# Chu's construction: A proof-theoretic approach

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**Abstract**

The essential interaction between classical and intuitionistic features in the system of linear logic is best described in the language of category theory. Given a symmetric monoidal closed category $\mathcal{C}$ with products, the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by a special case of the Chu construction. The main result of the paper is to show that the intuitionistic translations induced by Girard's trips determine the functor from the free $*$-autonomous category $\mathcal{A}$ on a set of atoms $\{P, P', \ldots\}$ to $\mathcal{C} \times \mathcal{C}^{op}$, where $\mathcal{C}$ is the free monoidal closed category with products and coproducts on the set of atoms $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair $P_O$, $P_I$ in $\mathcal{C}$ for each atom $P$ of $\mathcal{A}$).

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# 1 Preface

An essential aim of linear logic [16] is the study of the dynamics of proofs, essentially normalization (cut elimination), in a system enjoying the good proof-theoretic properties of *intuitionistic* logic, but where the dualities of *classical* logic hold. Indeed *classical linear logic* **CLL** has a denotational semantics and a game-theoretic semantics; proofs are formalized in a sequent calculus, but also in a system of *proof-nets* and in the latter representation cut elimination not only has the strong normalizability property, but is also confluent. Although Girard's main system of linear logic is *classical*, considerable attention in the literature has also been given to the system of *intuitionistic linear logic* **ILL**, where proofs are also formalized in a sequent calculus and in a natural deduction system. A better understanding of the relations between **CLL** and **ILL** is one of the goals to which the present work is intended as a contribution.

The fact that intuitionistic logic plays an important role in the architecture of linear logic is not surprising: as indicated in the introductory section of Girard's fundamental paper [16], a main source of inspiration for the system was its denotational semantics of coherent spaces, a refinement of Scott's semantics for the $\lambda$-calculus. Fundamental decisions about the system **CLL** were made so that **CLL** has a semantics of proofs in coherent spaces in the same way as intuitionistic logic has a semantics of proofs in Scott's domains. But linear logic is not just a refinement of intuitionistic logic, such as **ILL**: there are expectations that **CLL** may tell us something fundamental about classical logic as well, indeed, that through linear logic a deep level of analysis may have been reached from which the "unity of logic" can be appreciated [17]. Therefore the relations between classical and intuitionistic components of linear logic deserve careful investigation.

A natural points of view to look at this issue is *categorical logic.* It has been known for years that monoidal closed categories provide a model for *intuitionistic linear logic*, though a fully adequate formulation of the syntax and of the categorical semantics of **ILL** especially with respect to the exponentials, has required considerable subtlety and effort [4, 5, 6]. It is also well known that $*$-autonomous categories give a model for *classical linear logic* [3]. The appendix to [2] provides a method, due to Barr's student Chu, to construct $*$-autonomous categories starting from monoidal closed ones.

In our proof-theoretic investigation we encounter a special case of Chu's construction, namely $\mathbf{Chu}(\mathcal{C}, \top)$ where $\mathcal{C}$ is a symmetric monoidal closed category with terminal object $\top$. More specifically, given the free $*$-autonomous category $\mathcal{A}$ on a set of objects (propositional variables) $\{P, P', \ldots\}$ and given the symmetric monoidal closed category $\mathcal{C}$ with products, free on the set $\{P_O, P_I, P'_O, P'_I, \ldots\}$ (a pair $P_O$, $P_I$ in $\mathcal{C}$ for each atom $P$ of $\mathcal{A}$), the category $\mathcal{C} \times \mathcal{C}^{op}$ can be given the structure of a $*$-autonomous category by Chu's construction. Indeed, since the dualizing object is the terminal object, $\mathbf{Chu}(\mathcal{C}, \top)$ is just $\mathcal{C} \times \mathcal{C}^{op}$ and the pullback needed to internalize the homsets is in fact a product. Here the tensor product $(X, X^{op}) \otimes (Y, Y^{op})$ must be an object of the form $(X \otimes Y, (X \multimap Y^{op}) \times (Y \multimap X^{op}))$ and the identity of the tensor must be $(\mathbf{1}, \top)$. Dually, the *par* $(X, X^{op}) \wp (Y, Y^{op})$ is defined as $((X^{op} \multimap Y) \times (Y^{op} \multimap X), X^{op} \otimes Y^{op})$ and the identity of the *par* must be $(\top, \mathbf{1})$. Now since $\mathcal{A}$ is free, there is a functor $F$ of $*$-autonomous categories from $\mathcal{A}$ to $(\mathcal{C} \times \mathcal{C}^{op})$ taking $P$ to $(P_O, P_I)$. This is well-known, but so far no familiar construction had been shown to correspond to the functor $F$ given by the abstract theory. The main contribution of this paper is to show that a familiar proof-theoretic construction, namely *Girard's trips* [16] on a proof-net, represent the action of such a functor on the morphisms of $\mathcal{A}$. Of course one could state the same result using Danos-Regnier graphs, as it was done in [8], but as we shall see a simpler definition of orientations is possible in terms of Girard's trips.

The key idea is simple enough and may be illustrated as a logical translation of formulas and proofs in **CMALL** into formulas and proofs in **IMALL**. In the translation a **CMALL** sequent $S\colon \vdash \Gamma, A$ becomes *polarized*: a selected formula-occurrence $A$ is mapped to a *positive* formula-occurrence $A_O$ in the succedent of an intuitionistic sequent $S'$ (the *output* part of a logical computation); all other formula-occurrences $C$ in $\Gamma$ are mapped to *negative* $C_I$ in the antecedent of $S'$ (the *input* part). The polarized occurrences of an atom $A$ become $A_O$, $A_I$, just two copies of $A$. Negation changes the polarity. For other complex polarized formulas, the polarization of the immediate subformulas is uniquely determined – for instance, $(A \wp B)_I$ becomes $A_I \otimes B_I$ – except in the cases of $(A \wp B)_O$ and $(A \otimes B)_I$. In these cases we take the product (logically, the *with*) of two possible choices (the "switches" in a proof-net): for instance, $(A \wp B)_O$ is encoded as $(A_I \multimap B_O) \& (B_I \multimap A_O)$. The intuitive motivation is clear: $A \wp B$ has a reading simultaneously as the internalization of the function space $\mathrm{Hom}_{\mathcal{A}}(A^{\perp}, B)$ and of the function space $\mathrm{Hom}_{\mathcal{A}}(B^{\perp}, A)$. The fact that the translation is functorial here means, roughly, that it is defined independently on the formulas (objects) and on the proofs (morphisms) and that it admits the rule of Cut (composition of morphisms); it is also compatible with cut-elimination. In this form the result can be easily proved within the formalisms of the sequent calculi for **CMALL** and **IMALL**. However, when we ask questions about the *faithfulness* and *fullness* of such a functor, thus also asking questions about the identity of proofs in linear logic, we find it convenient to consider the more refined syntax of proof-nets.

On the other hand, proof-nets are also useful to highlight the geometric aspect of certain logical properties; indeed ideas related to the present result have already proved quite useful in the study of what is sometimes called the *géometrie du calcul* (*geometry of computations*). Our own investigation has been motivated by the desire to understand and clarify the notion of a proof-net and the present result appears to reward many collective efforts in this direction. Given a proof-structure, i.e., a directed graph where edges are labeled with formulas, a *correctness criterion* characterizes those proof-structures which correspond to proofs in the sequent calculus. Girard's original condition (*"there are no short trips"*) [16] is *exponential* in time on the size of the proof-structure, but other *quadratic* criteria were found soon after (among others, one was given in [7]). Thus it is natural to ask *what additional information is contained in the construction of Girard's trips other than the correctness of a proof-structure*. The beginning of an answer came in 1992, when Jacques van de Wiele and the author, inspired by Danos' notion of a *pure net*, defined the *trip translation*: every Girard's trip on a cut-free proof-net corresponds to a derivation in the fragment of intuitionistic linear logic with *times* and *linear implication.* But the significance of this result seemed limited by the fact that the treatment of cut was quite cumbersome and the result itself did not seem to extend beyond the multiplicative fragment. A better understanding of its significance – and, as we hope, the possibility of its generalization – has come only from an explicit effort to formulate the trip translation as a *functorial* operation. In this way it became evident that *classical multiplicative* linear logic has to be related to *intuitionistic multiplicative and additive* linear logic and the categorical result followed, for which we gratefully acknowledge the influence and the support of Martin Hyland.

There is a conspicuous literature on Chu's spaces and linear logic. Moreover Chu's construction is related to many other more concrete semantics that yield *full completeness* results for fragments of linear logic, from R. Loader's *Linear logical Predicates* [22] to various game-theoretic semantics (cfr. [26]). Clearly this is not the place to survey such a body of literature. It is impossible however not to mention the work of V. Pratt, who has advocated this direction of research for a long time (see, e.g., [25]) and has recently obtained (with Plotkin *et al.* [11]) a full completeness result for multiplicative linear logic (without units) with respect to Chu spaces. Chu's construction $\mathbf{Chu}(\mathcal{H}, \top)$ where $\mathcal{H}$ is a Heyting algebra, is also explicitly used by Anna Patterson in her thesis ([24], section 6.7.), to show that the *algebra of constructive duality* is a model of **CLL** with Mix (we are grateful to an anonymous referee for this reference).

Among the researchers who have worked on proof-nets and developed ideas related to the trip translation, we should mention F. Lamarche, who introduced the notion of *essential net* for intuitionistic linear logic [20] in the context of his research on the game-theoretic semantics [21]. Arnaud Fleury has considered trips and intuitionistic translations in a non-commutative context with explicit exchange rule [13], giving one of the most interesting and least understood developments in this area. Already in 1992-93 the consideration of trips as translations from classical to intuitionistic linear logic had suggested the possibility of giving a *linear time* correctness condition for proof-nets: after all, just one unsuccessful trip suffices to discard a proof-structure as incorrect, and just one successful trip, if appropriately translated to an intuitionistic derivation, suffices to test the correctness of a proof-net. However only in 1999 Murawski and Ong [23] were able to prove such a conjecture, making essential use of a result of Gabow and Tarjan [15].

When the languages and the aims of different scientific communities meet in a new theory and new structures are identified, the conceptual architecture may look different from the different points of view and it may be hard to say which structures are fundamental. From the point of view of categorical logic Chu's construction seems to suggest a fundamental status to monoidal closed categories with respect to the $*$-autonomous ones. However, such a view must accompanied by the warning that the correspondence established by our interpretation of Chu's construction is not an isomorphism, as the functor $F$ is *not full*, and its faithfulness at the moment is only a conjecture. Research towards a refinement of the present result, in particular with respect to the units and the additives is in progress, as well as towards its extension to the exponentials. Finally, further work is needed to spell out intriguing analogies between our version of Chu's construction and the *game theoretic semantics* of linear logic.

# 2 The trip translation

In this section, after the basic formal definitions we present our functorial translation from the sequent calculus for **CMALL** to that for **IMALL**, we state the categorical result and sketch the proof.

## 2.1 Languages, intuitionistic and classical MALL

The syntax of propositional *classical* Linear Logic **CLL** is given in Girard [16]: formulas are in "negation normal form", i.e., they are built from propositional constants $\mathbf{1}$, $\perp$, $\top$ and $\mathbf{0}$, atoms $P_i$ and negations of atoms $(P_i)^\perp$ using the connectives $\otimes$, $\wp$, $\&$ and $\oplus$ and the exponential operators $\mathbf{!}$ and $\mathbf{?}$; linear negation for nonatomic formulas is defined and linear implication is also defined, see in Table 0.

$$\bot^\bot =_d \mathbf{1} \quad \mathbf{1}^\bot =_d \bot \quad (A \otimes B)^\bot =_d A^\bot \wp B^\bot \quad (A \wp B)^\bot =_d A^\bot \otimes B^\bot$$

$$\top^\bot =_d \mathbf{0} \quad \mathbf{0}^\bot =_d \top \quad (A \& B)^\bot =_d A^\bot \oplus B^\bot \quad (A \oplus B)^\bot =_d A^\bot \& B^\bot$$

$$(\mathbf{!}A)^\bot =_d \mathbf{?}(A^\bot) \qquad (\mathbf{?}A)^\bot =_d \mathbf{!}(A^\bot)$$

$$A \multimap B =_d A^\bot \wp B$$

Table 0: Definition of linear negation and linear implication.

**CMALL** [**CMLL**] is the fragment of **CLL** without exponentials [without exponentials and additives]; **CMLL**$^-$ is **CMLL** without propositional constants.

Recall that the sequent calculus for propositional **CMALL** is defined by the axioms and rules given in Table 1:

**identity rules**

*logical axiom:* $\vdash A^\bot, A$ *cut:* $\dfrac{\vdash \Gamma, A^\bot \quad \vdash \Delta, A}{\vdash \Gamma, \Delta}$

**structural rules**

*exchange:* $\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$

**logical rules**

*one:* $\vdash \mathbf{1}$ *nil:* $\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}$ *times:* $\dfrac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$ *par:* $\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$

*true:* $\vdash \Gamma, \top$ *with:* $\dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$ *plus:* $\dfrac{\vdash \Gamma, A_i}{\vdash \Gamma, A_0 \oplus A_1}$ for $i = 0, 1$.

Table 1: The sequent calculus **CMALL**.

The propositional language of *intuitionistic* Linear Logic **ILL** is built from a set of propositional atoms $\{P_O,\ P_I,\ P'_O,\ P'_I,\ \ldots\}$, and the propositional constants $\mathbf{1}$, $\top$ and $\mathbf{0}$, using the connectives $\multimap$ (*linear implication*) and $\otimes$, & and $\oplus$ and the exponential $\mathbf{!}$. (We take the point of view that there is no symbol "$\perp$" for "multiplicative falsity" in **ILL**. Thus $P_O$ and $P_I$ may be regarded as "positive and negative atoms", respectively). Again **IMALL** is the fragment of **ILL** without exponentials, etc.

The sequent calculus for propositional **IMALL** has the axioms and rules given in Table 2.

**identity rules**

*logical axiom:*

$$A \vdash A$$

*cut:*

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

**structural rules**

*exchange:*

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

**logical rules**

*R one:*

$$\vdash \mathbf{1}$$

*L one:*

$$\frac{\Gamma \vdash B}{\Gamma, \mathbf{1} \vdash B}$$

*R times:*

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

*L times:*

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

*R implication:*

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

*L implication:*

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{A \multimap B, \Gamma, \Delta \vdash C}$$

*R true:*

$$\Gamma \vdash \top$$

*L zero:*

$$\mathbf{0}, \Gamma \vdash A$$

*R with:*

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$$

*L with:*

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_0 \& A_1 \vdash B}$$

for $i = 0, 1$.

*plus:*

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \oplus A_1}$$

for $i = 0, 1$.

*L plus:*

$$\frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \oplus B, \Gamma \vdash C}$$

Table 2: The sequent calculus **IMALL**.

When a derivation $\pi$ (in **CMALL** or **IMALL**) ends with a *cut* inference and the immediate subderivations are $\pi_1$, $\pi_2$, we use the notation $(\pi_1|\pi_2)$ for $\pi$.

## 2.2 The functorial trip translation for MALL

**Definitions 1.** (*Trip translation*) (i) The *trip translation* maps formulas of **CMALL** to pair of formulas of **IMALL**: $A \mapsto (A_O, A_I)$. The formulas of **CMALL** are first *polarized* and then translated into **IMALL** formulas. To say that a formula $A$ is polarized is to say that it is regarded either as an *output* $A_O$ (*positive polarization*) or as an *input* $A_I$ (*negative polarization*).

(ii) The *trip translation* maps polarized atoms $P_O$, $P_I$ to atomic formulas of **IMALL** (also denoted by $P_O$, $P_I$, respectively). For polarized constants and polarized complex formulas the trip translation is defined inductively according to the table in Table 3.

$$
\begin{array}{rcl@{\qquad}rcl}
(P^\perp)_O & = & P_I, \quad \text{for } P \text{ atomic}; & (P^\perp)_I & = & P_O; \\
\mathbf{1}_O & = & \mathbf{1} \qquad \mathbf{1}_I \ = \ \top & \bot_I & = & \mathbf{1} \qquad \bot_O \ = \ \top; \\
(A \otimes B)_O & = & A_O \otimes B_O, & (A \wp B)_I & = & A_I \otimes B_I; \\
(A \otimes B)_I & = & (A_O \multimap B_I) \ \& \ (B_O \multimap A_I); & (A \wp B)_O & = & (A_I \multimap B_O) \ \& \ (B_I \multimap A_O); \\
\top_O & = & \top \qquad \top_I \ = \ \mathbf{0}; & \mathbf{0}_O & = & \mathbf{0} \qquad \mathbf{0}_I \ = \ \top; \\
(A \& B)_O & = & A_O \& B_O, & (A \& B)_I & = & A_I \oplus B_I; \\
(A \oplus B)_O & = & A_O \oplus B_O, & (A \oplus B)_I & = & A_I \& B_I
\end{array}
$$

Table 3: Functorial trip translation, the propositions.

(iii) A *pointed sequent* $\vdash \Gamma, \mathbf{A}$ is a sequent with a selected formula occurrence, i.e., with a *switch* choosing one of its formulas. The chosen formula will be written in **boldface**.

(iv) The polarization of a sequent $\vdash \Gamma, \mathbf{A}$ is defined as follows: the selected formula $\mathbf{A}$ is regarded as an *output* $A_O$, all other formulas $C$ in $\Gamma$ are regarded as *inputs* $C_I$; we write $\Gamma_I$ to indicate this fact.

(v) The *trip translation* maps a pointed sequent $\mathrm{S} = \vdash \Gamma, \mathbf{A}$ of **CMALL** to a sequent $S' = \Gamma_I \vdash A_O$ of **IMALL**, where $\Gamma_I$ and $A_O$ are translated as in (ii). Notice that since a sequent $\vdash \Delta$ may be regarded as the *par* of all its formulas, by Table 3 the translation $\vdash (\wp(\Delta))_O$ of the pointed sequent $\vdash \wp(\boldsymbol{\Delta})$ is the product (*with*) of the translations of all the *"pointings"* of $\Delta$.

(vi) The *trip translation* maps sequent derivations of **CMALL** to sequent derivations in **IMALL** according to the definition in Table 4.

**Proposition 1.** *For any formula $A$ of* **CMALL**, *the translation satisfies* $(A^{\perp})_O = A_I$ and $(A^{\perp})_I = A_O$. *Therefore the translation of the* cut-*rule is well-defined.*

**Proof.** By induction on the logical complexity of $A$. ∎

**Theorem 1.** (i) *The trip translation maps a* **CMALL** *proposition $A$ to a pair of* **IMALL** *propositions $(A_O, A_I)$ and a* **CMALL** *derivations $\pi$ of $\vdash \Gamma$ to an* **IMALL** *derivation $\tilde{\pi}$ of $\vdash (\wp(\Gamma))_O$. If $\vdash \Gamma', \mathbf{A}$ is a pointing of $\vdash \Gamma$, then a branch of $\tilde{(\pi)}$ contains a derivation $\pi'$ of $\Gamma'_I \vdash A_O$.*

(ii) *The trip translation is functorial in the sense it preserves the cut rule. Namely, given a derivation $(\pi_1|\pi_2)$ ending with a cut*

$$\frac{\begin{array}{c}\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash A^{\perp}, \Delta\end{array}}{\vdash \Gamma, \Delta}$$

*and a pointing $\vdash \Xi, \mathbf{C}$ of $\vdash \Gamma, \Delta$ there is a unique pair of pointings of $\vdash \Gamma, A$ and of $\vdash A^{\perp}, \Delta$ such that*

$$(\pi_1|\pi_2)' = (\pi'_1|\pi'_2)$$

(iii) *Moreover, if $S$ is translated to $S'$ and $S$ reduces to $S_0$ by cut-elimination, then there exists $S'_0$ which is the translation of $S_0$ and $S'$ reduces to $S'_0$.* ∎

**Remarks 1.** (i) The concise notation of part (*ii*) of Theorem 1 can be spelt out as follows. Here $(\pi_1|\pi_2)'$ is the trip translation of $(\pi_1|\pi_2)$ restricted to the given pointing $\vdash \Xi, \mathbf{C}$ of $\vdash \Gamma, \Delta$. Depending on whether $C$ is in $\Gamma$ or in $\Delta$ we have either the pair of pointings $\{\vdash A, \Gamma', \mathbf{C}, \quad \vdash \Delta, \mathbf{A}^{\perp}\}$ or the pair $\{\vdash \Gamma, \mathbf{A}, \quad \vdash A^{\perp}, \Delta', \mathbf{C}\}$. Moreover, $\pi'_1$ and $\pi'_2$ are the trip translations of $\pi_1$ and $\pi_2$ restricted to the appropriate pointings, i.e., either $\pi'_1$ is a derivation of $A_I, \Gamma'_I \vdash C_O$ and $\pi'_2$ a derivation of $\Delta_I \vdash A_I$, or $\pi'_1$ is a derivation of $\Gamma_I \vdash A_O$ and $\pi'_2$ a derivation of $A_O, \Delta' \vdash C$. The theorem says that $(\pi_1|\pi_2)'$ is the same as the derivation $(\pi'_1|\pi'_2)$ which is obtained by applying *cut* to $\pi'_1$ and $\pi'_2$.

(ii) Notice that there is no anomaly in the $\perp_O$ rule. One pointing of $\vdash \Gamma, \perp$ selects $\perp$ and $\perp_O = \top$, thus one branch in the derivation $\tilde{\pi}$ of $\vdash (\wp(\Gamma, \perp))_O$

$$\vdash P^\perp, \mathbf{P} \quad\Longrightarrow\quad P_O \vdash P_O \qquad\qquad \vdash \mathbf{P}^\perp, P \quad\Longrightarrow\quad P_I \vdash P_I$$

$$cut : \frac{\vdash \Gamma, \mathbf{A} \qquad \vdash A^\perp, \Delta, \mathbf{C}}{\vdash \Gamma, \Delta, \mathbf{C}} \quad\Longrightarrow\quad cut : \frac{\Gamma_I \ \vdash \ A_O \qquad A_I^\perp, \Delta_I \ \vdash \ C_O}{\Gamma_I, \Delta_I \ \vdash \ C_O}$$

$$\otimes_O : \frac{\vdash \Gamma, \ \mathbf{A} \qquad \vdash \Delta, \ \mathbf{B}}{\vdash \Gamma, \Delta, \ \mathbf{A} \otimes \mathbf{B}} \quad\Longrightarrow\quad \otimes - R : \frac{\Gamma_I \ \vdash \ A_O \qquad \Delta_I \ \vdash \ B_O}{\Gamma_I, \Delta_I \ \vdash A_O \otimes B_O}$$

$$\otimes_I : \frac{\vdash \Gamma, \ \mathbf{A} \qquad \vdash B, \Delta, \mathbf{C}}{\vdash \Gamma, A \otimes B, \ \Delta, \ \mathbf{C}} \quad\Longrightarrow\quad \multimap - L : \frac{\dfrac{\Gamma_I \ \vdash \ A_O \qquad B_I, \ \Delta_I \ \vdash \ C_O}{\Gamma_I, A_O \multimap B_I, \Delta_I \ \vdash \ C_O}}{\Gamma_I, (A_O \multimap B_I) \& (B_O \multimap A_I), \Delta_I \ \vdash \ C_O}$$

$$\wp_O : \frac{\vdash \Gamma, A, \ \mathbf{B}}{\vdash \Gamma, \ \mathbf{A} \wp \mathbf{B}} \quad \text{and} \quad \frac{\vdash \Gamma, \ \mathbf{A}, \ B}{\vdash \Gamma, \mathbf{A} \wp \mathbf{B}} \quad\Longrightarrow\quad \multimap - R : \frac{\dfrac{\Gamma_I, A_I \ \vdash \ B_O}{\Gamma_I \ \vdash \ A_I \multimap B_O} \qquad \dfrac{\Gamma_I, B_I \ \vdash \ A_O}{\Gamma_I \ \vdash \ B_I \multimap A_O}}{\Gamma_I \ \vdash \ (A_I \multimap B_O) \& (B_I \multimap A_O)}$$

$$\wp_I : \frac{\vdash A, B, \Gamma, \ \mathbf{C}}{\vdash A \wp B, \Gamma, \ \mathbf{C}} \quad\Longrightarrow\quad \otimes - L : \frac{A_I, B_I, \Gamma_I \ \vdash \ C_O}{A_I \otimes B_I, \Gamma_I \ \vdash \ C_O}$$

$$\vdash \mathbf{1} \quad\Longrightarrow\quad \vdash \mathbf{1}$$

$$\perp_I : \frac{\vdash \Gamma, \mathbf{A}}{\vdash \perp, \Gamma, \ \mathbf{A}} \quad\Longrightarrow\quad \mathbf{1} - L : \frac{\Gamma_I \ \vdash \ A_O}{\mathbf{1}, \Gamma_I \ \vdash \ A_O}$$

$$\perp_O : \frac{\vdash \Gamma}{\vdash \Gamma, \ \boldsymbol{\perp}} \quad\Longrightarrow\quad \Gamma_I \vdash \top$$

$$\vdash \Gamma, \boldsymbol{\top} \quad\Longrightarrow\quad \Gamma_I \vdash \top \qquad\qquad \vdash \top, \Gamma, \ \mathbf{C} \quad\Longrightarrow\quad \mathbf{0}, \Gamma_I \vdash C_O$$

$$\&_O : \frac{\vdash \Gamma, \ \mathbf{A} \qquad \vdash \Gamma, \ \mathbf{B}}{\vdash \Gamma, \ \mathbf{A} \& \mathbf{B}} \quad\Longrightarrow\quad \& - R : \frac{\Gamma_I \ \vdash \ A_O \qquad \Gamma_I \ \vdash \ B_O}{\Gamma_I, \Delta_I \ \vdash A_O \& B_O}$$

$$\&_I : \frac{\vdash A, \Gamma, \ \mathbf{C} \qquad \vdash B, \Gamma, \ \mathbf{C}}{\vdash A \& B, \Gamma, \ \mathbf{C}} \quad\Longrightarrow\quad \oplus - L : \frac{A_I, \Gamma_I \ \vdash \ C_O \qquad B_I, \Gamma_I \ \vdash \ C_O}{A_I \oplus B_I, \Gamma_I \ \vdash C_O}$$

$$\oplus_O : \frac{\vdash \Gamma, \ \mathbf{A_i}}{\vdash \Gamma, \mathbf{A_0} \oplus \mathbf{A_1}} \quad\Longrightarrow\quad \oplus - R : \frac{\Gamma_I \ \vdash \ (A_i)_O}{\Gamma_I \vdash (A_0)_O \oplus (A_1)_O}$$

$$\oplus_I : \frac{\vdash \Gamma, A_i, \ \mathbf{C}}{\vdash \Gamma, A_0 \oplus A_1, \ \mathbf{C}} \quad\Longrightarrow\quad \& - L : \frac{\Gamma_I, (A_i)_I \ \vdash \ C_O}{\Gamma_I, (A_0)_I \& (A_1)_I \ \vdash \ C_O}$$

Table 4: Functorial trip translation, the proofs.

is given by the *axiom* $\Gamma_I \vdash \top$. Such a branch adds no information in addition to that contained in the derivation of $\vdash (\wp(\Gamma))_O$ which is also contained in the *other* branches of $\tilde{\pi}$.

**Facts.** (i) *To the trip translation there corresponds an obvious map in the opposite direction*, let us write it as $(\ )^c$; in proof-theoretic terms it amounts to *regarding* **IMALL** *as a fragment of* **CMALL**, namely:

- writing formulas in *"negation normal form"*, using De Morgan laws as in Table 0 and then rewriting $(P_O)^c = (P_I^{\perp})^c = P$, $(P_I)^c = (P_O^{\perp})^c = P^{\perp}$ for propositional letters $P$;
- writing proofs in the sequent calculus with *"right-hand sequents"* only.

(ii) Unfortunately, in general it is not true that $((A)_O)^c = A$, e.g., let $A$ be $\perp$.

# 3 Chu's Construction

We follow the categorical semantics for **IMLL** in [5]:

**Theorem 2.** *Let $A$ be the free $*$-autonomous category on a set of objects $\{P, P', \ldots\}$ and let $C$ be the symmetric monoidal closed category with products, free on the set of objects $\{P_O, P_I, P'_O, P'_I, \ldots\}$* (a pair $P_O$, $P_I$ for each atom $P$ of $A$).

(i) *We can give $C \times C^{op}$ the structure of a $*$-autonomous category thus:*

$$(X_O, X_I) \otimes (Y_O, Y_I) =_{df} (X_O \otimes Y_O, (X_O \multimap Y_I) \times (Y_O \multimap X_I))$$

$$\textit{with unit} \quad (\mathbf{1}, \top) \quad \textit{and involution} \quad (X_O, X_I)^{\perp} =_{df} (X_I, X_O)$$

*where $\mathbf{1}$ is the unit of $\otimes$ in $C$ and $\top$ the terminal object of $C$.*

*Therefore there is a functor $F$ from $A$ to $(C \times C^{op})$ sending an object $P$ to $(P_O, P_I)$. If $\pi : I \to \wp(\Gamma)$ is a morphism of $A$ represented as a proof-net $\mathcal{R}$ with conclusions $\Gamma$, then the morphism $(\mathbf{1}, \top) \to (\wp(\Gamma)_O, \wp(\Gamma)_I)$ is given by all the* Girard's trips *on $\mathcal{R}$, in the precise sense spelt out in* Proposition 3 *below.*

(ii) *If in addition $C$ has also coproducts, then $(C \times C^{op})$ has also products:*

$$(X_O, X_I)\&(Y_O, Y_I) =_{df} (X_O\&Y_O, X_I \oplus Y_I).$$

*The functor $F$ preserves also the structure of the products.*

**Sketch of proof.** In $C \times C^{op}$ define $X \wp Y =_{df} (X^\perp \otimes Y^\perp)^\perp$ and $X \multimap Y =_{df} X^\perp \wp Y$ and show that $X \multimap Y$ gives the closed structure for $\otimes$. By the definitions

$$(V, Y) \multimap (W, Z) = ((V \multimap W) \times (Z \multimap Y), V \otimes Z)$$

We exhibit a natural bijection of hom-sets

$$(U, X) \otimes (V, Y) \to (W, Z) \qquad \sim \qquad (U, X) \to (V, Y) \multimap (W, Z)$$

Indeed a map in $(U, X) \otimes (V, Y) \to (W, Z)$ consists of a map $a : U \otimes V \to W$ and of a pair of maps $b : Z \to U \multimap Y$, $c : Z \to V \multimap X$; a map in $(U, X) \to (V, Y) \multimap (W, Z)$ consists of a pair of maps $a' : U \to V \multimap W$, $b' : U \to Z \multimap Y$ and of a map $c' : V \otimes Z \to X$. But $a \mapsto a'$ and $c \mapsto c'$ by the natural bijections given by the symmetric monoidal closed structure of $C$. Similarly, there is a natural bijection $\eta$ between $Z \to U \multimap Y$ and $Z \otimes U \to Y$ and also a natural bijection $\eta'$ between $Z \otimes U \to Y$ and $U \to Z \multimap Y$; composing $\eta'$ and $\eta$ we get $b \mapsto b'$. Therefore $\multimap$ is a right adjoint for $\otimes$ in $C \times C^{op}$.

To see how the action of the functor is given by Girard's trips in the case of **CMLL** *without units*, the proof in [8], pp. 37-44, (briefly reviewed in the following section) shows that a Danos-Regnier graph on a cut-free proof-net $\mathcal{R}$ with conclusions $\Gamma, \mathbf{C}$ (with $\mathbf{C}$ the selected conclusion) determines an orientation (polarization) $\delta$ of the formulas in the proof-net and a *reduced translation* of $\mathcal{R}$ into a cut-free derivation $\pi^\delta$ of $\Gamma^\delta_I \vdash C^\delta_O$ *in* **IMLL**$^-$. (The same result holds if we start with a Girard trip, as indicated below; indeed each Girard trip uniquely determines a Danos-Regnier graph.)

Now it is easy to see that a derivation $\pi^\delta$ can also be obtained as follows: consider the **IMALL** derivation $F(\mathcal{R})$ given by the trip translation of (a sequentialization of) $\mathcal{R}$ and remove all &-*left* and &-*right* inferences, modifying the formulas in the derivation accordingly. The result is a family $\mathcal{F}$ of **IMLL**$^-$ derivations, a pair of derivations for each &-*right* application in

$F(\mathcal{R})$. Every derivation $\pi^\delta$ determined by some Danos-Regnier switching is equivalent *modulo permutations of inferences* with one derivation in $\mathcal{F}$ and, moreover, every derivation in $\mathcal{F}$ is equivalent to a derivation $\pi^\delta$ for some $\delta$ induced by a Danos-Regnier switching. (In the terminology of Proposition 4 below we have: $\mathcal{F} = \{\sigma_{s_1\mathcal{R}}[F(\mathcal{R})], \ldots, \sigma_{s_k\mathcal{R}}[F(\mathcal{R})]\}$ where $s_1, \ldots, s_k$ are all the Danos-Regnier switchings of $\mathcal{R}$.)

A simpler formulation of the above result could be given by mapping proof-nets for *classical* $\mathbf{MLL}^-$ to F. Lamarche and A. Tan's proof-nets for *intuitionistic* $\mathbf{MLL}^-$ (see [20, 26]). For an extension to **CMLL** *with units* and to **CMALL**, see Remarks 2 and 6 below. ∎

**Remarks 2.** (i) The functor $F$ is *not full*. For instance, $F(\perp) = (\top, \mathbf{1})$, and there is a morphism $F(\mathbf{1}) \to F(\perp)$ in $C \times C^{op}$, namely, the map $t \times t'$ : $(\mathbf{1}, \top) \to (\top, \mathbf{1})$, where $t' : \top \to \mathbf{1} =_{df} t : \mathbf{1} \to \top$; however, the free category $A$ does not have a morphism $\mathbf{1} \to \perp$, i.e., $\perp$ is not provable in **MLL**. The system $\mathbf{CLL} + \perp = \mathbf{1}$ has been studied, e.g., in [14], and perhaps this extension of linear logic deserves further consideration, but it is not the answer to our concern here. The task is rather to characterize a subcategory of $C \times C^{op}$ for which the functor $F$ is full.

(ii) Is the functor $F$ is faithful? This question raises the issue of the identity of proofs in linear logic. Proof-nets provide a solution to this problem for the multiplicative fragment without units, because in this fragment proof-nets represent sequent derivations up to permutations of inferences and the Church-Rosser property holds strictly (see the next section). Notice that in [5] cut reductions in **IMALL** correspond to $\beta\eta$ equality of the terms which express the maps in the free symmetric monoidal closed category and commutative conversions correspond to natural isomorphisms between them. If in $C \times C^{op}$ we consider terms up to $\beta\eta$ equivalence, then faithfulness is obtained by making the corresponding assumption about $A$, i.e., by stipulating that the morphisms of $A$ are represented by proof-nets *up to cut-elimination and $\eta$-equivalence*. For an extension to this result to **MLL** with units, see the note in section 4.4. However, in the case of proof-nets for **MALL** the notion of identity of proofs is not well-understood, and thus the issue of faithfulness must be left to further research.

# 4 Proof-nets, Trips and translations

In this section we summarize some results about proof-nets that illustrate the geometric connections between Girard's trips and translations into fragments of intuitionistic linear logic.

## 4.1 Proof-nets: basic definitions

**Definitions 2.** (i) Proof-structures for **CMALL** are directed graphs with at least one external point and where each vertex is typed and has the form indicated in Figure 1. The dashed line in a $\perp$-vertex is called an *attachment*. When a $\perp$-link is attached to an edge $A$, we may think of the attachment as resulting from an axiom $\perp$, $\mathbf{1}$ linked to a *times* link $\mathbf{1} \otimes A$, i.e., as an application of the isomorphism $A \sim A \otimes \mathbf{1}$.

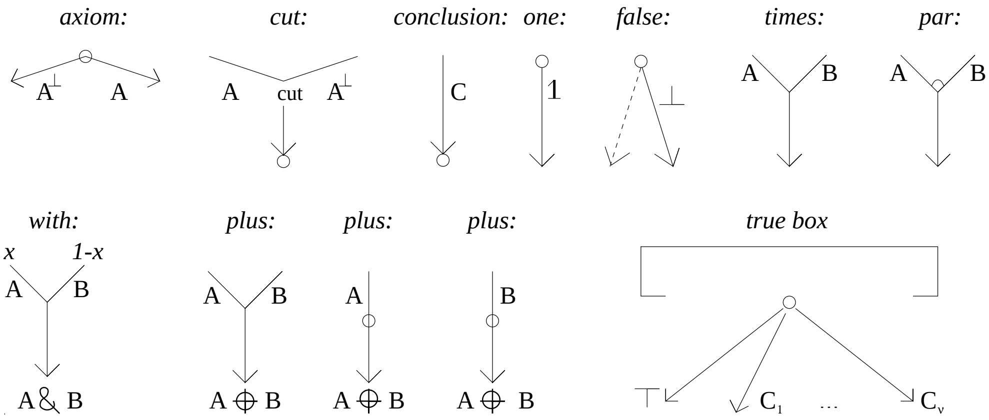


Figure 1: Links

(ii) On proof-structures for **CMLL** *Girard's trips* are defined according to the drawings in Figure 2; the choice of the form of a trip at a *par* link is called a *left* or *right switch*. *Proof-nets* for **MLL** are proof-structures satisfying Girard's *no-short-trip* condition, namely, that for every switching of the *par* links and every conclusion $C$, the trip starting at $C$ returns to $C$ after visiting each edge precisely twice (cf. [16])

(iii) Perhaps more familiar is the equivalent characterization of proof-nets for $\mathbf{MLL}^{-}$ in terms of *Danos-Regnier graphs* [10] (which is readily extended

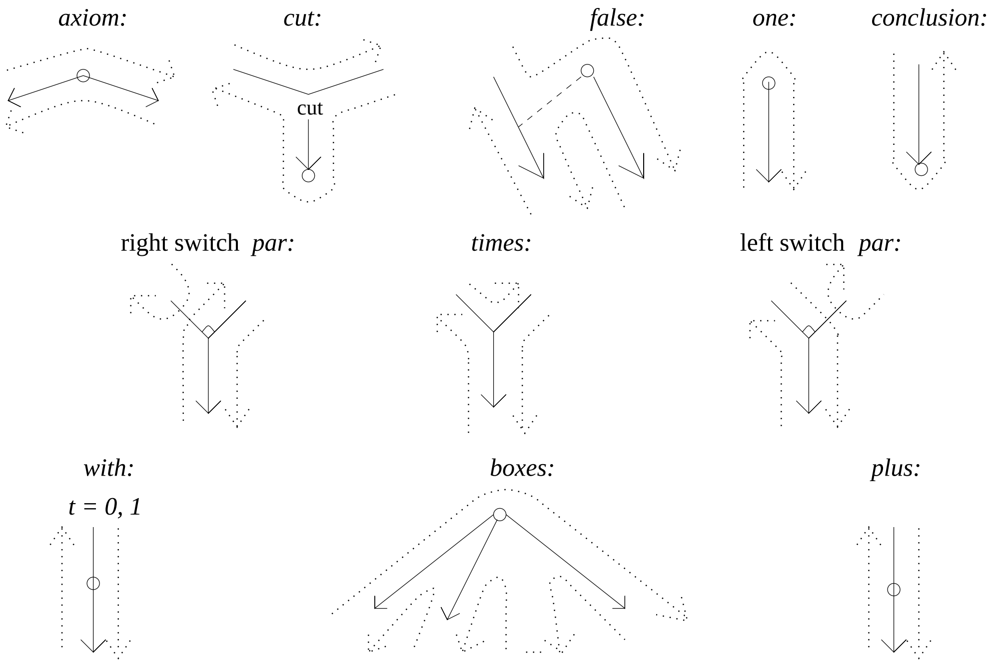


Figure 2: Girard's trips

to **MLL** *with units* using attachments as above). Given a proof-net R for $\mathbf{MLL}^-$, *a switching* $s$ *of* $\mathcal{R}$ *in the sense of Danos-Regnier* is an assigment to each *par* link in $\mathcal{R}$ of a choice of the *left* or *right* premise and, moreover, a "pointing" of the conclusions $\Gamma$ of $\mathcal{R}$. Given a proof-net $\mathcal{R}$ and a switching $s$ of it, the *Danos-Regnier* graph $s\mathcal{R}$ (determined by $s$) is the graph resulting from $\mathcal{R}$ by removing the edge which enters a *par* link from the premise which is *not* selected by $s$. The standard correctness condition for $\mathbf{MLL}^-$ proof-nets is the following: a proof-structure $\mathcal{R}$ for $\mathbf{MLL}^-$ is a proof-net if for every switching $s$ the Danos-Regnier graph $s\mathcal{R}$ is acyclic and connected (an undirected tree).

In the case of **CMALL** we have *true boxes* which behave like $n+1$-ary axioms. Moreover, a *boolean valued polynomial* is associated with each edge, a distinct *boolean variable* $x$ is associated with each *with* link, $x$ and $1-x$ being added to the polynomials associated with the left and right premise of the *with* link in question. The polynomial of the conclusion of a link is the sum of the polynomials of the premises. All the conclusions of the proof-net must have 1 as associated polynomial. A proof structure is *sliced* by substituting arbitrary boolean values for the variables and erasing the edges whose polynomial evaluates to 0; in a sliced proof-structure additive links are all unary. (For a more precise definition, of additive proof-nets, see Girard [18].) All said, a **CMALL** proof-structure is a proof-net if for every evaluation of the polynomials, the resulting slice has no short trip.

What matters here is that the following theorem can be proved:

**Girard's Theorem.** *There exists a 'context-forgetting' map* $(\ )^-$ *from sequent derivations in* **MALL** *to proof-nets with the following properties:*

(a) *Let* $\mathrm{d}$ *be a sequent derivation of* $\vdash \Gamma$*; then* $(\mathrm{d})^-$ *is a proof-net with conclusion* $\Gamma$;

(b) (Sequentialization) *If* $\mathcal{R}$ *is a proof-net with conclusion* $\Gamma$*, then there is a sequent calculus derivation* $\mathrm{d}$ *of* $\vdash \Gamma$ *such that* $\mathcal{R} = (\mathrm{d})^-$.

About proof-nets for $\mathbf{MLL}^-$ more can be proved (see [9], Theorem 2):

**Permutability of Inferences Theorem.** *Let* $\mathrm{d}$ *and* $\mathrm{d'}$ *be a pair of derivations of the same sequent* $\vdash \Gamma$ *in* $\mathbf{MLL}^-$. *Then* $(\mathrm{d})^- = (\mathrm{d'})^-$ *if and only if there exists a sequence of derivations* $\mathrm{d} = \mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_n = \mathrm{d'}$ *such that* $\mathrm{d}_i$ *and* $\mathrm{d}_{i+1}$ *differ only for a permutation of consecutive inferences.*

**Remark 3.** A corollary of the latter theorem is that the syntax of proof-nets for $\mathbf{MLL}^-$ solves the problem of identity of proofs in this fragment; for this reason proof-nets have found applications to coherence problems in category theory.

## 4.2 Trips and linear $\lambda$ terms

A trip in the sense of Girard induces the structure of a $\lambda$-term in any suitable graph with a selected external point (*pointed graph*):

**Theorem 3.** (J. van de Wiele) *Every* connected pointed graph *with vertices of incidence* 1 *and* 3 *corresponds to a linear* $\lambda$*-term (and vice versa). The correspondence is established in linear time by a* trip *starting with the selected external point.*

We perform a trip in the style of Girard according to the figure below; during the trip we determine whether a vertex is to be regarded as (1) a variable, (2) an application or (3) a $\lambda$-abstraction. Case (2) occurs when during the *second visit* to a vertex of incidence 3 the trip enters the vertex through *the same edge* from which it had exited after the first visit; case (3) occurs when the second visit is through the *other* edge; case (1) is that of an external vertex different from the selected one.

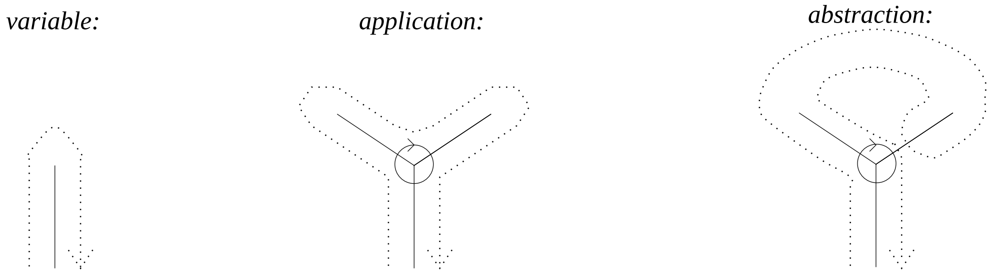


Figure 3: Linear $\lambda$-terms

The proof is by induction on the number of vertices. Notice that if we remove the first vertex of incidence 3 encountered during the trip then the resulting graph is disconnected in case (2) but remains connected in case (3). Different variables are assigned to different external points in case (1). Since linear $\lambda$-terms are always typable, van de Wiele's result also shows that every such connected pointed graph corresponds to a proof in the implicative fragment of intuitionistic linear logic.

## 4.3 Reduced translations from $\mathbf{CMLL}^-$ to $\mathbf{IMLL}^-$

Essentially the same technique applied to a *proof-net* for $\mathbf{CMLL}^-$ yields a translation of the proof-net into a derivation in the *implication and tensor* fragment of intuitionistic multiplicative linear logic. These translations shall be called *reduced trip translations* or simply *reduced translations*. Trips always have a *starting point* in a conclusion and the *order* of the passages across a link matters. To state our result we need a definition.

**Definitions 3.** (i) Given a proof-net $\mathcal{R}$ for $\mathbf{CMLL}^-$ and a selected conclusion $A$, we say that a Girard trip starting from $A$ is *covariant* on an edge if the *second passage* of the trip is in the same direction as the edge; otherwise, the trip is *contravariant* on the edge. Now a trip starting from $A$ induces an *input-output orientation* $\delta : \mathcal{R} \to \{I, O\}$ thus: an edge $X$ is an *output* $X_O$ or an *input* $X_I$ depending on whether the trip is covariant or contravariant on it.

**Theorem 4.** (Bellin and van de Wiele) (i) *Every* Girard's trip *on a* cut-free *proof-net for* **CMLL**$^-$ *starting from a selected conclusion corresponds to a sequent calculus derivation in* **IMLL**$^-$.

(ii) *Conversely, every sequent derivation in* **IMLL**$^-$ *corresponds to a trip on a proof-net.*

The following proposition follows almost immediately from the definition of orientation and the basic properties of trips.

**Proposition 2.** *Every orientation makes the selected conclusion an output, all other conclusions are inputs. Every link is oriented in one of the admissible ways indicated in Figure 4.* ∎

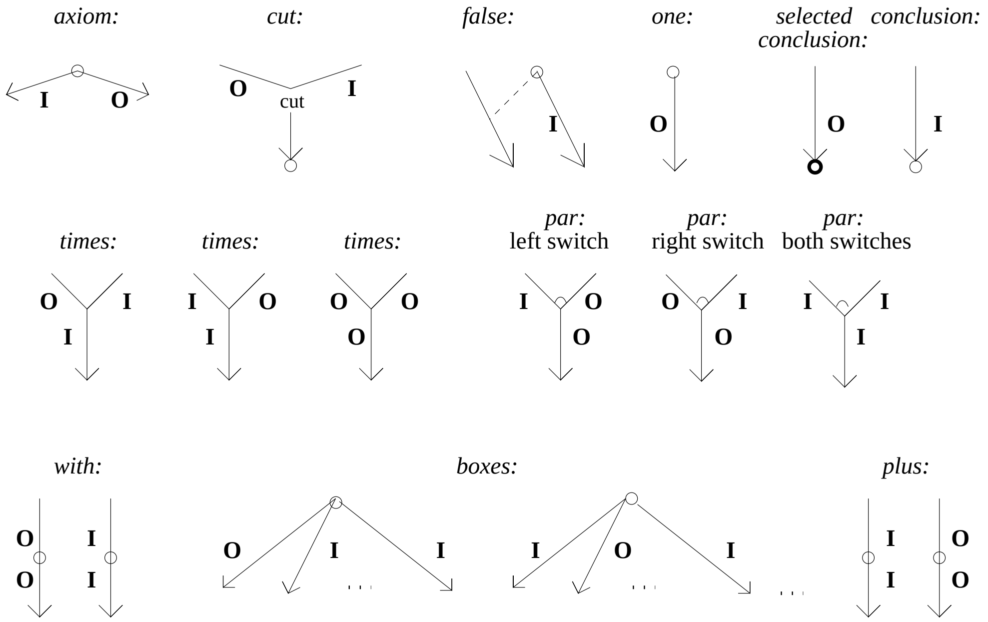


Figure 4: Admissible orientations.

Given an orientation $\delta : \mathcal{S} \to \{I, O\}$, the *formulas* in the proof-net are translated as follows:

$$\begin{array}{rcllrcl}
(P^\perp)_O & = & P_I, & \text{for } P \text{ atomic} & (P^\perp)_I & = & P_O; \\
(A \otimes B)_O & = & A_O \otimes B_O, & & (A \wp B)_I & = & A_I \otimes B_I; \\
(A \otimes B)_I & = & A_O \multimap B_I, & \text{if } \delta(A) = O, & & & \\
& = & B_O \multimap A_I, & \text{if } \delta(B) = O. & & & \\
(A \wp B)_O & = & A_I \multimap B_O, & \text{if } \delta(B) = O, & & & (\textit{right switch}) \\
& = & B_I \multimap A_O, & \text{if } \delta(A) = O. & & & (\textit{left switch})
\end{array}$$

For further details, see [8], pp. 37-44. ■

**Remarks 4.** (i) Theorem 4 could be stated in terms of *Danos-Regnier graphs*, as it was done in [8]; notice that every Girard's trip determines a unique Danos-Regnier graph [10]. Girard's trips allow us to give a more concise definition of orientation, but Danos and Regnier's characterization yields the refinements in Proposition 3 below.

(ii) Reduced translations of formulas are not *functorial*, in the sense that they depend not only on the given **CMLL** formula, but also on a trip on a given proof; i.e., the map on objects depends also on morphisms. Reduced translations of proofs are not functorial, in the sense that they may not be compatible with *cut*. Indeed, the orientation induced by a switching may be *computationally inconsistent*: e.g., consider the orientation on the cut formulas $A_O \otimes B_I$ and $A^\perp_O \wp B^\perp_I$ induced by a *left switch* on the *par* link.

(iii) The above result does not extend to full **CMLL**: let $\perp$ be the selected conclusion in the cut-free proof-net with conclusion $P$, $P^\perp$, $\perp$.

(iv) The above result does not extend to **CMALL**: consider the cut-free proof-net with conclusions $A \& B$, $A^\perp \otimes B^\perp$, $A \oplus B$.

However, reduced translations suffice to characterize the action of the functor $F$ of theorem 2 on a derivation in the fragment $\mathbf{MLL}^-$ in the following sense. Let $s$ be a switching *in the sense of Danos-Regnier* on a proof-net R for $\mathbf{MLL}^-$. Let $\mathbf{IMALL}^-$ be **IMALL** without *plus*. Consider the set of maps $\sigma : \mathbf{IMALL}^- \to \mathbf{IMLL}^-$ with the following properties:
(a) $\sigma$ acts on the propositions as follows:

$$(i) \qquad \sigma[(A_I \multimap B_O)\&(B_I - oA_O)] = \sigma[A_I - oB_O] \quad \text{or} \quad \sigma[B_I \multimap A_O]$$

$(ii)$ $$\sigma[(A_O \multimap B_I)\&(B_O \multimap A_I)] = \sigma[A_O \multimap B_I] \quad \text{or} \quad \sigma[B_O \multimap A_I]$$

(b) $\sigma$ acts on **IMALL**$^-$ derivations $\pi$ by removing all &-right and &-left inferences

Clearly, given such a $\sigma$ defined arbitrarily on propositions we do not know whether there is a proof $\pi$ of $\vdash (\wp(\Gamma))_O$ such that $\sigma(\pi)$ is a proof of $\vdash \sigma[(\wp(\Gamma))_O]$. Moreover, given any derivation $\pi$ in **IMALL**$^-$, $\sigma[\pi]$ needs not be a derivation in **IMLL**$^-$ (for instance, if the map on derivation removes a &-left inference with active formula $\sigma[A_O \multimap B_I]$ and the map on propositions yields $\sigma[B_O \multimap A_I]$). However, Danos-Regnier switchings allow us to define well-behaved maps $\sigma$ as functions of a proof-net and of a switchings.

**Proposition 3**. *Let $\mathcal{R}$ be a cut-free proof-net for* **MLL**$^-$ *with conclusions $\Gamma$ and let $s$ be a Danos-Regnier switching of $\mathcal{R}$. Let $\pi$ be the derivation of $\vdash (\wp(\Gamma))_O$ in* **IMALL**$^-$ *given by the Chu functor. Then there exists a map $\sigma_{s\mathcal{R}}$ such that $\sigma_{s\mathcal{R}}[\pi]$ is a derivation of $\sigma_{s\mathcal{R}}[(\wp(\Gamma))_O$ in* **IMLL**$^-$. *Moreover,*

$$\sigma_{s\mathcal{R}}[(A_I \multimap B_O)\&(B_I \multimap A_O)] = \sigma_{s\mathcal{R}}[A_I \multimap B_O] \quad \text{iff} \quad s(A\wp B) = \textit{right}.$$

■

**Remarks 5.** (i) It can be shown that and the map $\sigma_{s\mathcal{R}}$ depends only on the values of $s$ on the *par* links which in a Girard trip are reached from below, i.e., the *par* links whose conclusion is oriented as an "output" and which correspond to formula-occurrences of type (i) in $\pi$.

(ii) Every reduced translation of an $MLL-$ derivation $\pi$, *regarded as a derivation in* **MLL**$^-$ according to Fact (i) in section 2.2, contains the same information as $\pi$.

## 4.4 Chu's construction in MLL with units

As indicated in the Preface, one of the original motivations for this paper was to find a *functorial* definition of the trip translation, in view of a possible extension to the whole system **CLL** and given the fact that the *reduced trip translation* does not extend beyond **MLL**$^-$. We have now a functorial translation and a satisfactory explanation of its meaning in terms of Chu's construction. But what about extensions to **MLL** with units and **CMALL**?

As indicated in Remark 2, the problem with *fullness* may require a basic reformulation of the construction, e.g., the definition of a subcategory of

$C \times C^{op}$ for which the functor is full. Moreover, *faithfulness* for **CMALL** requires a reconsideration of additive proof-nets. On the other hand, the proof of *faithfulness* for **MLL** *with units* seems at hand, thanks to A. Tan's thesis [26], although we cannot spell out the details here.

**Remarks 6.** (**MLL** *with units*) (i) We do not know how to define proof-nets for **MLL** *with units* so as to extend the theorem on Permutability of Inferences to **MLL** with units, thus it is no longer true that the the proof-net representation solves the problem of identity of proofs in **MLL** with units (cfr. Remark 3). Any permutation of the *nil* rule with other inferences in a derivation $d$ results in a *rewiring* of $(d)^{-}$, i.e., in a modification of the 'attachment' of the corresponding $\perp$-link. (Of course, this problem would not occur in the system **MLL** *with the axiom* $\perp = \mathbf{1}$.) Therefore the obvious way to characterize the identity of proofs for **MLL** *with units* is to give explicit equations between proof-nets.

(ii) A similar problem occurs for the representation of proofs in **IMLL**: in fact the systems of Natural Deduction or Sequent Calculi with term-assignments for **ILL** in [4, 5, 6] are given together with an axiomatic characterization of the identity of proofs in the form of an *equational theory* of terms. Similarly, Lamarche's proof-nets for **ILL** [20] require a theory of *rewiring* already in the case of **MLL** *with units*. This work has been done in Chapter 6 of A. Tan's thesis [26]: after a careful definition of the correspondence between sequent calculus with term assignments and proof-nets for **IMLL**, the process of *rewiring* is defined so that it does preserves the correctness criterion, it does not affect the (equivalence classes of) terms which the proof-net interprets, it is strongly normalizing and confluent and, moreover, the process of cut-elimination, *incorporating unit rewirings*, remains strongly normalizing and confluent.

(iii) Rewiring in **CMLL** proof-nets is also defined in such a way that it preserves the correctness criterion. Since classical proof-nets may have several conclusions, it is not obvious how to define a canonical element in each equivalence class of proof-nets.

(iv) Let us consider the again action of the functor $F : \mathbf{CMLL} \to \mathbf{IMALL}$ *on the units*. If $\pi$ is a proof of $\vdash \Gamma, \perp\!\!\!\perp$, then $F(\pi) = \Gamma_I \vdash \top$, an *axiom*, i.e., the proof $\pi$ is *erased*. It follows that a single *reduced translation*, regarded as a **CMLL** proof, no longer contains the same information as the original proof (cfr. Remark 5.(ii)).

(v) Considering the definition of reduced translations from Danos-Regnier graphs, we may follow the hint of Definition 2.(i) and define the orientation as if an attachment resulted from an *axiom* $\perp, \mathbf{1}$ where the edge $\mathbf{1}$ enters a *times* link with conclusion $\mathbf{1} \otimes A \sim A$. The definition extends without problems when the orientation of $\perp$ is $\perp_I$: indeed the corresponding **IMLL** proof-net has a **1**-link with a suitable attachment. If the orientation is $\perp_O$ we may no longer have a coherent orientation for the edge $A$ (in the case where we would give the different orientations $A_O$, $(\mathbf{1} \otimes A)_I$): but this is still fine, because $F(\perp_O) = \top$ and we may certainly take an axiom $\top, A_I, A_O$ in the reduced translation.

(vi) Finally, let us consider the effect of rewiring of **CMLL** proof-nets $\mathcal{R} \mapsto \mathcal{R}'$ on a reduced trip translation of $\mathcal{R}$. Given a $\perp$ link in $\mathcal{R}$, the rewiring in question may

(1) preserve the orientation $\perp_I$ or
(2) preserve the orientation $\perp_O$ or
(3) change an orientation $\perp_I$ into $\perp_O$ or
(4) vice versa.

In case (1) the effect of the rewiring is either null or a rewiring in **IMLL** as described in [26]. In case (2) the effect is either null or a commutation of a $\top$-axiom, in accordance with standard equations between **IMALL**-proofs. Only cases (3) and (4) do reserve some surprises; e.g., in case (iii) the **IMLL** proof-net resulting from a switching $s\mathcal{R}'$ may be obtained from the **IMLL** proof-net corresponding to $s\mathcal{R}$ only through some complicated *"surgery"*.

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