box$ -category such that all morphisms are interpretations of natural deduction proofs.*

(ii) If the interpretation of two natural deduction proofs is equal in all $\mathbf{IK}_\Box$ -categories, then the two proofs are equal modulo proof-normalisation in natural deduction.

Both statements can be easily shown by constructing an $\mathbf{IK}_\Box$ -category $\mathcal{C}$ out of the natural deduction proofs.

Now to model the system with $\Diamond$'s we first define:

Definition 5. *A* **IK** *-category consists of a cartesian closed category, together with a monoidal endofunctor (to model $\Box$) and another endofunctor $\Diamond$ that is strong with respect to $\Box$. This means that there is a natural transformation $\alpha_{A,B}: \Box A \times \Diamond B \to \Diamond(A \times B)$ satisfying some commutative diagrams.*

The soundness theorem has the same shape, which means that its proof simply gets extended with the rule for possibility.

Theorem 6. *Let $\mathcal{C}$ be any* **IK** *-category. Then there is a canonical interpretation $[\![\ ]\!]$ of* **IK** *in $\mathcal{C}$ such that:*

- *a formula A is mapped to an object $[\![A]\!]$ of $\mathcal{C}$;*
- *a natural deduction proof ψ of B using formulae $A_1, \ldots, A_n$ as hypotheses is mapped to a morphism $[\![\psi]\!]$ from $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ to $[\![B]\!]$;*
- *each two natural deduction proofs ψ and ψ of B using formulae $A_1, \ldots, A_n$ as hypotheses which are equal (modulo normalisation of proofs) are mapped to the same morphism, in other words $[\![\phi]\!] = [\![\psi]\!]$.*

Describing the new modality rule, consider a proof Φ

$$
\dfrac{\begin{array}{cccccc} \Gamma & & \Gamma & \Gamma & [A_1 \cdots A_n, B] \\ \vdots\, \phi_1 & & \vdots\, \phi_n & \vdots\, \phi & \vdots\, \psi \\ \Box A_1 & \cdots & \Box A_n & \Diamond B & C \end{array}}{\Diamond C}\,\Diamond
$$

By induction hypothesis, let $f_1, \ldots, f_n, f, g$ be the interpretations of $\phi_1, \ldots, \phi_n, \phi, \psi$, respectively. Then the interpretation of Φ is

$$(\Diamond g) \circ \alpha_{A_1,\ldots,A_n,B} \circ (f_1 \times \cdots \times f_n \times f) \circ \Delta$$

where $\alpha_{A_1,\ldots,A_n,B}$ is inductively defined and Δ is the diagonal map.

Completeness follows exactly as before:

Theorem 7. *(i) There exists an* **IK** *-category such that all morphisms are interpretations of natural deduction proofs.*

(ii) If the interpretation of two natural deduction proofs is equal in all **IK** *-categories, then the two proofs are equal modulo proof-normalisation in natural deduction.*

Interesting categorical models need to be investigated. A first example of an application of these ideas appears in Rowan Davies paper[D96], but he did not discuss categorical semantics.

5 Conclusions

We discussed a problem with the formulation of a Natural Deduction version of modal system **K** and an (old) solution to it. We showed how this version of the system can be used to provide a Curry-Howard or "proofs-as-types" correspondence for the system. We then described categorical semantics for the system. Each of these steps is relatively straightforward, but they haven't been put together before, as far as we know. Putting these together is just the first step to clarify other issues, such as the limits of the Curry-Howard paradigm and the widespread use of intensional notions in functional programming.

6 Appendix: Variations on the same Theme

We now discuss the two other solutions for the problem of Natural Deduction for system **K**. These other systems also pose some problems of their own.

6.1 Fitch-style Modal Natural Deduction

Borghuis' work ([B98]) is based on Pure Type Systems, Barendregt's elegant systematization of higher-order lambda-calculi. This framework is very expressive, but since we are not interested (at least in this note) on higher-order systems, some of the complications of Borghuis system can be avoided.

The main idea of Fitch-style ND proofs is that to prove an implication, say $A \to B$, we go into a box, which is called a *subordinate proof* where we assume that the antecedent of the proposed implication, A is true. If we can derive B under this assumption, then we can discharge the assumption, by exiting the box and adding $A \to B$ to the original proof. One is also allowed to reiterate any formulas from the main proof into a subordinate proof, this simply corresponds to many uses of the same assumption. This device makes it possible to write proofs in a linear order, but does not change the intuitive meaning of the implication rule.

Having invented a box for dealing with implication, it's now natural to invent a different kind of box to deal with the modality $\Box$. Fitch adds a new kind of subordinate proof, a *strict subordinate proof* to his system in his book[F52]. A strict subordinate proof requires no hypothesis, and more importantly "reiteration" in a strict subordinate proof is restricted to formulas of a certain form. For system **K**, one is only allowed to reiterate formulas of the form $\Box A$, ie modal formulas and their reiteration appears in the subordinate proof without the $\Box$ operator, ie as simply A. This is the so-called **K**-*import rule*. A formula imported into a strict subordinate proof does not count as a hypothesis of that proof.

To export proofs from the box, we have to be a bit more careful: a conclusion A can be exported if it was derived by means of a *categorical* strict subordinate proof. A categorical strict subordinate proof has to satisfy:

1. All of its assumptions have been discharged;
2. The conclusion lies directly inside the modal interval;
3. There are no nested subordinate proofs that are still "open".

Having briefly described Fitch-style modal natural deduction, we want to simplify Borghuis type theory to deal simply with the *propositional* modal logic **K**. We also want a *constructive* propositional basis, so we shall not use encodings of operators. The following is a simplification (no higher-order logic, no dependency on contexts, no classical modals) of Borghuis system for $\mathbf{IK}_\Box$.

We have ground types, implications $A \to A$, conjunctions $A \wedge A$, and modal types $\Box A$. We omit disjunctions and falsum, for the time being. The type theory has judgements

$$\Gamma_k | \dots | \Gamma_1 | \Gamma_0 \vdash M : A$$

meaning that (logically) M is a proof of A where a context has k-many compartments, where k is the maximum modal depth of formulas in the derivation. We have the following raw terms:

$$M ::= x \mid \lambda a\colon A.M \mid MM \mid \mathtt{Box}(M) \mid \mathtt{Unbox}(M)$$

The typing rules are as follows

$$\frac{}{\Gamma|\Gamma_0, x\colon A \vdash x\colon A}$$

$$\frac{\Gamma|\Gamma_0, a\colon A \vdash M\colon B}{\Gamma|\Gamma_0 \vdash \lambda a\colon M\colon A \to B} \qquad \frac{\Gamma|\Gamma_0 \vdash M\colon A \to B \qquad \Gamma|\Gamma_0 \vdash N\colon A}{\Gamma|\Gamma_0 \vdash MN\colon B}$$

$$\frac{\Gamma|\Gamma_0 \vdash M\colon \Box A}{\Gamma|\Gamma_0|\_ \vdash \mathtt{Unbox}(M)\colon A} \qquad \frac{\Gamma|\_ \vdash M\colon A}{\Gamma \vdash \mathtt{Box}(M)\colon \Box A}$$

Usual meta-theoretical results hold for this calculus. In particular we have

Theorem 8 (Borghuis). *Subject Reduction, Strong Normalization (SN) and the Church-Rosser property hold for the calculus above.*

The proof detailed in Borghuis thesis proceeds by defining an erasing mapping from the modal lambda-calculus to the simply typed λ-calculus and proving properties of this erasing.

Fitch-style Natural Deduction does not emphasize the importance of the normalization procedure. Clearly one can translate from Fitch-ND proofs to Gentzen-ND proofs, normalize them and translate back into Fitch-style. But it would be interesting to see whether one can say something about the normalization procedure within the Fitch-style system.

As we saw, we can provide a categorical semantics for Gentzen-like modal systems. Thus we can translate Fitch-style ND proofs into Gentzen-style ND ones and this provides categorical semantics for the Fitch-style system. But a more direct route would be preferred. Especially because Fitch-style does seem to accommodate many more logical systems than the Gentzen-style. But we do not know, presently, how to describe direct semantics for Fitch-style natural deduction proofs.

6.2 Dual-context Modal Natural Deduction

The third calculus follows a pattern exploited recently by Girard, Miller, Barber and Plotkin: contexts are divided into two zones, one where assumptions have modal ($\Box$) types and the other where assumptions have the usual (intuitionistic) types. We have judgements of the form

$$\Gamma|\Delta \vdash M\colon A$$

meaning that (logically) M is a proof of the sequent $\Box\Gamma, \Delta \vdash A$. We have the following raw terms:

$$M ::= x \mid a \mid \lambda a{:}\,A.M \mid MM \mid \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a} \mid \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ M$$

The typing rules are the usual typing rules for the simply typed λ-calculus plus the following rules for the modality:

$$\frac{}{\Gamma|\Delta, x{:}\,A \vdash x{:}\,A} \qquad \frac{}{\Gamma, x{:}\,A|\Delta \vdash x{:}\,\Box A}$$

$$\frac{\Gamma|\_ \vdash \vec{N}{:}\,\Box\vec{A} \qquad \_\,|\vec{a}{:}\,\vec{A} \vdash M{:}\,A}{\Gamma|\_ \vdash \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a}{:}\,\Box A} \qquad \frac{\Gamma|\Delta \vdash M{:}\,\Box A \qquad \Gamma, x{:}\,A|\Delta \vdash N{:}\,B}{\Gamma|\Delta \vdash \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N{:}\,B}$$

where by $\vec{a}{:}\,\vec{A}$ we mean $a_1{:}\,A_1, a_2{:}\,A_2, \ldots a_n{:}\,A_n$.

Note that unlike our previous work on S4 here the right-hand side of the context does not mean without a box. Actually the left-side means with at least one box, possibly more. So there might be boxed formulas on the right-hand side of the context.

For the reduction rules we consider only β-rules and commuting conversions. We have one new β-rule, namely

$$\texttt{Unbox}\ (\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a})\ \texttt{for}\ \Box x\ \texttt{in}\ R \ \rightsquigarrow\ R[\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a}/x]$$

and the following commuting conversions:

$$\begin{array}{rl} (\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)R & \rightsquigarrow \\ \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ (NR) & \end{array}$$

$$\begin{array}{rl} \texttt{Unbox}\ (\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)\ \texttt{for}\ y\ \texttt{in}\ R & \rightsquigarrow \\ \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ (\texttt{Unbox}\ N\ \texttt{for}\ y\ \texttt{in}\ R) & \end{array}$$

$$\begin{array}{rl} \texttt{Box}\ M\ \texttt{with}\ \vec{N}, \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N, \vec{R}\ \texttt{for}\ \vec{b}, a, \vec{a} & \rightsquigarrow \\ \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ \texttt{Box}\ M\ \texttt{with}\ \vec{N}, N, \vec{R}\ \texttt{for}\ \vec{b}, a, \vec{a} & \end{array}$$

Traditional syntactic results like subject reduction, strong normalisation and confluence hold for $\mathbf{IK}_\Box$.

Proposition 9 (Subject Reduction). *Assume* $\Gamma|\Delta \vdash M{:}\,A$ *and* $M \rightsquigarrow N$*. Then also* $\Gamma|\Delta \vdash N{:}\,A$*.*

For strong normalisation we define a translation from our calculus into DIML[GdPR98] which preserves reductions. This translation is given in the following definition.

Definition 10. *For each term M define a DIML-term $(M)^D$ by induction over the structure of M as follows:*

$$
\begin{array}{rcl}
(a)^D & = & a \\
(x)^D & = & \Box x \\
(\lambda a.M)^D & = & \lambda a.(M)^D \\
(MN)^D & = & (M)^D(N)^D \\
(\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a})^D & = & \texttt{let}\ (\vec{N})^D\ \texttt{be}\ \Box\vec{x}\ \texttt{in}\ \Box(M)^D[\vec{x}/\vec{a}] \\
(\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)^D & = & \texttt{let}\ (M)^D\ \texttt{be}\ \Box x\ \texttt{in}\ (N)^D
\end{array}
$$

This translation preserves typing:

Lemma 11. *Assume $\Gamma|\Delta \vdash M{:}A$ in the $\mathbf{IK}_\Box$ calculus. Then $\Gamma|\Delta \vdash (M)^D{:}A$ in DIML.*

Proof. Easy induction over the definition of $(M)^D$. $\Box$

Strong normalisation follows now directly:

Theorem 12. *The $\mathbf{IK}_\Box$ calculus is strongly normalising.*

Proof. One shows that if $M \leadsto N$, then $(M)^D \leadsto^+ (N)^D$. Hence $\nu(M) \leq \nu((M)^D) < \infty$, where $\nu(M)$ is the length of the longest reduction sequence of M if M is strongly normalising and is ∞ otherwise. $\Box$

Confluence cannot be inferred directly from confluence of DIML but needs to be shown separately.

Theorem 13. *The $\mathbf{IK}_\Box$ calculus is confluent.*

Proof. As we have already shown strong normalisation it suffices to show local confluence. It is now easy to see that all critical pairs can be completed. $\Box$

Although it is clear that this system is equivalent (as far a provability is concerned) to the two other Natural Deduction formulations discussed, it has (at least) one main problem: We do not know how to syntactically extend it to deal with the possibility operator. Moreover, we do not know how to model it categorically, as there seems to be too little structure to prove it equivalent to the categorical model presented.

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