Categorical Proof Theory of Co-Intuitionistic Linear Logic.

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Summary. To provide a categorical semantics for co-intuitionistic logic, one has to face the fact, noted by Tristan Crolard, that the definition of co-exponents as adjuncts of co-products does not work in the category **Set**, where co-products are disjoint unions. Following the familiar construction of models of intuitionistic linear logic with exponent **!**, we build models of co-intuitionistic logic in symmetric monoidal closed categories with additional structure, using a variant of Crolard's term assignment to co-intuitionistic logic in the construction of a free category.

1 Preface

This paper sketches a categorical semantics for co-intuitionistic logic, advancing a line of proof-theoretic research developed in [1, 2, 3, 4, 7]. Co-intuitionistic logic, also called *dual-intuitionistic* [22, 31, 32], may be superficially regarded as completely determined by the duality, as in its lattice-theoretic semantics. A *co-Heyting algebra* is a (distributive) lattice $\mathcal{C}$ such that its opposite $\mathcal{C}^{op}$ is a Heyting algebra. In a Heyting algebra *implication* $B \to A$ is defined as the right adjoint of *meet*, so in a co-Heyting algebra $\mathcal{C}$ *co-implication* (or *subtraction*) $A \smallsetminus B$ is defined as the left adjoint of *join*:

$$\frac{C \wedge B \ \leq \ A}{C \ \leq \ B \to A} \qquad\qquad \frac{A \ \leq \ B \vee C}{A \smallsetminus B \ \leq \ C}$$

A *bi-Heyting* algebra is a lattice that has both the structure of Heyting and of a co-Heyting algebra. The logic of bi-Heyting algebras was introduced by Cecylia Rauszer [27, 28] (called *Heyting-Brouwer logic*), who defined also its Kripke semantics; a category-theoretical approach to the topic is due to Makkai, Reyes and Zolfaghari [25, 29]. The suggestion by F. W. Lawvere to use co-Heyting algebras as a logical framework to treat the topological notion of boundary has not been fully explored yet (but see recent work by Pagliani [26]).

Early research showed that the extension of first order intuitionistic logic with subtraction yields an *intermediate* logic of *constant domains* [21]. In a rich and interesting paper [16] T. Crolard showed, essentially by Joyal's argument, that Cartesian closed categories with exponents and co-exponents are degenerate; in fact even the topological models of bi-intuitionistic logic, i.e., *bi-topological spaces*, are degenerate. Crolard's motivations are mainly computational: he studies bi-intuitionistic logic in the framework of the *classical* $\lambda\mu$ calculus, to provide a type-theoretic analysis of the notion of *coroutine*; then he identifies a subclass of *safe coroutines* that can be typed constructively [17]. From our viewpoint, on one hand Crolard's work opens the way to a "bottom up" approach to safe coroutines, independent of the $\lambda\mu$ calculus, i.e., co-intuitionistic coroutines. On the other hand, the question arises whether the collapse of algebraic and topological models may be avoided by building the intuitionistic and co-intuitionistic sides separately, starting from distinct sets of elementary formulas, and then by joining the two sides with mixed connectives (mainly, two negations expressing the duality): this is our variant of bi-intuitionistic logic, presented in [2, 4].

Both of these tasks were advocated by this author and pursued within a project of "logic for pragmatics" with motivations from linguistics and natural language representation [2, 3, 4, 8]. In the characterization of the logical properties of "illocutionary acts", such as *asserting*, *making hypotheses* and *conjectures* one finds in natural reasoning forms of duality that can be related to intuitionistic dualities. For co-intuitionistic logic Crolard's term assignment has been adapted to a sequent-style natural deduction setting with single-premise and multiple-conclusions. For (our variant of) bi-intuitionistic logic Kripke semantics has been given (both in **S4** and in *bi-modal* **S4**) and a sequent calculus has been proposed where sequents are of the form

$$\Gamma \; ; \; \Rightarrow \; A \; ; \; \Upsilon \qquad \text{or} \qquad \Gamma \; ; \; C \; \Rightarrow \; ; \; \Upsilon$$

where Γ and A are intuitionistic (assertive) formulas and C and Υ co-intuitionistic (hypothetical).

But from the viewpoint of category theory a crucial remark by Crolard shows that already in *co-intuitionistic* logic there is a problem: namely, co-exponent are trivial in the category **Set**. Indeed the categorical semantics of intuitionistic disjunction is given by *coproducts* [19], which in **Set** are represented by *disjoint unions*. On the other hand the categorical semantics of subtraction is given by *co-exponents*. The co-exponent of A and B is an object B_A together with an arrow $\ni_{A,B}$: $B \to B_A \oplus A$ such that for any arrow $f : B \to C \oplus B$ there exists a unique $f_* : B_A \to C$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[dr, "\ni_{A,B}"'] & C \oplus A \\
& B_A \oplus A \arrow[u, "f_* \oplus id_A"']
\end{tikzcd}$$

It follows that

> *in the category of sets, the co-exponent B_A of two sets A and B is defined if and only if $A = \emptyset$ or $B = \emptyset$* (see [16], Proposition 1.15).

The proof is instructive: in **Set**, the coproduct $\oplus$ is *disjoint union*; thus if $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \oplus 1_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute.

Thus to have a categorical semantics of co-exponents we need categories where a different notion of disjunction is modelled. The connective *par* of linear logic is a good candidate and a treatment of *par* is available in *full intuitionistic linear logic* (**FILL**) [15, 23], with a proof-theory and a categorical semantics. The *multiple-conclusion* consequence relation of **FILL** and its term assignment have given motivation and inspiration to our work, as a calculus where a distinct term is assigned to each formula in the succedent. The language of **FILL** has tensor ($\otimes$), linear implication ($\multimap$) and *par* ($\wp$) and a main proof-theoretic concern has been the compatibility between *par* and intuitionistic linear implication (as it is in bi-intuitionistic logic). However to construct categorical models of co-intuitionistic logic it suffices to notice that in *monoidal categories par* can be modelled by the given monoidal operation and co-exponents as the *left adjoint* of *par*. The main task then is the to model Girard's exponential *why not?*: in this way a categorical semantics for co-intuitionistic logic can be recovered by applying the dual of Girard's translation of intuitionistic logic into linear logic, namely:

$$\begin{aligned}
(p)^\circ &= \mathrm{p}\\
(\mathbf{f})^\circ &= \mathbf{0}\\
(C \curlyvee D)^\circ &= ?(C^\circ \oplus D^\circ) = ?(C^\circ)\wp?(D^\circ)\\
(C \setminus D)^\circ &= C^\circ \setminus (?D^\circ)\\
(E \vdash C_1, \ldots, C_n)^\circ &= ?(E^\circ) \vdash ?(C_1^\circ), \ldots, ?(C_n^\circ))
\end{aligned}$$

where $\mathbf{0}$ is the identity of $\oplus$ and we use "$\setminus$" both in linear and in non-linear co-intuitionistic logic.

The task amounts to dualizing Nick Benton, Gavin Bierman, Valeria de Paiva and Martin Hyland's well-known semantics for intuitionistic linear logic [10]. This may be regarded as a routine exercise, except that one has to provide a term assignment suitable for the purpose. In this task we build on a term assignment to multiplicative co-intuitionistic logic, which has been proposed as an abstract *distributed calculus* dualizing the linear λ calculus [1, 3, 7]: in our view such a dualization underlies the translation of the linear λ-calculus into the π-calculus (see [9]).

As a matter of fact, Nick Benton's *mixed Linear and Non-Linear* logic [11] may give us not only an easier approach to modelling the exponentials but also the key to a categorical semantics of (our version of) bi-intuitionistic logic: indeed, by dualizing the linear part of Benton's system we may obtain both a proof-theoretic and a category theoretic framework for mixed co-intuitionistic linear and intuitionistic logic and thus also for bi-intuitionistic logic - of course, we need to use the exponential *why not?* and dualize Girard's translation. But then a categorical investigation of *linear cointuitionistic logic* and of the *why not?* exponential is a preliminary step in this direction and has an independent interest.

2 Proof Theory

As *co-intuitionistic linear logic* may be quite unfamiliar, we sketch an intuitive explanation of its proof theory. We think of co-intuitionistic logic as being about *making hypotheses* [2, 4]. It has a consequence relation of the form

$$H \vdash H_1, \ldots, H_n. \tag{1}$$

Suppose H is a hypothesis: which (disjunctive sequence of) hypotheses H_1 *or* ... *or* H_n follow from H? Since the logic is *linear*, commas in the meta-theory stand for Girard's *par* and the structural rules Weakening and Contraction are not allowed. A relevant feature, which we shall not discuss here, is that the consequence relation may be seen as *distributed*, i.e., we may think of the alternatives H_1, ..., H_n in (1) as lying in different *locations* [1, 7].

The main connectives are *subtraction* $A \smallsetminus B$ (*possibly* A *and not* B) and Girard's *par* $A \wp B$. Natural Deduction inference rules for *subtraction* (in a sequent form) are as follows.

$$\smallsetminus\text{-}intro\ \frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \smallsetminus D, \Delta} \qquad \smallsetminus\text{-}elim\ \frac{H \vdash \Delta, C \smallsetminus D \qquad C \vdash D, \Upsilon}{H \vdash \Delta, \Upsilon}$$

Notice that in the $\smallsetminus$-*elimination* rule the evidence that D may be derivable from C given by the *right premise* has become *inconsistent with the hypothesis* $C \smallsetminus D$ in the left premise; in the conclusion we drop D and we *set aside* the evidence for the inconsistent alternative. Namely, such evidence is not destroyed, but rather stored somewhere for future use.

If the *left premise* of $\smallsetminus$-*elimination*, deriving $C \smallsetminus D$ or Δ from H, has been obtained by a $\smallsetminus$-*introduction*, this inference has the form

$$\frac{H \vdash \Delta_1, C \qquad D \vdash \Delta_2}{H \vdash \Delta_1, \Delta_2, C \smallsetminus D}.$$

Then the pair of *introduction/elimination* rules can be eliminated: using the *removed evidence* that D with Υ are derivable from C (*right premise* of the $\smallsetminus$-*elim.*) we can conclude that $\Delta_1, \Delta_2, \Upsilon$ are derivable from H. This is, in a nutshell, the principle of normalization (or *cut-elimination*) for subtraction.

The *storage* operation is made explicit in the rules for the **?** operator of linear logic. Here an entire derivation d of $?\Delta$ from C (where $?\Delta = ?D_1, \ldots, ?D_n$) is set aside; what is accessible now is something like a non-logical axiom of the form $?C \vdash ?\Delta$. However in the process of normalization the derivation d may be recovered to be *used*, *discarded* or *copied* in the interaction of a *storage* rule with *dereliction*, *weakening* or *contraction*: all of this is conceptually clear, thanks to J-Y. Girard - and mathematically analysed in the *geometry of interaction.*

The rules of sequent-style natural deduction **co-ILL** for co-Intuitionistic Linear Logic are given in Table 1.

assumption $A \vdash A$	*substitution* $\dfrac{E \vdash \Gamma, A \qquad A \vdash \Delta}{E \vdash \Gamma, \Delta}$
$\perp$-I $\dfrac{E \vdash \Gamma}{E \vdash \Gamma, \perp}$	$\perp$-E $\perp \rhd$
$\smallsetminus$-I $\dfrac{E \vdash \Gamma, C \qquad D \vdash \Delta}{E \vdash \Gamma, C \smallsetminus D, \Delta}$	$\smallsetminus$-E $\dfrac{H \vdash \Upsilon, C \smallsetminus D \qquad C \vdash D, \Delta}{H \vdash \Upsilon, \Delta}$
$\wp$-I $\dfrac{E \vdash \Gamma, C_0, C_1}{E \vdash \Gamma, C_0 \wp C_1}$	$\wp$-E $\dfrac{H \vdash \Upsilon, C_0 \wp C_1 \qquad C_0 \vdash \Gamma_0 \qquad C_1 \vdash \Gamma_1}{H \vdash \Upsilon, \Gamma_0, \Gamma_1}$
dereliction $\dfrac{E \vdash \Gamma, C}{E \vdash \Gamma, ?C}$	*storage* $\dfrac{H \vdash \Upsilon, ?C \qquad C \vdash ?\Delta}{H \vdash \Upsilon, ?\Delta}$
weakening $\dfrac{E \vdash \Gamma}{E \vdash \Gamma, ?C}$	*contraction* $\dfrac{E \vdash \Gamma, ?C, ?C}{E \vdash \Gamma, ?C}$

Table 1. Natural Deducton for **co-ILL**

2.1 From Crolard's classical coroutines to co-intuitionistic ones

Crolard [17] provides a term assignment to the subtraction rules in the framework of Parigot's $\lambda\mu$-calculus, typed in a sequent-style natural deduction system. The $\lambda\mu$-calculus provides a typing system for functional programs with *continuations* and a computational interpretation of classical logic (see, e.g., [18, 30]).

In the type system for the $\lambda\mu$ calculus sequents may be written in the form $\Gamma \vdash t : A \mid \Delta$, with contexts $\Gamma = x_1 : C_1, \ldots, x_m : C_m$ and $\Delta = \alpha_1 : D_1, \ldots, \alpha_n : D_n$, where the x_i are *variables* and the α_j are *μ-variables* (or *co-names*). In addition to the rules of the simply typed lambda calculus, there are *naming rules*

$$\frac{\Gamma \vdash t : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]t : \bot \mid \alpha : A, \Delta}[\alpha] \qquad\qquad \frac{\Gamma \vdash t : \bot \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.t : A \mid \Delta}\mu$$

whose effect is to "change the goal" of a derivation and which allow us to represent the familiar *double negation rule* in Prawitz Natural Deduction.

Crolard extends the $\lambda\mu$ calculus with introduction and elimination rules for subtraction:[1]

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{make-coroutine}(t, \beta) : A \setminus B \mid \beta : B, \Delta} \setminus I$$

$$\frac{\Gamma \vdash t : A \setminus B \mid \Delta \qquad \Gamma,\ x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\ t\ \texttt{with}\ x \mapsto u : C \mid \Delta} \setminus E$$

The reduction of a redex of the form

$$\frac{\dfrac{\Gamma \vdash t : A \mid \Delta}{\Gamma \vdash \texttt{make-coroutine}(t, \beta) : A \setminus B \mid \beta : B, \Delta}\setminus\text{-I} \qquad \Gamma, x : A \vdash u : B \mid \Delta}{\Gamma \vdash \texttt{resume}\ (\texttt{make-coroutine}(t, \beta))\ \texttt{with}\ x \mapsto u\ \ : C \mid \beta : B, \Delta} \setminus\text{-E}$$

is as follows:

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma, x : A \vdash u\ B \mid \Delta}{\Gamma \vdash u[t/x] : B \mid \Delta}\textit{substitution}}{\Gamma \vdash [\beta]u[t/x] : \bot \mid \beta : B, \gamma : C, \Delta'}[\beta]}{\Gamma \vdash \mu\gamma.[\beta]u[t/x] : C \mid \beta : B, \Delta'}\mu$$

Working with the full power of classical logic, if a constructive system of bi-intuitionistic logic is required, then the implication right and subtraction left rules must be restricted; this can be done by considering *relevant dependencies*.[2] Crolard is able to show that the term assignment for such a restricted logic is a calculus of *safe coroutines*, described as terms in which no coroutine can access the local environment of another coroutine.

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[1] Actually in Crolard [17] the introduction rule is given in the more general form of $\setminus$-*introduction* with two sequent premises (which we use below) and more general *continuation contexts* occur in place of β; the above formulation is logically equivalent and suffices for our purpose.

[2] For instance, in the derivation of the right premise $\Gamma, x : A \vdash u : B \mid \Delta$ of a subtraction elimination ($\setminus E$), there should be no relevant dependency between the formula B and the assumptions in Γ, but only between B and A. Similar issues arise in **FILL**, see [23] and [5], section 4.

Crolard's work suggests the possibility of defining co-intuitionistic coroutines directly, independently of the typing system of the $\lambda\mu$-calculus. Since μ-variable abstraction and the μ-rule are devices to change the "actual thread" of computation, the effect of removing such rules is that all "threads" of computation are simultaneously represented in a multiple conclusion sequent, but variables y that are temporarily inaccessible in a term N are being replaced by a term $\mathtt{y}(M)$ by the substitution $N[y := \mathtt{y}(M)]$, where M contains a free variable x which is accessible in the current context. This is the approach pursued in [2, 3, 4] leading to the present categorical presentation.

2.2 A dual linear calculus for $\mathbf{MNJ}^{\smallsetminus\wp\perp}$

We present the grammar and the basic definitions of our dual linear calculus for the fragment of linear co-intuitionistic logic with subtraction and disjunction.

Definition 1. *We are given a countable set of* free variables *(denoted by x, y, z ...), and a countable set of* unary functions *(denoted by* $\mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots$*).*

(i) *Terms and p-terms are defined by the following grammar. Let $R = M, P$:*

$$\begin{aligned} M, N \ &:= x \mid \mathtt{x}(M) \mid \mathtt{connect\ to}(R) \mid M \wp N \mid \mathtt{casel}(M) \mid \mathtt{caser}(M) \mid \\ &\quad \mid \mathtt{mkc}(M, \mathtt{x}) \mid \mathtt{store}(M_1, \ldots, M_n, \mathtt{y}_1, \ldots, \mathtt{y}_n, \mathtt{x}, N) \mid [M] \mid [M, N] \\ P \ &:= \mathtt{postp}(\mathtt{y} \mapsto N, M) \ \textit{and} \ \mathtt{postp}(M). \end{aligned}$$

Definition 2. (i) *The* free variables $FV(\ell)$ *in a term are defined as follows:*

$$\begin{aligned} FV(x) &= \{x\} \\ FV(\mathtt{x}(M)) &= FV(M) \\ FV(\mathtt{connect\ to}(R)) &= FV(R) \\ FV(M \wp N) &= FV(M) \cup FV(N) \\ FV(\mathtt{casel}(M)) = FV(\mathtt{caser}(M)) &= FV(M) \\ FV(\mathtt{mkc}\,(M, \mathtt{x})) &= FV(M) \\ FV(\mathtt{store}(M_1, \ldots, M_n, \mathtt{y}_1, \ldots, \mathtt{y}_n, \mathtt{x}, N)) &= ((FV(M_1) \cup \ldots \cup FV(M_n)) \smallsetminus \{x\}) \cup FV(N) \\ FV(\,[M]\,) \ = \ FV(M) &\quad FV(\,[M, N]\,) \ = \ FV(M) \cup FV[N] \\ FV(\mathtt{postp}(\mathtt{x} \mapsto N, M) &= (FV(N) \smallsetminus \{x\}) \cup FV(M). \\ FV(\mathtt{postp}(M) &= FV(M). \end{aligned}$$

(ii) *A* computational context $\mathcal{S}_x$ *is a set of terms and p-terms containing the free variable x and no other free variable. We may represent a computational context as a list $\kappa = P_1, \ldots, P_M \mid M_1, \ldots, M_n$ of p-terms and terms.*

Definition 3. *Substitution of a term t for a free variable x in a in a list of terms κ is defined as follows:*

$$\begin{array}{rcl}
x[x:=M]=M, & & y[x:=M]=y \textit{ if } x\neq y;\\
\texttt{connect to}(N)[x:=M] & = & \texttt{connect to}(N[x:=M])\\
\texttt{postp}(N)[x:=M] & = & \texttt{postp}(N[x:=M])\\
\texttt{y}(N)[x:=M] & = & \texttt{y}(N[x:=M]);\\
(N_0 \wp N_1)[x:=M] & = & (N_0[x:=M]\,)\wp(\,N_1[x:=M])\\
\texttt{casel}(N)[x:=M] & = & \texttt{casel}(N[x:=M]),\\
\texttt{caser}(N)[x:=M] & = & \texttt{caser}(N[x:=M]);\\
\texttt{mkc}(N,\texttt{y})[x:=M] & = & \texttt{mkc}(N[x:=M],\texttt{y}),\\
\texttt{store}(\overline{N},\overline{\texttt{y}},\texttt{z},N)[x:=M] & = & \texttt{store}(\overline{N}[x:=M],\overline{\texttt{y}},\texttt{z},N[x:=M)\\
\texttt{postp}(\texttt{y}\mapsto(N_1),N_0)[x:=M] & = & \texttt{postp}(\texttt{y}\mapsto(N_1[x:=M]),N_0[x:=M]).\\
[\,R\,][x:=M]=\big[R[x:=M]\big] & & [R_0,R_1][x:=M]=\big[R_0[x:=M],R_1[x:=M]\big]
\end{array}$$

Definition 4. *β-reduction of a* redex $\mathcal{R}ed$ *in a* computational context $\mathcal{S}_x$ *is defined as follows.*
(i) *If $\mathcal{R}ed$ is a term N of the following form, then the reduction is local and consists of the rewriting $N \rightsquigarrow_\beta N'$ in $\mathcal{S}_x$ as follows:*

$$\texttt{postp}(\texttt{connect to}(R) \rightsquigarrow_\beta \; [\,].$$
$$\texttt{casel }(N_0 \wp N_1) \rightsquigarrow_\beta \; N_0; \qquad \texttt{caser }(N_0 \wp N_1) \rightsquigarrow_\beta \; N_1.$$

(ii) *If $\mathcal{R}ed$ is a term with principal operator* `store`*, then the reduction is global and consists of the following rewriting:*

$$\begin{array}{l}
\texttt{store}(N_1,\dots,N_n,\texttt{y}_1,\dots,\texttt{y}_n,\texttt{z},[M]) \mid \texttt{y}_1(\texttt{z}([M])),\dots,\texttt{y}_n(\texttt{z}([M])) \quad \rightsquigarrow_\beta\\
\qquad\qquad\qquad\qquad\qquad\qquad \mid N_1[z:=M],\dots N_n[z:=M].
\end{array}$$

$$\begin{array}{l}
\texttt{store}(\overline{N},\overline{\texttt{y}},\texttt{z},\texttt{connect to}(R)) \mid\\
\quad \mid \texttt{y}_1\big(\texttt{z}(\texttt{connect to}(R))\big),\dots,\texttt{y}_n\big(\texttt{z}(\texttt{connect to}(R))\big) \;\rightsquigarrow_\beta\\
\qquad\qquad \rightsquigarrow_\beta \;\mid\; \underbrace{\texttt{connect to}(R),\dots,\texttt{connect to}(R)}_{n\ times}.
\end{array}$$

$$\begin{array}{l}
\texttt{store}(\overline{N},\overline{\texttt{y}},\texttt{z},[M_0,M_1]) \mid \texttt{y}_1(\texttt{z}([M_0,M_1])),\dots,\texttt{y}_n(\texttt{z}([M_0,M_1])) \;\rightsquigarrow_\beta\\
\qquad \rightsquigarrow_\beta \; \texttt{store}(\overline{N},\overline{\texttt{y}},\texttt{z},M_0),\texttt{store}(\overline{N},\overline{\texttt{y}},\texttt{z},M_1) \mid\\
\qquad \mid \; [\texttt{y}_1(\texttt{z}(M_0)),\texttt{y}_1(\texttt{z}(M_1))]\dots[\texttt{y}_n(\texttt{z}(M_0)),\texttt{y}_n(\texttt{z}(M_1))].
\end{array}$$

(iii) *If $\mathcal{R}ed$ has the form $\texttt{postp}(\texttt{z}\mapsto N,\texttt{mkc}(M,\texttt{y}))$, then $\mathcal{S}_x$ has the form*

$$\mathcal{S}_x \;=\; \mathcal{R}ed, \quad \kappa, \quad \zeta_{\texttt{y}}, \quad \xi_{\texttt{z}}$$

where $\zeta_{\texttt{y}} = \zeta[y:=\texttt{y}(M)]$ and $\xi_{\texttt{z}} = \xi[z:=\texttt{z}((M\to\texttt{y}))]$ and neither $\texttt{y}(M)$ nor $\texttt{z}((M\to\texttt{y}))$ occurs in κ. Then a reduction *of $\mathcal{R}ed$ transforms the computational context as follows:*

$$\mathcal{S}_x \quad\rightsquigarrow\quad \kappa, \quad \zeta\,[y:=N[z:=M]]\,, \quad \xi[z:=M]. \tag{2}$$

Thus for $\zeta = S_1,\dots,S_k$ and $\xi = R_1,\dots,R_m$ we have:

$$\begin{array}{rcl}
\xi[z:=M] & = & R_1[z:=M],\dots,R_m[z:=M];\\
\zeta\,[y:=N[z:=M]] & = & S_1\,[y:=N[z:=M]]\,,\dots,S_k\,[y:=N[z:=M]]\,.
\end{array}$$

Remark 1. Here are some informal explanations about our calculus and notations.

(i) In a "multi-conclusion logical computation" a term M witnesses a "thread" of a logical computation which can be continued; on the contrary a p-term P sits in the "control area", waiting to become active in a computation at some point. Moreover a term $\mathtt{y}(M)$ denotes a variable y that has become bound because of an operation in which the term M is active; thus $\mathtt{y}(M)$ is an input which is no longer accessible. Later in the computation such an input may become active again in a term R and ready for a substitution for a term N. We denote such an operation as $R[y := \mathtt{y}(M)] \rightsquigarrow R[y := N]$.

(ii) When a logical computation is *stored*, its threads $N_1, \ldots, N_n$ are set aside in the control area, but the "guarding terms" $\mathtt{y}_1, \ldots, \mathtt{y}_n$ associated with them remain active; also the free variable z occurring in the N_i becomes inaccessible and is substituted with $\mathtt{z}(M)$, where M is the term active in the storage operation. If $M = [M']$ then the computation is reactivated with threads $N_i[z := M']$ and the guarding terms $\mathtt{y}_i$ destroyed. If $M = [M_0, M_1]$ then both of the "guarding terms" and the `store` term are copied; if $M =$ `connect to`(R) then the stored computation and the guarding terms are destroyed and the outputs are all connected to the term R.

(iii) A term "*make-coroutine*" $\mathtt{mkc}(M, \mathtt{y})$ jumps from the term M to an input y which becomes inaccessible and thus is substituted by a term $\mathtt{y}(M)$ throughout the computational context. On the other hand a "*postpone*" term $\mathtt{postp}(\mathtt{z} \mapsto N, M')$ stores some *threads* of the computation from z to N (possibly a list of terms). As a consequence the input z becomes inaccessible and is substituted by a term $\mathtt{z}(M')$ throughout the computational context. If M' is $\mathtt{mkc}(M, \mathtt{y})$, then we can reactivate the stored threads N and free the variables z and y in the computational context. The variable z is substituted by M wherever it occurs, i.e., as $\xi[z := M]$. Moreover the threads N are connected to M through the substitution $N[z := M]$ and the variable y is substituted by $N[z := M]$. Here we see a calculus with binding and substitution implemented as "global effects" in a co-intutionistic calculus through terms originally conceived by Tristan Crolard [17] as extensions of the $\lambda\mu$ calculus.

The term assignment to **co-ILL** in a sequent calculus notation is given in table 2. Sequents are of the form

$$x : E \triangleright \overline{P} \mid \overline{M} : \Gamma$$

where

- the area of the succedent to the left of "|" may be called *"control area"*;
- $\overline{P} = P_1, \ldots, P_m$ is a sequence of p-terms;
- $\overline{M} : \Gamma$ stands for $M_1 : C_1, \ldots, M_n : C_n$, where $\Gamma = C_1, \ldots, C_n$;
- writing $\kappa : \Gamma$ for $\overline{P} \mid \overline{M} : \Gamma$ and $\zeta : \Delta$ for $\overline{Q} \mid \overline{N} : \Delta$, then $\kappa : \Gamma, \zeta : \Delta$ stands for $\overline{P}, \overline{Q} \mid \overline{M} : \Gamma, \overline{N} : \Delta$.

The term assignment for the corresponding system of Natural Deduction is given below in the discussion of the categorical models.

3 Categorical Semantics

We recall the definition of a symmetric monoidal category.

axiom
$$x : A \rhd \;\mid x : A$$

cut
$$\frac{v : E \vdash \overline{P} \mid \overline{M} : \Gamma, M : A \qquad x : A \vdash \overline{Q} \mid \overline{N} : \Delta}{x : E \vdash \overline{P}, \overline{Q}[x := M] \mid \overline{M} : \Gamma, \overline{N}[x := M] : \Delta}$$

$\perp$-R
$$\frac{x : E \vdash \overline{\pi} \mid \overline{M} : \Gamma}{x : E \vdash \overline{\pi} \mid \overline{M} : \Gamma, \mathtt{connect\ to}(R) : \perp}$$
where $R \in \overline{\pi} \cup \overline{M}$

$\perp$-L
$$x : \perp \rhd \mathtt{postp}(x) \mid$$

Write $\kappa : \Gamma$ for $\overline{P} \mid \overline{M} : \Gamma$ and $\zeta : \Delta$ for $\overline{Q} \mid \overline{N} : \Delta$.

$\setminus$-R
$$\frac{v : E \rhd \kappa : \Gamma, M : C \qquad x : D \rhd \zeta : \Delta}{v : E \rhd \kappa : \Gamma, \zeta[x := \mathtt{x}(M)] : \Delta, \mathtt{mkc}(M, \mathtt{x}) : C \setminus D}$$

$\setminus$-L
$$\frac{x : C \rhd \overline{P} \mid M : D, \overline{M} : \Delta}{z : C \setminus D \rhd \overline{P}[x := \mathtt{x}(z)], \mathtt{postp}(x \mapsto M, z) \mid \overline{M}[x := \mathtt{x}(z)] : \Delta}$$

$\wp$-R
$$\frac{x : E \rhd \overline{\pi} \mid \overline{M} : \Gamma, M_0 : C_0,\ M_1 : C_1}{x : E \rhd \overline{\pi} \mid \overline{M} : \Gamma, M_0 \wp M_1 : C_0 \wp C_1}$$

$\wp$-L
$$\frac{x_0 : C_0 \rhd \kappa : \Gamma_0 \qquad\qquad x_1 : C_1 \rhd \zeta : \Gamma_1}{z : C_0 \wp C_1 \rhd \kappa[x_0 := \mathtt{casel}(z)] : \Gamma_0, \zeta[x_1 := \mathtt{caser}(z)] : \Gamma_1}$$

dereliction
$$\frac{x : E \rhd \kappa : \Gamma, M : C}{x : E \rhd \kappa : \Gamma, [\,M\,] : ?C}$$

storage
$$\frac{x : C \rhd \overline{P} \mid \overline{N} : ?\Delta}{z : ?C \rhd \overline{P}, \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, z) \mid \overline{\mathtt{y}}(z) : ?\Delta}$$

weakening
$$\frac{x : E \rhd \kappa : \Gamma}{x : E \rhd \kappa : \Gamma, \mathtt{connect\ to}(R) : ?C}$$
where $R \in \kappa$.

contraction
$$\frac{x : E \vdash \kappa : \Gamma, M : ?C, N : ?C}{x : E \vdash \kappa : \Gamma, [\,M, N\,] : ?C}$$

Table 2. Term assignment to sequent calculus for **co-ILL**

Definition 5. *A* symmetric monoidal category (SMC) $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$*, is a category* $\mathbb{C}$ *equipped with a bifunctor* $\bullet : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *with a neutral element* 1 *and natural isomorphisms* α, λ, ρ *and* γ*:*

1. $\alpha_{A,B,C,} : A \bullet (B \bullet C) \xrightarrow{\sim} (A \bullet B) \bullet C$*;*

2. $\lambda_A : 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A : A \bullet 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \bullet B \xrightarrow{\sim} B \bullet A.$

which satisfy the following coherence diagrams.

$$\begin{tikzcd}
A \bullet (B \bullet (C \bullet D)) \arrow[r, "\alpha_{A,B,C\bullet D}"] \arrow[d, "id_A \bullet \alpha_{B,C,D}"'] & (A \bullet B) \bullet (C \bullet D) \arrow[r, "A\bullet B{,}C{,}D"] & (((A \bullet B) \bullet C) \bullet D \\
A \bullet ((B \bullet C) \bullet D) \arrow[rr, "\alpha_{A,B\bullet C,D}"'] & & (A \bullet (B \bullet C)) \bullet D \arrow[u, "\alpha_{A,B,C} \bullet id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet (B \bullet C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \bullet id_C"'] & A \bullet (B \bullet C) \arrow[r, "\gamma_{A,B\bullet C}"] & (B \bullet C) \bullet A \arrow[d, "\alpha B{,}C{,}A"] \\
(B \bullet A) \bullet C \arrow[r, "\alpha_{B,A,C}"'] & B \bullet (A \bullet C) \arrow[r, "id_B \bullet \gamma_{A,C}"'] & B \bullet (C \bullet A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet (1 \bullet B) \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "id_A \bullet \lambda_B"'] & & (A \bullet 1) \bullet B \arrow[dl, "\rho_A \bullet id_B"] \\
& A \bullet B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \bullet B \arrow[r, "\gamma_{A,B}"] \arrow[dr, "id_{A\bullet B}"'] & B \bullet A \arrow[d, "\gamma_{B,A}"] \\
& A \bullet B
\end{tikzcd}$$

$$\begin{tikzcd}
A \bullet 1 \arrow[rr, "\gamma_A{,}1"] \arrow[dr, "\rho_a"'] & & 1 \bullet A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}$$

The following equality is also required to hold: $\lambda_1 = \rho_1 : \bot \bullet \bot \to 1.$

Given a *signature* Sg, consisting of a collection of types σ_i and a collection of *sorted function symbols* $f_j : \sigma_1, \dots, \sigma_n \to \tau$ and given a SMC $(\mathbb{C}, \bullet, 1, \alpha, \lambda, \rho, \gamma)$, a *structure* $\mathcal{M}$ for Sg is an assignment of an object $[\![\sigma]\!]$ of $\mathbb{C}$ for each type σ and of a morphism $[\![f]\!] : [\![\sigma_1]\!] \bullet \dots \bullet [\![\sigma_n]\!] \to [\![\tau]\!]$ for each function $f : \sigma_1, \dots, \sigma_n \to \tau$ of Sg.

The types of terms in context $\Delta = [M_1 : \sigma_1, \dots, M_n : \sigma_n]$ are interpreted as $[\![\sigma_1, \sigma_2, \dots, \sigma_n]\!] = (\dots([\![\sigma_1]\!] \bullet [\![\sigma_2]\!]) \dots) \bullet [\![\sigma_n]\!]$; left associativity is also intended for concatenations of type sequences Γ, Δ. Thus the obvious functions $\mathtt{Split}(\Gamma, \Delta) : [\![(\Gamma, \Delta)]\!] \to [\![(\Gamma]\!] \bullet [\![\Delta)]\!]$ and $\mathtt{Join}(\Gamma, \Delta) : [\![(\Gamma]\!] \bullet [\![\Delta)]\!] \to [\![(\Gamma, \Delta)]\!]$ are defined by suitable combinations of α, λ, ρ and their inverses; similarly for $\mathtt{Split}_n(\Gamma_1, \dots, \Gamma_n) : [\![(\Gamma_1, \dots, \Gamma_n)]\!] \to [\![\Gamma_1]\!] \bullet \dots \bullet [\![\Gamma_n)]\!]$

The semantics of terms in context is then specified by induction on terms:

$$[\![x : \sigma \rhd x : \sigma]\!] =_{df} id_\sigma$$

$$[\![x : \sigma \rhd f(M_1, \dots, M_n) : \tau]\!] =_{df} [\![x : \sigma \rhd M_1 : \sigma_1]\!] \bullet \dots \bullet [\![x : \sigma \rhd M_n : \sigma_n]\!]; f$$

The *Exchange right* rule is handled implicitly by symmetry in the model:

$$[\![x : \sigma \triangleright \overline{M} : \Gamma, N : \tau, M : \sigma]\!] = [\![x : \sigma \triangleright \overline{M} : \Gamma, M : \sigma, N : \tau]\!]; \alpha^{-1}_{\Gamma,\tau,\sigma} \bullet \gamma_{\tau,\sigma}; \alpha_{\Gamma,\sigma,\tau}$$

One then proves by induction on the type derivation that substitution in the term calculus corresponds to composition in the category:

Lemma 1. *Let $x : \sigma \triangleright \overline{M} : \Gamma, M : \tau$ and $y : \tau \triangleright \overline{N} : \Delta$ be derivable terms in context, then*

$$\begin{array}{l}[\![x : \sigma \triangleright \overline{M} : \Gamma, \overline{N}[y := M]\,]\!] = \\ \qquad [\![x : \sigma \triangleright \overline{M} : \Gamma, M : \tau]\!]; id_\Gamma \bullet [\![y : \tau \triangleright \overline{N} : \Delta]\!]; \mathtt{Join}(\Gamma, \Delta)\end{array}$$

Let M be a structure for a signature Sg in a SMC $\mathbb{C}$. Given an equation in context for Sg

$$x : \sigma \triangleright \overline{M} : \Gamma, M = N : \sigma$$

we say that the structure *satisfies* the equation if the morphisms assigned to $x : \sigma \triangleright \overline{M} : \Gamma, M$ and to $x : \sigma \triangleright \overline{M} : \Gamma, N : \sigma$ are equal. Then given an algebraic theory $Th = (Sg, Ax)$, a structure $\mathcal{M}$ for Sg is a *model* for Th if it satisfies all the axioms in Ax.

Lemma 2. *Let $\mathbb{C}$ be a SMC, Th an algebraic theory and $\mathcal{M}$ a model of Th in $\mathbb{C}$. Then $\mathcal{M}$ satisfies the equations in context in Table 3.*

$$\frac{z : D \triangleright \overline{M} : \Gamma, M : \sigma}{z : D \triangleright \overline{M} : \Gamma, M = M : \sigma}\,\mathit{Refl} \qquad \frac{z : D \triangleright \overline{M} : \Gamma, M = N : \sigma}{z : D \triangleright \overline{M} : \Gamma, N = M : \sigma}\,\mathit{Symm}$$

$$\frac{z : D \triangleright \overline{M} : \Gamma, M = N : \sigma \qquad z : D \triangleright \overline{M} : \Gamma, N = P : \sigma}{z : D \triangleright \overline{M} : \Gamma, M = P : \sigma}\,\mathit{Trans}$$

$$\frac{z : D \triangleright \overline{M} : \Gamma, M = N : \sigma \qquad x : \sigma \triangleright \overline{M} : \Gamma, P = Q : \tau}{z : D \triangleright \overline{M} : \Gamma, P[x := M] = Q[x := N] : \tau}\,\mathit{Subst}$$

Table 3.

3.1 Analysis of the rules of co-intuitionistic linear logic

We work with symmetric monoidal categories satisfying *the dual condition to closure*, namely, with monoidal categories of the form $(\mathbb{C}, \bullet, \setminus, 1, \alpha, \lambda, \rho, \gamma)$ such that for all objects A in $\mathbb{C}$, the functor $A \bullet -$ has a *left* adjoint $- \setminus A$. We call such monoidal categories *left closed*.

Given a symmetric monoidal category $\mathbb{C}$, its opposite is also symmetric monoidal. If $\mathbb{C}$ is closed, i.e., $A \bullet -$ has a right adjoint, then certainly $\mathbb{C}^{op}$ has a left adjoint. It is well-known that in a symmetric monoidal closed category $\mathbb{C}$ we can construct a model of *multiplicative intuitionistic linear logic*, hence it is certainly not surprising that a model of multiplicative co-intuitionistic linear logic may be constructed in $\mathbb{C}^{op}$. The point of the exercise that follows, however, is to check that the dual linear calculus given above in Section 2.2 is indeed suitable for the construction of such an interpretation. We consider the rules for each connective in turn.

Linear disjunction Par

3.1.1. *Par introduction.* The introduction rule for Par is of the form

$$\frac{x : D \rhd \kappa : \Theta, M_0 : A, M_1 : B}{x : D \rhd \kappa : \Theta, M_0 \wp M_1 : A \wp B} \wp \text{ I}$$

This suggests an operation on Hom-sets of the form

$$\Phi_{D,\Theta} : \mathbb{C}(D, \Theta \bullet A \bullet B) \to \mathbb{C}(D, \Theta \bullet A \wp B)$$

natural in Θ and D. Given $e : D \to \Theta \bullet A \bullet B$, $d : D' \to D$ and $h : \Theta \to \Theta'$, naturality yields

$$\Phi_{D',\Theta'}(d; e; h \bullet id_A \bullet id_B) = d; \Phi_{D,\Theta}(e); h \bullet id_{A \wp B}$$

In particular, letting $e = id_\Theta \bullet id_A \bullet id_B$, $d : D \to \Theta \bullet A \bullet B$ and $h = id_\Theta$ we have

$$\Phi_{D,\Theta}(b) = b; \Phi_\Theta(id_\Theta \bullet id_A \bullet id_B)$$

Writing **PAR** for $\Phi_\Theta(id_\Theta \bullet id_A \bullet id_B)$ we have $\Phi_{D,\Theta}(b) = b; \mathbf{PAR}$. We define

$$[\![x : D \rhd \overline{Q} : \Theta, M \wp N : A \wp B]\!] =_{df} [\![x : D \rhd \overline{Q} : \Theta, M : A, N : B]\!]; \mathbf{PAR}.$$

3.1.2. *Par elimination.* The Par elimination rule has the form

$$\frac{z : D \rhd \kappa : \Upsilon, N : A \wp B \qquad x : A \rhd \zeta : \Gamma \qquad y : B \rhd \xi : \Delta}{z : D \rhd \kappa : \Upsilon, \zeta[x := \texttt{casel}\, N] : \Gamma, \xi[y := \texttt{caser}\, N] : \Delta} \wp \text{ E}$$

This suggests an operation on Hom-sets of the form

$$\Psi_{D,\Upsilon,\Gamma,\Delta} : \mathbb{C}(D, \Upsilon \bullet A \wp B) \times \mathbb{C}(A, \Gamma) \times \mathbb{C}(B, \Delta) \to \mathbb{C}(D, \Upsilon \bullet \Gamma \bullet \Delta)$$

natural in $D, \Upsilon, \Gamma, \Delta$. Given morphisms $g : D \to \Upsilon \bullet A \wp B$, $e : A \to \Gamma$ and $f : B \to \Delta$ and also $a : D' \to D$, $p : \Upsilon \to \Upsilon'$, $c : \Gamma \to \Gamma'$ and $d : \Delta \to \Delta'$ naturality yields

$$\begin{aligned}\Psi_{D',\Upsilon',\Gamma',\Delta'}\big((a; g; p \bullet id_{A \wp B}), (e; c), (f; d)\big) = \\ a; \Psi_{D,\Upsilon,\Gamma,\Delta}(g, e, f); p \bullet c \bullet d; \texttt{Join}(\Upsilon', \Gamma', \Delta').\end{aligned}$$

In particular, setting $e = id_A,\ f = id_B$ and also $a = id_D,\ p = id_\Upsilon$, we get

$$\Psi_{D,\Upsilon,\Gamma,\Delta}(g,c,d) = \Psi_{D,\Upsilon,\Gamma,\Delta}(g, id_A, id_B); id_\Upsilon \bullet c \bullet d; \mathtt{Join}(\Upsilon,\Gamma,\Delta)$$

Writing $(g)^*$ for $\Psi_{D,\Upsilon}(g, id_A, id_B)$ we define

$$[\![z : D \triangleright \kappa : \Upsilon, \zeta[x := \mathtt{casel}\ N] : \Gamma, \xi[y := \mathtt{caser}\ N] : \Delta]\!] =_{df}$$
$$[\![z : D \triangleright \kappa : \Upsilon, N : A \wp B]\!]^*; id_\Upsilon \bullet [\![x : A \triangleright \zeta : \Gamma]\!] \bullet [\![y : B \triangleright \xi : \Delta]\!]; \mathtt{Join}(\Upsilon,\Gamma,\Delta).$$

3.1.3. *Equations in context.* We have equations in context of the form

$$\begin{array}{c} \wp - \beta\ \textbf{rule:} \\ \dfrac{z : D \triangleright \kappa : \Theta, M_0 : A, M_1 : B \qquad x : A \triangleright \zeta : \Gamma \quad y : B \triangleright \xi : \Delta}{\begin{array}{c} z : D \triangleright \kappa : \Theta, \zeta[x := \mathtt{casel}\,(M_0 \wp M_1)] = \zeta[x := M_0] : \Gamma \\ z : D \triangleright \kappa : \Theta, \xi[y := \mathtt{caser}\,(M_0 \wp M_1)] = \xi[y := M_1] : \Delta \end{array}} \end{array} \tag{3}$$

Let $q : D \to \Theta \bullet A \bullet B$, $m : A \to \Gamma$ and $n : B \to \Delta$. Then to satisfy the above equation in context we need that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "q"] & \Theta \bullet A \bullet B \arrow[r, "id_\Theta \bullet m \bullet n"] \arrow[d, "\wp"'] & \Theta \bullet \Gamma \bullet \Delta \\ & \Theta \bullet A \wp B \arrow[r, "*"'] & \Theta \bullet A \bullet B \arrow[u, "id_\Theta \bullet m \bullet n"'] \end{tikzcd}$$

We make the assumption that the above decomposition is unique. Moreover, supposing Θ empty and $m = id_A$, $n = id_B$, $q = id_A \bullet id_B$ we obtain $(id_A \bullet id_B; \mathbf{PAR})^* = id_A \bullet id_B$ and similarly $(id_{A \wp B})*; \mathbf{PAR} = id_{A \wp B}$; hence we may conclude that there is a natural isomorphism

$$\frac{D \to \Gamma \bullet A \bullet B}{\overline{D \to \Gamma \bullet A \wp B}}$$

so we can identify $\bullet$ and $\wp$. Finally we see that the following η equation in context are also satisfied:

$$\begin{array}{c} \wp - \eta\ \textbf{rule:} \\ \dfrac{z : D \triangleright \kappa : \Upsilon, M : A \wp B \qquad x : A \triangleright x : A \qquad y : B \triangleright y : B}{z : D \triangleright \kappa : \Upsilon, \mathtt{casel}(M) \wp \mathtt{caser}(M) = M : A \wp B} \end{array} \tag{4}$$

Linear subtraction

3.1.1. *Subtraction introduction.* The introduction rule for subtraction has the form

$$\frac{x : D \triangleright \kappa :\ \Gamma, M :\ A \qquad y : B \triangleright \zeta :\ \Delta}{x : D \triangleright \kappa :\ \Gamma, \zeta[y := \mathtt{y}(M)] : \Delta, \mathtt{mkc}(M, \mathtt{y}) : A \smallsetminus B} \smallsetminus \mathrm{I}$$

This suggests a natural transformation with components

$$\Phi_{D,\Gamma,\Delta} : \mathbb{C}(D, \Gamma \bullet A) \times \mathbb{C}(B, \Delta) \to \mathbb{C}(D, \Gamma \bullet \Delta \bullet A \smallsetminus B)$$

natural in D, Γ, Δ. Taking morphisms $e : D \to \Gamma \bullet A$, $f : B \to \Delta$ and $a : D' \to D$, $c : \Gamma \to \Gamma'$, $d : \Delta \to \Delta'$, by naturality we have

$$\Phi_{D',\Gamma',\Delta'}\,((a; e; c \bullet id_A), (f; d)) = a; \Phi_{D,\Gamma,\Delta}(e, f); c{\bullet}d{\bullet}id_{A \smallsetminus B}; \mathtt{Join}(\Gamma', \Delta', A{\smallsetminus}B)$$

In particular, taking $a = id_D$, $c = id_\Gamma$, $d : B \to \Delta$ and $f = id_B$ we have:

$$\Phi_{D,\Gamma,\Delta}(e, d) = \Phi_{D,\Gamma}(e, id_B); id_\Gamma \bullet d \bullet id_{A \smallsetminus B}; \mathtt{Join}(\Gamma, \Delta, A \smallsetminus B)$$

Writing $\mathbf{MKC}^B_{D,\Gamma}(e)$ for $\Phi_{D,\Gamma}(e, id_B)$, $\Phi_{D,\Gamma,\Delta}(e, d)$ can be expressed as the composition

$$\mathbf{MKC}^B_{D,\Gamma}(e); id_\Gamma \bullet d \bullet id_{A \smallsetminus B}$$

where $\mathbf{MKC}^B_{D,\Gamma}$ is a natural transformation with components

$$\mathbf{MKC}^B_{D,\Gamma} : \mathbb{C}(D, \Gamma \bullet A) \times \mathbb{C}(B, B) \to \mathbb{C}(D, \Gamma \bullet B \bullet A \smallsetminus B)$$

so we make the definition

$$\begin{aligned} &[\![x : D \triangleright \kappa :\ \Gamma, \zeta[y := \mathtt{y}(M)], \mathtt{mkc}(M, \mathtt{y}) :\ A \smallsetminus B]\!] =_{df} \\ &\quad \mathbf{MKC}^B_{D,\Gamma}[\![x : D \triangleright \kappa : \Gamma, M : A]\!]; id_\Gamma \bullet [\![y : B \triangleright \zeta : \Delta]\!] \bullet id_{A \smallsetminus B}; \mathtt{Join}(\Gamma, \Delta, A \smallsetminus B) \end{aligned}$$

3.1.2. *Subtraction elimination.* The subtraction elimination rule has the form

$$\frac{x : D \triangleright \overline{M} : \Gamma, M : A \smallsetminus B \qquad y : A \triangleright \overline{N} : \Delta, N : B}{x : D \triangleright \overline{M} : \Gamma, \overline{N}[y := Y(M)], \mathtt{postp}(y \mapsto N, M)} \wp\ \mathrm{E}$$

This suggests a natural transformation with components

$$\Psi_{D,\Gamma,\Delta} : \mathbb{C}(D, \Gamma \bullet (A \smallsetminus B)) \times \mathbb{C}(A, \Delta \bullet B) \to \mathbb{C}(D, \Gamma \bullet \Delta \bullet 1)$$

natural in D, Γ, Δ. Given $e : D \to \Gamma \bullet (A \smallsetminus B)$, $f : A \to \Delta \bullet B$ and also $a : D' \to D$, $c : \Gamma \to \Gamma'$, $d : \Delta \to \Delta'$, naturality yields

$$\Psi_{D',\Gamma',\Delta'}\,((a; e; f \bullet id_{A \smallsetminus B}), (f; c \bullet id_B)) = a; \Phi_{D,\Gamma,\Delta}(e, f); c \bullet d; \mathtt{Join}(\Gamma', \Delta')$$

In particular, taking $a : D \to \Gamma \bullet (A \smallsetminus B)$, $e = id_{\Gamma \bullet (A \smallsetminus B)}$, $c = id_\Gamma$, $d : id_\Delta$, we obtain

$$\Psi_{D,\Gamma,\Delta}(a, f) = a; \Phi_{D,\Gamma,\Delta}(id_{\Gamma \bullet (A \smallsetminus B)}, f); \mathtt{Join}(\Gamma, \Delta)$$

Writing $\mathbf{POSTP}(f)$ for $\Phi_{D,\Gamma,\Delta}(id_{\Gamma \bullet (A \smallsetminus B)}, f)$ we define

$$[\![x : D \triangleright \overline{M} : \Gamma, \overline{N}[y := \mathtt{Y}(M)], \mathtt{postp}(y \mapsto N, M)]\!] =_{df}$$
$$[\![x : D \triangleright \overline{M} : \Gamma, M : A \smallsetminus B]\!]; id_\Gamma \bullet \mathbf{POSTP}[\![y : A \triangleright \overline{N} : \Delta, N : B]\!]; \mathtt{Join}(\Gamma, \Delta)$$

3.1.3. *Equations in context.* We have equations in context of the form

$$\boxed{\begin{array}{c}\smallsetminus - \beta \ \textbf{rule:}\\ \dfrac{x : D \triangleright \overline{M} : \Gamma, M : A \qquad y : B \triangleright \overline{N} : \Delta \qquad z : A \triangleright \overline{L} : \Lambda, L : B}{x : D \triangleright \overline{M} : \Gamma, \left[\overline{N}' : \Delta, \mathbf{red}\overline{L}' : \Lambda\right] = \left[\overline{N}[y := L[z ::= M]], \overline{L}[z := M]\right]}\end{array}} \tag{5}$$

where $\overline{N}' = \overline{N}[y := Y(M)]$, $\mathbf{red} = \mathtt{postp}\,(z \mapsto L, \mathtt{mkc}(M, Y))$ and $\overline{L}' = \overline{L}[z := \mathtt{mkc}(M, Y)]$.

Given morphisms $n : D \to \Gamma \bullet A$ and $m : A \to \Delta \bullet B$, for these equations to be satisfied we need the following diagram to commute:

$$\begin{tikzcd}
D \arrow[r, "n"] \arrow[d, "\mathbf{MKC}^B(n)"'] & \Gamma \bullet A \arrow[d, "id_\Gamma \bullet m"] \\
\Gamma \bullet (A \smallsetminus B) \bullet B \arrow[r, "\mathbf{POSTP}(m) \bullet id_B"'] & \Gamma \bullet \Delta \bullet B
\end{tikzcd}$$

in particular, taking $n = id_A$ we have

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "\mathbf{MKC}^B(id_A)"'] & \Delta \bullet B \\
(A \smallsetminus B) \bullet B \arrow[ur, "\mathbf{POSTP}(m) \bullet id_B"'] &
\end{tikzcd}$$

Assuming the above decomposition to be unique, we can show that the η equation in context is also satisfied:

$$\boxed{\begin{array}{c}\smallsetminus - \eta \ \textbf{rule}\\ \dfrac{z : D \triangleright \overline{N} : \Delta, M : A \smallsetminus B \qquad x : A \triangleright x : A \qquad y : B \triangleright y : B}{z : D \triangleright \overline{N} : \Delta, \left[\mathtt{mkc}(X(M), Y) : A \smallsetminus B, \mathtt{postp}(x \mapsto Y(x), M)\right] = M : A \smallsetminus B}\end{array}} \tag{6}$$

and conclude that there is a natural isomorphism between the maps

$$\begin{array}{c} A \to \Delta \bullet B \\ \hline\hline A \smallsetminus B \to \Delta \end{array}$$

i.e., that $\smallsetminus$ is the left adjoint to the bifunctor $\bullet$.

Unit

3.1.1 *Unit rules.* The introduction and elimination rules for the unit $\perp$ are

$$\frac{x : D \triangleright \kappa : \Gamma}{x : D \triangleright \kappa : \Gamma, \texttt{connect to}(R) : \perp}\ \perp\ \textit{introduction} \qquad \text{where } R \in \kappa.$$

$$\perp\ \textit{elimination} \qquad x : \perp \triangleright \texttt{postp}(x)$$

The elimination rule is interpreted by a unique map $\langle\rangle : \perp \to 1$.

The introduction rule requires a natural transformation with components

$$\Phi_{D,\Gamma} : \mathbb{C}(D, \Gamma) \to \mathbb{C}(D, \Gamma \bullet 1)$$

natural in D and Γ. Given morphisms $e : D \to \Gamma$, $d : D' \to D$ and $c : \Gamma \to \Gamma'$, naturality yields

$$\Phi_{D',\Gamma'}(d; e; c) = d; \Phi_{D,\Gamma}(e); c.$$

Letting $d : D \to \Gamma$ and $e = id_\Gamma$, $c = id_{\Gamma \bullet \perp}$ we have

$$\Phi_{D,\Gamma}(d) = d; \mathbf{Bot}_\Gamma$$

where we write $\mathbf{Bot}_\Gamma$ for $\Phi_\Gamma(id_\Gamma)$. We define

$$[\![x : D \triangleright \kappa : \Gamma, \texttt{connect to}(x) : \perp]\!] =_{df} [\![x : D \triangleright \kappa : \Gamma]\!]; \mathbf{Bot}_\Gamma.$$

3.1.2. *Equations in context.* We may assume the operation $\mathbf{Bot}_\Gamma$ to be compatible with the generalized associativity and commutativity properties of $\bullet$, so that for $\Gamma = C_1, \ldots, C_n$ we have

$$\Phi_{C_1 \bullet \ldots C_i \bullet \perp \bullet \ldots \bullet C_n}(id_\Gamma) : C_1, \ldots, C_i, \perp, \ldots, C_n \quad = \quad \Phi_\Gamma(id_\Gamma) : \Gamma, \perp$$

for all $i \leq n$. Together with naturality of $\mathbf{Bot}_\Gamma$ these yield the equations in context

$$\boxed{\begin{array}{c} \dfrac{x : D \triangleright \kappa : \Gamma}{x : D \triangleright \kappa : \Gamma, \big[\texttt{connect to}(R_i) = \texttt{connect to}(R_j)\big]} \\ R_i, R_j \in \kappa \\[2ex] \dfrac{\dfrac{x : D \triangleright \kappa : \Gamma}{y : E \triangleright \zeta : \Gamma'}}{y : E \triangleright \zeta, \big[\texttt{connect to}(R_i) = \texttt{connect to}(R_j)\big]} \\ R_i \in \kappa, R_j \in \zeta \end{array}} \qquad (7.0)$$

that correspond to the *rewiring properties* of $\perp$-links in the proof-net representation by [13, 14]. Moreover the equation in context

$$\boxed{\begin{array}{c} \perp - \beta\ \textbf{rule} \\ \dfrac{x : D \triangleright kappa : \Gamma \quad y : \perp \triangleright \texttt{postp}(y)}{x : D \triangleright [\kappa : \Gamma, \texttt{postp}(\texttt{connect to}(R)\,) = \kappa : \Gamma]} \\ \text{where } R \in \kappa. \end{array}} \qquad (7)$$

requires that for any $m : D \to \Gamma$ the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "m;\mathbf{Bot}"] \arrow[dr, "m"'] & \Gamma \bullet \bot \arrow[d, "id_\Gamma \bullet \langle\rangle;\lambda_A"] \\ & \Gamma. \end{tikzcd}$$

Assuming that this decomposition is unique and taking $m = id_A$ we have that $\mathbf{Bot}_A; id_A \bullet \langle\rangle; \lambda_A = id_A$. Arguing as before, we see that there is a natural isomorphism

$$\frac{D \to \Gamma \bullet 1}{\overline{D \to \Gamma \bullet \bot}}$$

(so we identify $\bot$ and 1) and that the following equation in context is satisfied:

$$\boxed{\begin{array}{c} \bot - \eta \textbf{ rule:} \\ \dfrac{z : D \triangleright \kappa :\ \Gamma, M : \bot \qquad\qquad x : \bot \triangleright \texttt{postp}(x)}{z : D \triangleright \kappa :\ \Gamma, \big[\texttt{connect to}(\texttt{postp}(M)) : \bot = M : \bot\big]} \end{array}} \tag{8}$$

Let $\mathcal{L}$ be the signature having

- the types given by the following grammar on a collection of ground types γ:
$$A \ := \ \gamma \mid \bot \mid A \wp A \mid A \setminus A$$
- a collection of sorted function symbols including $\texttt{connect to}(-)$, $\texttt{postp}(-)$, $\wp(-,-)$, $\texttt{casel}(-)$, $\texttt{caser}(-)$, $\texttt{mkc}(-,-)$, $\texttt{postp}(-,-)$.

We have proved the following

Theorem 1. *Let $\mathcal{T} = (\mathcal{L}, \mathcal{A})$ be a theory with signature $\mathcal{L}$ having as axioms the equations in context in Table 3 and in (3) - (8). Let $(\mathbb{C}, \bullet, 1, \setminus, \alpha, \lambda, \rho, \gamma)$ be a symmetric monoidal left-closed category and $\mathcal{M}$ a structure for $\mathcal{L}$ in $\mathbb{C}$. Then $\mathcal{M}$ satisfies the equations in $\mathcal{A}$.*

Moreover, define the *syntactic category* as the category $\mathcal{C}$ which has the formulas of multiplicative co-intuitionistic linear logic as objects and typed terms of the form $x : E \triangleleft \kappa : \Gamma$ (*modulo* renaming of the variable x) as morphisms. Set $x : E \triangleright \kappa : \Gamma \ = \ y : E \triangleright \zeta : \Gamma$ iff $\kappa = \zeta[y := x]$ is derivable from equations in context in Table 3 and in (3) - (8). Then we have

Theorem 2. *The syntactic category is a symmetric monoidal left-closed category.*

From this fact the *categorical completeness* theorem follows.

4 Extension to co-intuitionistic linear logic with coproducts and exponential

Let $\mathcal{L}^{\oplus}$ be $\mathcal{L}$ extended with additive disjunction $\oplus$ and the familiar functions $\texttt{inl} : A \to A \oplus B$, $\texttt{inr} : B \to A \oplus B$ and $\texttt{case} : A \oplus B \times (A \to C) \times (B \to C) \to C$. Then it is easy to extend the above result to show that if $\mathbb{C}$ has also the structure of coproducts, then a structure for $\mathcal{L}^{\oplus}$ in $\mathbb{C}$ satisfies also the theory $\mathcal{T}^{\oplus}$ where $\mathcal{A}$ is extended with familiar equations in context for `inl`, `inr` and `case`. We shall not pursue this extension here.

The extension of $\mathcal{T}$ to a theory with the exponential ? (*why not?*) is less simple. On one hand, one can dualize Benton, Bierman, De Paiva, Hyland's definition of a linear category [10, 12] and obtain in this way a sound and complete categorical semantics for co-intuitionistic linear logic. The construction of weakly distributive categories with storage operators based on proof-nets by Blute, Cockett and Seely [13] provides a categorical model for both exponentials **!** and **?**. On the other hand, the semantics for the exponential **!** can recovered in the context of Nick Benton's treatment of *Linear Non Linear* logic [11]. After dualizing the linear part of **LNL** one should be able to recover the semantics for **?** and at the same time obtain a framework where the duality of intuitionistic and co-intuitionistic logic can be studied. We leave the development of this approach to future work and focus on the categorical semantics of the multiplicative and exponential **?** fragment of co-intuitionstic linear logic.

4.1 Co-intuitionistic linear categories

We begin by dualizing the definition of a linear category [10, 12].

Definition 6. *A* dual linear category $\mathbb{C}$ *consists of*

1. *A symmetric monoidal left-closed category together with*
2. *a symmetric co-monoidal monad* $(?, \eta, \mu, n_{-,-}, n_{\perp})$ (namely, the functor **?** is co-monoidal with respect to $\wp$ and the linear transformation η, μ are co-monoidal) *such that*
   (i) - *each free ?-algebra* $(?A, \mu_A)$ *carries naturally the structure of a commutative* $\wp$*-monoid* (i.e., for each $(?A, \mu_A)$ there are distinguished monoidal natural transformations $i_A : \perp \to ?A$ and $c_A : ?A \wp ?A \to ?A$ which form a commutative monoid and are algebra morphisms)*;*
   (ii) - *whenever* $f : (?A, \mu_A) \to (?B, \mu_B)$ *is a morphism of free algebras, then it is also a monoid morphism.*

Remark 2. By Maietti, Maneggia de Paiva and Ritter (see [24], Prop. 25), condition 2 (ii) is equivalent to the requirement that μ is a monoidal morphism.

(i) To say that the functor **?** is symmetric co-monoidal means that it comes equipped with a *comparison* natural transformation $\mathtt{n}_{A,B} : ?(A \wp B) \to ?A \wp ?B$ and a morphism $\mathtt{n}_{\perp} : ?\perp \to \perp$, satisfying

$$\begin{tikzcd} ?(\bot \wp A) \arrow[r, "\mathtt{n}_{\bot,A}"] \arrow[d, "?\lambda_A"] & ?\bot \wp ?A \arrow[d, "\mathtt{n}_\bot \wp id_{?A}"] \\ ?A \arrow[r, "\lambda^{-1}_{?A}"'] & \bot \wp ?A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} ?(A \wp \bot) \arrow[r, "\mathtt{n}_{A,\bot}"] \arrow[d, "?\rho_A"] & ?A \wp ?\bot \arrow[d, "id_{?A} \wp \mathtt{n}_\bot"] \\ ?A \arrow[r, "\rho^{-1}_{?A}"'] & ?A \wp \bot \end{tikzcd}$$

$$\begin{tikzcd} ?((A \wp B) \wp C) \arrow[r, "\mathtt{n}_{A \wp B, C}"] \arrow[d, "?\alpha_{A,B,C}"'] & ?(A \wp B) \wp ?C \arrow[r, "\mathtt{m}_{A,B} \wp id_C"] & (?A \wp ?B) \wp ?C \arrow[d, "\alpha^{-1}_{?A,?B,?C}"] \\ ?(A \wp (B \wp C)) \arrow[r, "\mathtt{n}_{A, B \wp C}"'] & ?A \wp ?(B \wp C) \arrow[r, "id_{?A} \wp \mathtt{n}_{B.C}"'] & ?A \wp (?B \wp ?C) \end{tikzcd}$$

$$\begin{tikzcd} ?(A \wp B) \arrow[r, "\mathtt{n}_{A,B}"] \arrow[d, "?\gamma_{A,B}"] & ?A \wp ?B \arrow[d, "\gamma_{?A,?B}"'] \\ ?(B \wp A) \arrow[r, "\mathtt{n}_{B,A}"] & ?B \wp ?A \end{tikzcd} \quad \text{and naturality} \quad \begin{tikzcd} ?(A \wp B) \arrow[r, "n_{A,B}"] \arrow[d, "?(f \wp g)"'] & ?A \wp ?B \arrow[d, "?f \wp ?g"] \\ ?(A' \wp B') \arrow[r, "n_{A',B'}"] & ?A' \wp ?B' \end{tikzcd}$$

(ii) To say that η and μ are co-monoidal is to say that the following diagrams commute:

$$\begin{tikzcd} A \wp B \arrow[r, "\eta_A \wp \eta_A"] \arrow[d, "\eta_{A \wp B}"'] & ?A \wp ?B \\ ?(A \wp B) \arrow[ur, "n_{A.B}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bot \arrow[r, "\eta_\bot"] \arrow[dr, "id"'] & ?\bot \arrow[d, "n"] \\ & \bot \end{tikzcd}$$

$$\begin{tikzcd} ??(A \wp B) \arrow[r, "\mu_{A \wp B}"] \arrow[d, "?n_{A,B}"'] & ?(A \wp B) \arrow[dd, "n_{A,B}"] \\ ?(?A \wp ?B) \arrow[d, "n_{?A,?B}"'] & \\ ??A \wp ??B \arrow[r, "\mu_A \wp \mu_B"'] & ?A \wp ?B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} ??\bot \arrow[r, "?n_\bot"] \arrow[d, "\mu_\bot"'] & ?\bot \arrow[d, "n_\bot"] \\ ?\bot \arrow[r, "n_\bot"'] & \bot \end{tikzcd}$$

(iii) To say that the natural transformations $i_A : \bot \to ?A$ and $c_A : ?A \wp ?A \to ?A$ are monoidal means that they are compatible with the comparison maps, i.e., that the following diagrams commute:

$$\begin{tikzcd} \bot \arrow[r, "id"] \arrow[d, "i_\bot"'] & \bot \\ ?\bot \arrow[ur, "n_\bot"'] & \end{tikzcd} \quad \begin{tikzcd} \bot \wp \bot \arrow[r, "\lambda = \rho"] & \bot \\ ?\bot \wp ?\bot \arrow[u, "n_\bot \wp n_\bot"] \arrow[r, "c_\bot"'] & ?\bot \arrow[u, "n_\bot"'] \end{tikzcd} \quad \begin{tikzcd} \bot \wp \bot \arrow[r, "i_A \wp i_B"] \arrow[d, "\lambda = \rho"'] & ?A \wp ?B \\ \bot \arrow[r, "A \wp B"'] & ?(A \wp B) \arrow[u, "n_{A,B}"'] \end{tikzcd}$$

$$\begin{tikzcd}
(?A \wp ?A) \wp (?B \wp ?B) \arrow[rr, "c_A \wp c_B"] & & ?A \wp ?B \\
(?A \wp ?B) \wp (?A \wp ?B) \arrow[u, "iso"] & & \\
?(A \wp B) \wp ?(A \wp B) \arrow[u, "n_{A,B} \wp n_{A,B}"] \arrow[rr, "c_{A \wp B}"'] & & ?(A \wp B) \arrow[uu, "n_{A,B}"']
\end{tikzcd}$$

where *iso* is the canonical isomorphism derived from symmetry and associativity;

$$\begin{tikzcd}
\bot \arrow[r, "i_A"] & ?A \\
?\bot \arrow[u, "n_\bot"] \arrow[r, "?i_A"'] & ??A \arrow[u, "\mu_A"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
?A \wp ?A \arrow[r, "c_A"] & ?A \\
??A \wp ??A \arrow[u, "\mu_A \wp \mu_A"] & \\
?(?A \wp ?A) \arrow[u, "n_{?A,?A}"] \arrow[r, "?c_A"'] & ??A \arrow[uu, "\mu_A"']
\end{tikzcd}$$

(*iv*) Finally for the free algebra morphisms to be monoid morphism we require that the following diagrams commute:

$$\begin{tikzcd}
?A \wp ?A \arrow[r, "c_A"] & ?A \\
??A \wp ??A \arrow[u, "\mu_A \wp \mu_A"] \arrow[r, "c_{?A}"'] & ??A \arrow[u, "A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bot \arrow[r, "i_A"] \arrow[dr, "i_{?A}"'] & ?A \\
 & ??A \arrow[u, "\mu_A"']
\end{tikzcd}$$

4.2 Term and equations in context

To sketch a proof that a dual linear category is a model of co-intuitionistic linear logic with storage operator **?** we give the term in context and the equation in context relevant to the dereliction, weakening, contraction and storage rules. These conditions are dual to those in Figures 4.1-4.5 in G. M. Bierman's thesis [12], pp. 112-142. Since in our context the exponential rules for dereliction, weakening and contraction do not involve `let` constructions, some of these conditions result immediately from properties of substitution.

In a Natural Deduction setting, the dereliction, weakening and contraction rules for the exponential **?** are *introduction rules* and therefore coincide with the *right* sequent calculus rules in Table 2. On the contrary, the storage rule is an *elimination rule* and its Term in Context rule is given in Table 4. There are three Equations in Context expressing "β reductions" for the storage operator in Table 5. Finally there are *Categorical Equations in Context* in Table 6.

$$\frac{v : E \rhd \kappa : \Gamma, M :?C \qquad x : C \rhd \overline{Q} \mid \overline{N} :?\Delta}{v : E \rhd \kappa : \Gamma, \overline{Q}[x := \mathtt{x}(M)], \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(M)) : ?\Delta}$$

$$\frac{x : E \rhd \kappa : \Gamma, M : C}{x : E \rhd \kappa : \Gamma, [\, M\,] :?C}\ \textit{dereliction}$$

$$\frac{x : E \rhd \kappa : \Gamma}{x : E \rhd \kappa : \Gamma, \mathtt{connect\ to}(R) :?C}\ \textit{weakening} \qquad \frac{x : E \vdash \kappa : \Gamma, M :?C, N :?C}{x : E \vdash \kappa : \Gamma, [\, M, N\,] :?C}\ \textit{contraction}$$

where $R \in \kappa$.

Table 4. Term in context judgement for the **?** storage operator

Dereliction - Storage:

$$\frac{v : E \rhd \kappa : \Gamma, M : C \qquad x : C \rhd \overline{Q} \mid \overline{N} :?\Delta}{v : E \rhd \kappa : \Gamma, \Big[\overline{Q}[x := \mathtt{x}(\,[M]\,)], \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(\,[M]\,)) : ?\Delta = \\ = \overline{Q}[x := M] \mid \overline{N}[x := M] : ?\Delta\Big]}$$

Contraction - Storage:

$$\frac{v : E \rhd \kappa : \Gamma, M_0 :?C, M_1 :?C \qquad x : C \rhd \overline{Q} \mid \overline{N} :?\Delta}{v : E \rhd \kappa : \Gamma, \Big[\overline{Q}[x := \mathtt{x}([M_0, M_1])], \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, [M_0, M_1]) \mid \overline{\mathtt{y}}(\mathtt{x}(\,[M_0, M_1]\,)) : ?\Delta = \\ = \overline{Q}[x := M_0], \overline{Q}[x := M_1],\ \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, M_0), \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, M_1]) \mid \\ \langle[\overline{\mathtt{y}}(\mathtt{x}(M_0)), \overline{\mathtt{y}}(\mathtt{x}(M_1))]\rangle : ?\Delta\Big]}$$

where $?\Delta = ?D_1, \ldots ?D_m$ and $\langle[\overline{\mathtt{y}}(\mathtt{x}(M_0)), \overline{\mathtt{y}}(\mathtt{x}(M_1))]\rangle :?\Delta$ stands for $[\mathtt{y}_1(\mathtt{x}(M_0)), \mathtt{y}_1(\mathtt{x}(M_1))] : ?D_1, \ldots, [\mathtt{y}_m(\mathtt{x}(M_0)), \mathtt{y}_m(\mathtt{x}(M_1))] : ?D_m$

Weakening - Storage:

$$\frac{v : E \rhd \kappa : \Gamma \qquad x : C \rhd \overline{Q} \mid \overline{N} :?\Delta \qquad R \in \kappa}{v : E \rhd \kappa : \Gamma, \Big[\overline{Q}[x := \mathtt{x}(\mathtt{connect\ to}(R))], \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, \mathtt{connect\ to}(R)) \mid \\ \overline{\mathtt{y}}(\mathtt{x}(\mathtt{connect\ to}(R)) : ?\Delta = \mid \langle[\overline{\mathtt{connect\ to}}(R)]\rangle : ?\Delta\Big]}$$

where $?\Delta = ?D_1, \ldots ?D_m$ and $\mid \langle[\overline{\mathtt{connect\ to}}(R)]\rangle :?\Delta$ stands for $\mid \mathtt{connect\ to}(R) : ?D_1, \ldots, \mathtt{connect\ to}(R) : ?D_m$

Table 5. Equations in context for the **?** storage operator

Monad:

$$z :?A \rhd \Big[\mathtt{store}([\,[x]\,], \mathtt{y}, \mathtt{x}, z), \mathtt{store}(x', \mathtt{y}', \mathtt{x}', t) \mid \mathtt{y}'(\mathtt{x}'(t)) : ?A \;= x :?A\Big]$$
$$\text{where } t = \mathtt{y}(\mathtt{x}(z)) :??A$$

Algebra 1

$$\frac{v : E \rhd \kappa : \Gamma, M :?C \qquad x : C \rhd \overline{P} \mid \overline{N} :?\Delta}{v : E \rhd \kappa : \Gamma, \overline{P}[x := \mathtt{x}(M)],}$$
$$\Big[\mathtt{store}(\langle \overline{N}, \mathtt{connect\ to}(R)\rangle, \langle \overline{\mathtt{y}}, \mathtt{y}\rangle, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(M)) : ?\Delta, \mathtt{y}(\mathtt{x}(M)) : ?A =$$
$$= \mathtt{store}(\overline{N}, \overline{\mathtt{y}}, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(M)), \mathtt{connect\ to}(R')\Big]$$
$$\text{where } R \in \overline{P} \cup \overline{N} \text{ and } R' \in \overline{P}[x := \mathtt{x}(M)] \cup \overline{\mathtt{y}}(\mathtt{x}(M))$$

Algebra 2

$$\frac{v : E \rhd \kappa : \Gamma, M :?C \qquad x : C \rhd \overline{P} \mid \overline{N} :?\Delta, N_0 :?A, N_1 :?A}{v : E \rhd \kappa : \Gamma, \overline{P}[x := \mathtt{x}(M)],}$$
$$\Big[\mathtt{store}(\langle \overline{N}, [N_0, N_1]\rangle, \langle \overline{\mathtt{y}}, \mathtt{y}\rangle, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(M)) : ?\Delta, \mathtt{y}(\mathtt{x}(M)) : ?A =$$
$$= \mathtt{store}(\langle \overline{N}, N_0, N_1\rangle, \langle \overline{\mathtt{y}}, \mathtt{y}_0, \mathtt{y}_1\rangle, \mathtt{x}, M) \mid \overline{\mathtt{y}}(\mathtt{x}(M)) : ?\Delta, [\mathtt{y}_0(\mathtt{x}(M)), \mathtt{y}_1(\mathtt{x}(M))] : ?A\Big]$$

Monoid 1

$$\frac{v : E \rhd \kappa : \Gamma, M :?C \qquad R \in \kappa}{v : E \rhd \kappa : \Gamma, \Big[\, [M, \mathtt{connect\ to}(R)] : ?C = M : ?C\,\Big]}$$

Monoid 2

$$\frac{v : E \rhd \kappa : \Gamma, M :?C \qquad R \in \kappa}{v : E \rhd \kappa : \Gamma, \Big[\, [\mathtt{connect\ to}(R), M] : ?C = M : ?C\,\Big]}$$

Monoid 3

$$\frac{v : E \rhd \kappa : \Gamma, M_0 : ?C, M_1 : ?C}{v : E \rhd \kappa : \Gamma, \Big[\, [M_0, M_1] : ?C = [M_1, M_0] : ?C\,\Big]}$$

Monoid 4

$$\frac{v : E \rhd \kappa : \Gamma, M_0 : ?C, M_1 : ?C, M_2 : ?C}{v : E \rhd \kappa : \Gamma, \Big[\, [[M_0, M_1], M_2] : ?C = [M_0, [M_1, M_2]] : ?C\,\Big]}$$

Table 6. Categorical Equations in context

The key decision, discussed at length in G. M. Bierman's thesis [12] pp. 127-131, arises in the analysis of the *Dereliction-Storage* reduction given by the equation in context in Table 5. By repeating for the rules of *dereliction* and *storage* the kind of analysis done for *par*, *subtraction* and *unit*, we see that in order to model the *storage* rule we need a natural transformation $\Phi_{E,\Gamma} : \mathbb{C}(E, \Gamma \bullet ?A) \times \mathbb{C}(A, ?\Delta) \to \mathbb{C}(E, \Gamma \bullet ?\Delta)$. By naturality considerations this is given by its action $\Phi_\Gamma(id_{\Gamma \bullet ?A}, d) =_{df} d^*$ on morphisms $d : A \to ?\Delta$. Similarly, for the *dereliction* rule we need a natural transformation $\Psi : \mathbb{C}(\_, A) \to \mathbb{C}(\_, ?A)$ and by applying Yoneda's Lemma we see that its action is given by a morphism $\eta_A : A \to ?A$.

We can certainly define a functor $? : \mathbb{C}(A, \Gamma) \to \mathbb{C}(?A, ?\Gamma)$ by $f \mapsto (f; \eta_\Gamma)^*$. Now by the equation in context for *dereliction-storage* we have that following the diagram commutes:

$$\begin{tikzcd} ??A & ??A \arrow[l, "(\eta_A)^*"'] \\ & ?A \arrow[ul, "?\eta_A"] \arrow[u, "\eta_{?A}"'] \end{tikzcd}$$

Assuming the above decomposition to be unique, we have $(\eta_A)^* = id_{??A}$ and thus the derivations

$$\eta_{?A} : \ x :?A \triangleright [x] :??A \quad \text{and} \quad ?\eta_A : \ z :?A \triangleright \mathtt{store}([\,[x]\,], \mathtt{y}, \mathtt{x}, z) \mid \mathtt{y}(\mathtt{x}(z)) :??A$$

must be identified. Now it can be shown that identifying $?\eta_A$ and $?\eta_A$ forces the functor **?** to be *idempotent*: $??f = ?f$. In order to avoid such collapse, the functor **?** is only assumed to be a **K** modality, and the properties of **S4** are given by the natural transformations $\eta : A \to ?A$ and $\mu :??A \to ?A$ of the *monad* $(?, \eta, \mu)$. Here μ_A is given by the proof $z :??A \triangleright \mathtt{store}(x, \mathtt{y}, \mathtt{x}, z)$ and the commutative diagram required by the definition of a monad

$$\begin{tikzcd} ?A & \\ ??A \arrow[u, "\mu_A"] & ?A \arrow[l, "?\eta_A"'] \arrow[ul, "id_{?A}"'] \end{tikzcd}$$

yields the equation in context of Table 6 identifying $id_{?A} : x :?A \triangleright x :?A$ with

$$\cfrac{\cfrac{\begin{array}{c} ?\eta_A : \\ \cfrac{x : A \triangleright [x] :?A}{x : A \triangleright [\,[x]\,] :??A} \end{array}}{z :?A \triangleright \mathtt{store}([\,[x]\,], \mathtt{y}, \mathtt{x}, z) \mid t :??A} \qquad \begin{array}{c} \mu_A : \\ z' :??A \triangleright \mathtt{store}(x', \mathtt{y}', \mathtt{x}', z') \mid \mathtt{y}'(\mathtt{x}'(t)) :?A \end{array}}{z :?A \triangleright \mathtt{store}([\,[x]\,], \mathtt{y}, \mathtt{x}, z), \mathtt{store}(x', \mathtt{y}', \mathtt{x}', t) \mid \mathtt{y}'(\mathtt{x}'(t)) : ?A}$$

$$\text{where } t = \mathtt{y}(\mathtt{x}(z)) :??A$$

Further details are left to the reader.

5 Conclusion.

In order to provide a categorical semantics for *co-intuitionistic logic* - given that as remarked by Tristan Crolard [16] co-exponents in the category **Set** are trivial - we have given a categorical semantics for *intuitionistic multiplicative and exponential* co-intuitionistic linear logic, from which our desired results follows by dualizing J-Y. Girard's embedding of intuitionistic logic into intuitionistic linear logic.

In this task we started from a term assignment to multiplicative co-intuitionistic logic, which has been proposed as an abstract *distributed calculus* dualizing the linear λ calculus [1, 3, 7]: in our view such dualization underlies the translation of the linear λ-calculus into the π-calculus (see [9]). Our *dual distributed calculus* is itself a restriction to a co-intuitionistic consequence relation of Crolard's term assignment to *subtraction* in the framework of the $\lambda\mu$-calculus: to *subtraction introduction* and *elimination* rules and to their β reduction *global operations of binding* and *global substitution* are assigned; these operations may appear as notationally awkward at first sight but are forced on us by the removal of the μ-rule and of the μ-variable abstraction used in Crolard's approach.

Thus our work required a lengthy exercise on well-known results by Benton, Bierman, Hyland and de Paiva[10, 12], with the considerable help given by Blute, Cockett, Seely and Trimble's work [13, 14]. To assess the merits and advantages of our work we need to evaluate the syntax for the exponential rules: here again the *storage* rule may appear notationally quite heavy, but it is a straightforward implementation of the act of storing. Moreover, the role of terms encoding the stored terms - which may be seen as a form of generalized axioms - evokes the rather mysterious notion of *guarded functoriality* appearing in the categorical proof-theory of classical logic [6]. On the other hand the advantages of working in the dual system are completely evident in the treatment of *dereliction* and *contraction*, where the awkward `let` operations and related naturality conditions are replaced by simple operations on lists. Finally, the treatment of *weakening* is also completely standard, thanks also to Blute, Cockett, Seely and Trimble's work [13, 14] on the notion of *rewiring*.

References

1. G. Bellin. On the π calculus and distributed calculi for co-intuitionistic logic, conference paper presented at the LAM-CONCUR'11 Workshop, Aachen, 2011.
2. G. Bellin. Assertions, Hypotheses, Conjectures, Expectations: Rough-sets semantics and proof-theory. In: *Advances in Natural Deduction*, Kluwer 2012
3. G. Bellin. A Term Assignment for Dual Intuitionistic Logic, conference paper presented at the LICS'05-IMLA'05 Workshop, Chicago, IL, June 30, 2005
4. G. Bellin and C. Biasi. Towards a logic for pragmatics. Assertions and conjectures. In: *Journal of Logic and Computation*, 14, 4, 2004, pp. 473-506.

5. G. Bellin. Subnets of Proof-nets in multiplicative linear logic with MIX, *Mathematical Structures in Computer Science* vol.7, pp.663-699 (1997).
6. G. Bellin, M. Hyland, E.Robinson and C. Urban. Categorical Proof Theory of Classical Propositional Calculus *Theoretical Computer Science* Vol. 364, 2, November 2006, pp. 146-165.
7. G. Bellin and A. Menti. On the π-calculus, λP Systems and distributed calculi for co-intuitionistic logic, forthcoming on *Fundamenta Informaticae* 2012.
8. G. Bellin and C. Dalla Pozza. A pragmatic interpretation of substructural logics. In *Reflection on the Foundations of Mathematics (Stanford, CA, 1998)*, Essays in honor of Solomon Feferman, W. Sieg, R. Sommer and C. Talcott eds., ASL, Lecture Notes in Logic, Volume 15, 2002, pp. 139-163.
9. G. Bellin and P. J. Scott. On the Pi-calculus and linear logic, *Theoretical Computer Science* 135, pp. 11-65, 1994.
10. P. N. Benton, G. M. Bierman, J. M. E. Hyland and V. C. V. dePaiva. A term calculus for Intuitionistic Linear Logic. In: *Typed Lambda Calculi and Applications*, Lecture Notes in Computer Science, Volume 664, 1993, pp.75-90.
11. P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models, *Computer Science Logic*, Lecture Notes in Computer Science, 1995, Volume 933, 1995, pp.121-135.
12. G. M. Bierman. On Intuitionistic Linear Logic, PhD Thesis, University of Cambridge Computer Laboratory, Technical Report No. 346, 1994.
13. R. F. Blute, J. R. B. Cockett and R. A. G.Seely. **!** and **?** as tensorial strength, *Mathematical Structures in Computer Science*, 6, pp.313-351, 1996.
14. R. F. Blute, J. R. B. Cockett, R. A. G.Seely and T. H. Trimble. Natural Deduction and Coherence for Weakly Distributive Categories, *Journal of Pure and Applied Algebra*, 113, pp. 229-296, 1996.
15. T. Brauner and V. de Paiva. Cut-elimination for Full Intuitionistic Logic, University of Cambridge Computer Laboratory, Technical Report 395 and BRICS, 1996.
16. T. Crolard. Subtractive logic, in *Theoretical Computer Science* 254,1-2, 2001, pp. 151-185.
17. T. Crolard. A Formulae-as-Types Interpretation of Subtractive Logic. In: *Journal of Logic and Computation*, vol.14(4), 2004, pp. 529-570
18. P-L. Curien. Abstract Machines, Control, and Sequents. G. Barthe, P. Dybjer, L. Pinto, J. Saraiva (Eds.): APPSEM 2000, Springer LNCS 2395, 2002
19. J. Lambek and P. J. Scott. *Introduction to higher order categorical logic* Cambridge University Press, 1986.
20. F. W. Lawvere, Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes. In A. Carboni, M.C. Pedicchio and G. Rosolini (eds.), *Category Theory (Como 1990)*, Springer Lecture Notes in Math. 1488, 1991, pp. 279 - 297.
21. D. Gabbay. Applications of trees to intermediate logics, *Journal of Symbolic Logic*, 37, 1972, pp. 135-38.
22. R. Goré. Dual Intuitionistic Logic Revisited, In TABLEAUX00: Automated Reasoning with Analytic Tableaux and Related Methods, LNAI 1847:252-267, 2000. Springer.
23. M. Hyland and V. de Paiva. Full intuitionsitic linear logic (extended abstract), *Annals of Pure and Applied Logic*, 64, pp. 273-91, 1993
24. M. Maietti, P. Maneggia, V. de Paiva ad E. Ritter. Relating Categorical Semantics for Intuitionistic Linear Logic, *Applied Categorical Structures*, 13, pp. 1-36. 2005.

25. M. Makkai and G. E. Reyes. Completeness results for intuitionistic and modal logic in a categorical setting, *Annals of Pure and Applied Logic*, 72, 1995, pp.25-101.
26. P. Pagliani. Intrinsic co-Heyting boundaries and information incompleteness in Rough Set Analysis. In: Polkowski, L., Skowron, A. (eds.) RSCTC 1998. Springer LNCS, vol. 1424, pp. 123-130, 2009.
27. C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, in *Fundamenta Mathematicae*, 83, 1974, pp. 219-249.
28. C. Rauszer. Applications of Kripke Models to Heyting-Brouwer Logic, in *Studia Logica* 36, 1977, pp. 61-71.
29. G. Reyes and H. Zolfaghari, Bi-Heyting algebras, Toposes and Modalities, in *Journal of Philosophical Logic*, 25, 1996, pp. 25-43.
30. P. Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. Mathematical Structures in Computer Science, 11:207–260, 2001.
31. L. Tranchini. Natural Deduction for Dual Intuitionistic Logic, *Studia Logica* (online version: 20 June 2012)
32. H. Wansing. Constructive negation, implication and co-implication, *Journal of Applied Non-Classical Logics* 18, 2008, pp. 341-364.