Proof Nets

for Multiplicative and Additive Linear Logic

by

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Abstract

This paper extends the theory of proof nets from Multiplicative Linear Logic to Multiplicative and Additive Linear Logic, by removing the additive boxes. This yields uniqueness of normal form for that fragment and a classification of permutability of inferences in the calculus of sequents for classical Linear Logic.

0. Introduction.

> *'A mathematical proof must be perspicuous'. Only a structure whose reproduction is an easy task is called a 'proof'.* (Wittgenstein)

This paper studies the general proof theory of linear logic and direct logic, *resource-aware* systems that restrict the iterated use of assumptions in proofs.[1] More specifically, it contributes to the theory of *proof nets*, developed by J-Y. Girard, and offers a solution to the technical problem of *additive boxes*. More generally, it aims at the following goals:

(I) *a simple and convincing explanation of proof nets, as a multiple conclusion natural deduction system for linear logic;*

(II) *a formalism for linear logic enjoying uniqueness of the normal form.*

Historical remark. The study of proof nets for classical linear logic is rooted in the tradition of Gentzen and Prawitz in an obvious and important way. Prawitz [1965] provides a definitive treatment of Gentzen's natural deduction systems, at least for intuitionistic logic. Prawitz [1971] gives a strong normalization theorem for second order logic and highlights the significance of normalization theorems by presenting the following conjecture:

Two formal derivations represent the same proof if and only if they reduce to the same normal form.

Gentzen and Prawitz's natural deduction for intuitionistic logic enjoys uniqueness of the normal form, *modulo* permutations of the $\vee$- and $\exists$-eliminations with other inferences. A many-one map from sequent calculus derivations to natural deduction derivations is defined in the fragment without $\vee$ and $\exists$ and several efforts have been made to generalize such results (e.g., Zucker [1974], Pottinger [1977], Ungar [1987]). It appears that a multiple conclusion natural deduction system is needed in classical logic, where a deduction $A \vdash B$ is regarded as *essentially the same as* a deduction of $\neg B \vdash \neg A$. Now proof nets appear as a significant breakthrough in the definition of such systems.

Less well known are the connections between proof nets and proof-search algorithms in the *matrix-methods*, which have been pursued D. Prawitz [1970], Bibel [1981], P. Andrews [1981], Ketonen and Weyhrauch [1984], L. A. Wallen [1989]. In particular, Ketonen and Weyhrauch introduced direct logic, a Contraction-free system in which Weakening is unrestricted, i.e., deductive resourced can be wasted but never reused. The logical connectives have an *intensional* interpretation, according to the terminology of relevance logic. A decision procedure for direct logic is described in Ketonen and Weyhrauch [1984] and implemented in the proof-checker `EKL` at the Stanford Artificial Intelligence Laboratory; the notion of a proof net is implicitly contained in that procedure (for corrections, comments and descriptions of the implementation see Bellin and Ketonen [1989?]).

[1] I wish to express my gratitude to my teachers S. Feferman, J. Ketonen and G. Kreisel. Thanks also to J-Y. Girard for interesting and challenging conversations.

The beginning and first development of linear logic is documented in the fundamental paper Girard [1987]. The distinction is made between *multiplicative* and *additive* connectives (corresponding to the *intensional* and *extensional* connectives of relevance logic) and the *exponential* operators, the main innovation of the system, are introduced. For a broad explanation of the meaning and the purpose of linear logic, see Girard [1989]. In Girard [1987] a concise language is introduced, and an efficient calculus of sequents for linear logic is presented. Also the notion of a proof net is introduced (section 1, pp. 28-47) for the *multiplicative fragment* (without propositional constants) of linear logic and extended to the whole system by the device *boxes*. A map is given from sequent calculus derivations to proof nets, which is proved to be onto (*sequentialization theorem*). A strong normalization theorem is proved for proof nets and the notion of a *slicing of the boxes* is described, which is Church-Rosser.

A *proof structure* is a set of formula occurrences connected by certain relations (*links*). *Proof nets* are the proof structures that represent correct linear proofs: they are characterized by certain consistency conditions on the structure as a whole. In Ketonen and Weyhrauch [1984] the global condition is *acyclicity*, where a cycle is defined as a chain of conjunctive subformulas satisfying certain properties. In Girard [1987] an interesting kind of algebraic invariant is introduced for the multiplicative fragment without propositional constants, the *no short trip* restriction (*ibid.*, pp.30-41). The condition presented in Danos and Regnier [1989] appears to be related to that of Ketonen and Weyhrauch. Ketonen and Weyhrauch's condition appears also in Abramsky [1991], where an interesting type-theoretic interpretation is given, essentially focused on the dynamics of computation, i.e., on cut elimination. Boxes are retained in Abramsky [1991] as appropriate to characterize *lazy evaluations.*

In Girard [1987] boxes are introduced in correspondence with the propositional constants $\perp$ and $\top$, the additive conjunction $\sqcap$, the universal quantifier $\forall$, with the exponential **!** and the (restricted) weakening rule. Then the task is set of removing these boxes (see especially *op.cit.* section 7, pp. 93-97). A method is outlined for the solution of the problem by *slicing* boxes; the contraction steps for the normalization of slices are given. Thus the problem is whether slices are a viable representation of linear derivations: 'Each slice is in itself logically incorrect, but it is expected that the total family of slices has a logical meaning. However, there is no characterization of sets of slices which are the slices of a proof-net." *ibidem,* p. 95.

Carrying this program on, Girard [1987a] and [1991] achieve the elimination of $\forall$ boxes.

Remarks. (*i*) The *global* verification of consistency conditions is a feature of natural deduction systems, in contrast with sequent calculi, where correctness of local links and global consistency are verified *locally.* The restriction on eigenvariables is an example in Gentzen and Prawitz's natural deduction. In our presentation we will make the global constraints explicit as much as possible.

(*ii*) Systems of sequent calculi and of natural deduction can be regarded as formalisms in their own right. In this case it is debatable whether the calculus of proof nets is a perspicuous method of proof, if the verification of any of the present consistency conditions is part of the formal representation of the deductive process (it would seem that the 'no short trip' condition is less perspicuous than the acyclicity condition). Alternatively, a proof net can be regarded as the result of an *inductive* inferential process, as formalized in a sequent calculus derivation. Hence a proof net would always be given together with some derivation in the calculis of sequents (or other global inductive process). This view is indeed suggested by Girard's remark that the soundness condition is only 'an abstract notion (just like, say, semantical soundness), whose importance lies in its relation to linear sequent calculus' (Girard [1987], p.33).

(*iii*) It is possible to abstract from Girard's proofs some consistency conditions in terms of properties of the *subnets* (or *empires*) in a proof net. The justification of such conditions is particularly natural, as they correspond to elementary properties of inductively defined structures, and the presentation of the Sequentialization Theorem is improved. Still, it is convenient to use other conditions in the proof of the Cut Elimination Theorem.

A consistency condition defining proof nets must highlight some interesting feature of inductively generated structures, and in particular, of their behavior under cut elimination. The proposal of alternative conditions should therefore be welcome.

(*iv*) A proof structure with boxes represents a logical deduction as a collection of separate structures, in a certain ordering. On the other hand, the fact that the computation of correctness of a *multiplicative* structure can be done with large degree of parallelism has been regarded as very promising and significant feature of the new calculus (cf. Girard [1986]). It would seem interesting from the point of view of parallelism to eliminate boxes and to replace them with more flexible forms of *synchronization* between different processes of computation.

(*v*) The elimination of boxes is also a main step towards a representation of deductions where *normal forms are unique*. To this purpose, we must eliminate *commutative reductions* (familiar in connection with the $\vee$- and $\exists$-elimination rules in natural deduction). More generally, we must find a representation of proofs such that two derivations that differ only in the following sequence of inferences are mapped to the same structure:

$$\dfrac{\vdash \Delta, D \qquad \dfrac{\vdash C, \Gamma, A \qquad \vdash C, \Gamma, B}{\vdash C, \Gamma, A \sqcap B}}{\vdash \Delta, D \otimes C, \Gamma, A \sqcap B} \qquad (\sqcap/\otimes)$$

$$\dfrac{\dfrac{\vdash \Delta, D \vdash C, \Gamma, A}{\vdash \Delta, D \otimes C, \Gamma, A} \qquad \dfrac{\vdash \Delta, D \vdash C, \Gamma, B}{\vdash \Delta, D \otimes C, \Gamma, B}}{\vdash \Delta, D \otimes C, \Gamma, A \sqcap B} \qquad (\otimes/\sqcap)$$

(*vi*) In a $\sqcap$ box the conclusions of two structures $\mathcal{S}'$ and $\mathcal{S}''$ are 'identified' in some sense; because of commutative reductions, several choices may be possible for such an identification. When slicing a $\sqcap$ box we must retain the identifications within the resulting family of slices in some form. Now it seems that the most elegant solution is a representation where *all correct choices are open*, and only the *possibility* of some identification is required for logical correctness.

(*vii*) The $\bot$ boxes confront us with the 'disturbances' caused by irrelevance on the geometric symmetries of proof structures. A $\bot$ symbol introduced by the $\bot$ rule

$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

bears no *relevant* connection to the occurrences in Γ (or to the axioms above them in the derivation). A box for $\bot$ should be sliced into a disconnected configuration of the form

$$\begin{array}{cc} \mathcal{S} & \\ & \bot \\ \Gamma & \end{array}$$

However, the presence of a $\bot$ symbol in a proof structure is justified by its 'attachement' to some subtructure which contains proper axioms among its topmost formulas. Unlike in direct logic, (cf. Ketonen and Weyhrauch [1984], Bellin and Ketonen [1989?]), such attachments create new connections that must be compatible with the global consistency conditions of *multiplicative* proof nets. To represent such attachments as ordinary links is unsatisfactory, since such links represent an undesirable modification of meaning: they exhibit a 'relevant connection' where there is none. The most satisfactory solution would be to use slices, and leave all correct attachments open, but also to make the computation of slices as *multiplicative* structures possible, by representing attachments is some way. (An alternative approach has been considered in Solitro [1990], that reduces the $\bot$ boxes to the **!** boxes and the $\top$ axiom.)

(*viii*) A similar discussion applies to $\top$ axiom

$$\vdash \top, X_1, \ldots, X_n.$$

The $\top$ box is sliced as a disconnected configuration

$$\top \quad X_1 \ldots X_n.$$

The main difference is due to the meaning of $\top$ (expressed in the property of its dual: $\mathbf{0} \multimap A$, *ex falso quodlibet*). Indeed $\top$ acts like an 'eraser': a derivation of the form

$$\frac{\vdash \top, \Gamma, C \qquad \begin{array}{c} \cdots\cdots \\ \vdash D, \Delta \end{array}}{\vdash \top, \Gamma, C \otimes D, \Delta}$$

reduces to

$$\vdash \top, \Gamma, C \otimes D, \Delta.$$

The above property represents *garbage collection* (cf. section 3.5.). It can also be described as *propagation of irrelevance*: if one (multiplicative) conjunct is irrelevant in a proof then the other also may be regarded as irrelevant. When a $\top$ box is sliced into a disconnected set of occurrences, a formula occurrence in a slice is said to be *irrelevant* if there is no axiom above it. Propagation of irrelevance is implicit in the definition of *path* by Ketonen and Weyhrauch [1984] in the context of a search procedure for cut-free proofs in direct logic (cf. remark (ii) in section 2.3.); it is used to improve the efficiency of the Proof Checker EKL, based on direct logic. The issue becomes more problematic in linear logic ('we are forced to make *bricolage* on such details', Girard [1987], p. 95), where the distinction between $\perp$ and $\mathbf{0}$ is introduced and propagation of irrelevance is considered an explicit rule of contraction (*zero commutation*).

(*ix*) We will not consider the problem of eliminating **!** boxes.

(*x*) We propose to slice an additive proof structure into parts (*quasi structures*) such that the multiplicative conditions can still be tested within each of them.

Linear logic is mostly concerned with the computational aspects of reasoning, but is also of significance to the study of parallelism (cf. Girard [1986], Asperti [1987], Marti-Oliet and Meseguer [1989], Brown [1989], Gehlot and Gunter [1990]). A motivation for our work is to extend some syntactical features of the *multiplicative fragment* that seem relevant for parallelism to the *additive fragment* as far as possible.

A strong normalization result for first order linear logic, with some form of Church-Rosser property, could be transfered to classical logic; this may improve our understanding of a well-known problem of general proof theory (cf. Gandy [1980]).

In this paper we will not consider applications of the theory of proof nets *as such*. In Bellin [1990] and [1990a] proof nets are used for proof-transformations — applied there to the Infinite Ramsey Theorem.

1. Preliminaries.

We refer the reader to Girard [1989] for a broad explanation of the meaning and the purpose of linear logic. In particular, the motivations of the syntax and the terminology are presented there.

1.1. Language.

The following fragments of classical linear logic **LL** are considered:

MLL$^-$: multiplicative linear logic without rules for propositional constants;

MLL: multiplicative linear logic with rules for propositional constants;

MALL$^-$: multiplicative and additive linear logic without rules for propositional constants;

MALL: multiplicative and additive linear logic with rules for propositional constants.

The above system are considered in the propositional and first order cases. $\mathbf{NCMLL}^-$, **NCMLL**, $\mathbf{NCMALL}^-$, **NCMALL** are the (cyclic) noncommutative corresponding systems, also considered in the propositional and first order cases.

The language for the *propositional* systems **MLL** and **NCMLL** of Commutative and Noncommutative Multiplicative Linear Logic contain the propositional constants $\mathbf{1}$, $\perp$, the unary connective $(\ )^\perp$ (*linear negation*), the binary connectives $\otimes$ (*times*) and $\sqcup$ (*par*). The following axioms hold in both logics:

$$A^{\perp\perp} = A, \qquad \mathbf{1}^\perp = \perp, \qquad \perp^\perp = \mathbf{1}.$$

In **MLL** we have

$$(A \otimes B)^\perp = A^\perp \sqcup B^\perp \qquad (A \sqcup B)^\perp = A^\perp \otimes B^\perp$$

and in **NCMLL** we have

$$(A \otimes B)^\perp = B^\perp \sqcup A^\perp \qquad (A \sqcup B)^\perp = B^\perp \otimes A^\perp.$$

The systems **MALL** and **NCMALL** extend the previous fragments to Additive linear logic: there are new logical constants $\top$ and $\mathbf{0}$, new connectives $\sqcap$ (*with*) and $\oplus$ (*plus*), satisfying

$$\top^\perp = \mathbf{0}, \qquad \mathbf{0}^\perp = \top$$

and moreover

$$(A \sqcap B)^\perp = A^\perp \oplus B^\perp \qquad (A \oplus B)^\perp = A^\perp \sqcap B^\perp.$$

The fragments $\mathbf{MLL}^-$ and $\mathbf{MALL}^-$ have the same language as **MLL** and **MALL**, respectively, but no axioms or rules for the propositional constants $\mathbf{1}$, $\perp$, $\top$ and $\mathbf{0}$.

The languages for the *first order* systems **MLL**, **NCMLL**, **MALL**, **NCMALL** are defined as expected, using the quantifiers $\bigwedge$ (*every*) and $\bigvee$ (*some*). We have the axioms

$$(\bigwedge x.A)^\perp = \bigvee x.(A^\perp) \qquad (\bigvee x.A)^\perp = \bigwedge x.(A^\perp).$$

The above axioms allow us to reduce the scope of linear negation: henceforth in our syntax *linear negation will be applied only to atomic formulas.*

The language for the full system **LL** of (classical) linear logic contains in addition the propositional operators **!** (*of course!*) and **?** (*why not?*), called *exponentials* (or *modalities*).

1.2. Sequent Calculus.

We list the rules of inference of the calculus of sequents for linear logic (Girard [1987]) and for direct logic (Ketonen and Weyhrauch [1984]), for convenience of the reader.

Sequent Calculus for LL.

Logical Axioms: $\vdash A^{\perp}, A$

Cut Rule:
$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

Exchange Rules:

Non Commutative
$$\frac{\vdash B, \Gamma, A}{\vdash A, B, \Gamma}$$

Commutative
$$\frac{\vdash \Gamma, B, A, \Delta}{\vdash \Gamma, A, B, \Delta}$$

1 *Axiom:* $\vdash \mathbf{1}$

$\perp$ *Rule:*
$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

$\otimes$ *Rule:*
$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B}$$

$\sqcup$ *Rule:*
$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \sqcup B}$$

$\top$ *Axioms:* $\vdash \top, X_1, \ldots, X_p$ for $p \geq 0$

$\sqcap$ *Rule:*
$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \sqcap B}$$

$\oplus$ *Rule:*
$$\frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_0 \oplus A_1} \quad \text{for } i = 0 \text{ or } 1$$

Dereliction:
$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \mathbf{?}A}$$

Weakening:
$$\frac{\vdash \Gamma, \perp}{\vdash \Gamma, \mathbf{?}A}$$

Contraction:
$$\frac{\vdash \Gamma, \mathbf{?}A, \mathbf{?}A}{\vdash \Gamma, \mathbf{?}A}$$

! *Rule:*
$$\frac{\vdash \mathbf{?}\Gamma, A}{\vdash \mathbf{?}\Gamma, \mathbf{!}A}$$

where $\mathbf{?}\Gamma$ consists only of formulas of the form $\mathbf{?}C$ or $\perp$.

Sequent Calculi for Direct Logic: The system of sequent calculus for the logic **DL** consists of the rules for $\mathbf{MLL}^{-}$, namely Logical Axioms and rules for multiplicative conjunction and disjunction, and also on the rule of *Mingle*; the calculus of sequents for $\mathbf{DL}^{+}$ has the rules for $\mathbf{MLL}^{-}$, and in addition unrestricted *Weakening*.

Mingle
$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

Weakening
$$\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

1.3. Links and Proof Structures.

A *link* is a $m+n$-ary relation between formula occurrences. For each link if $X_1, \ldots,$ X_{m+n} are in the link, then $X_1, \ldots, X_m$ are called the *premises* and $X_{m+1}, \ldots, X_{m+n}$ are called the *conclusions* of the link. If $m = 0$, the link is an *axiom* link. Links are graphically represented as follows. The *logical axioms* and the **1** *axioms* are

$$\overline{P, P^{\perp}} \qquad \overline{\mathbf{1}}.$$

The Cut, $\otimes$-, $\sqcup$- $\oplus$-, $\bigwedge$- and $\bigvee$-links are

$$\frac{X \quad X^{\perp}}{} \qquad \frac{X \quad Y}{X \otimes Y} \qquad \frac{X \quad Y}{X \sqcup Y} \qquad \frac{X}{X \oplus Y} \qquad \frac{Y}{X \oplus Y} \qquad \frac{X(a)}{\bigwedge x.X} \qquad \frac{X(t)}{\bigvee x.X}$$

respectively. A variable a occurring in the premise of a $\bigwedge$ link is called *eigenvariable*, or *parameter*, with the convention that each eigenvariable is associated with one and only one $\bigwedge$ link. Logical and **1** axioms, Cut, $\otimes$- and $\sqcup$- links are *multiplicative* links; the $\top$ axiom and the $\oplus$-link are *additive*. The first order links $\bigwedge$ and $\bigvee$ should also be regarded as *additive* (but we will not consider here the issue of multiplicative quantifiers).

When considering the exponentials, we add the links for **?**, i.e., *Dereliction*, *Weakening* and *Contraction*:

$$\frac{A}{?A} \qquad \frac{\perp}{?A} \qquad \frac{?A \quad ?A}{?A}.$$

A *proof structure* $\mathcal{S}$ for [first order] $\mathbf{MLL}^{-}$ consists of

(i) a nonempty set of *formula-occurrences* (i.e., a multiset of formulas) together with

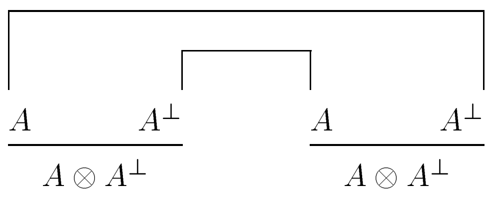
(ii) a set of *logical axioms* and *multiplicative links* [and first order links], satisfying the properties

(1) every formula-occurrence in $\mathcal{S}$ is the conclusion of one and only one link;

(2) every formula-occurrence in $\mathcal{S}$ is the premise of at most one link;

(3) an axiom link $\overline{X_1, \ldots, X_n}$ is identified with $\overline{X_{\sigma(1)}, \ldots, X_{\sigma(n)}}$, where σ is any cyclic permutation, and a Cut link $\underline{X \quad X^{\perp}}$ is identified with $\underline{X^{\perp} \quad X}$.

Notice that a link $\frac{X \quad Y}{X \circ Y}$ is *different from* $\frac{Y \quad X}{Y \circ X}$. For example, a proof structure in $\mathbf{MLL}^{-}$ is



We write $X \prec Y$ if the formula occurrence X occurs *above* the formula occurrence Y in $\mathcal{S}$ (with the obvious meaning of *above*).

It is convenient to put a marker for the missing conclusion of a Cut link: Girard [1987] introduces the special symbol **CUT**. Such a symbol is *not* a formula occurrence; we will call it a *ghost* (to reduce the number of different uses of the word 'cut'). The premises of a Cut are called *cut formulas*, as usual. The formula occurrences of $\mathcal{S}$ which are not premises of any link are called the *conclusions of* $\mathcal{S}$.

A *proof structure* $\mathcal{S}$ for $\mathbf{NCMLL}^{-}$ is defined like a proof structure for $\mathbf{MLL}^{-}$, with in addition

(iii) a cyclic order l of the conclusions and of the cut formulas.

The ordering l is extended to all the formulas in $\mathcal{S}$ by letting $X \,\mathrm{l}\, Y$ if and only if for some U and V such that $X \preceq U$, $Y \preceq V$, either (i) $U \circ V$ is a formula occurrence in $\mathcal{S}$ or (ii) U and V are conclusions or cut formulas and $U \,\mathrm{l}\, V$.

Remark. The above syntax follows Girard [1987a], but with the technical modification of the Weakening link. The following syntax for the propositional constants is a minor departure from the tradition (see Remark (v), section 1.5):

A *proof structure* for **MLL** or **NCMLL** is defined as for $\mathbf{MLL}^{-}$ or $\mathbf{NCMLL}^{-}$, respectively, allowing also $\overline{\mathbf{1}}$ axioms and of $\perp$ links of the form

$$\frac{A}{A, \perp,}$$

where $A \neq \perp$.

It is convenient to introduce *proof structures with non-logical axioms*, written as

$$\overline{X_1, \ldots, X_n}$$

for $n > 0$. These correspond to *sequent axioms* or *initial sequents* in sequent calculus.

Let $m : \mathcal{S} \to \mathcal{S}'$ be any injective map of proof structures (regarded as sets of formula occurrences). We say that *m preserves the links*

$$\overline{X_1, \ldots, X_n} \qquad \frac{A}{B} \qquad \frac{A \quad B}{C}$$

if we have

$$\overline{m(X_1), \ldots, m(X_n)} \qquad \frac{m(A)}{m(B)} \qquad \frac{m(A) \quad m(B)}{m(C)}$$

Finally, let $C_1, \ldots, C_n$ the set of conclusions of $\mathcal{S}$. We say that *m preserves the conclusions* if $m(C_1), \ldots, m(C_n)$ are all the conclusions of $\mathcal{S}'$.

An injective map $\iota : \mathcal{S} \to \mathcal{S}'$ is an *embedding* if it preserves links and, moreover, X and $\iota(X)$ are occurrences of the same formula, for every X in $\mathcal{S}$. A subset $\mathcal{S}'$ of $\mathcal{S}$ is a *substructure* if the identity is an embedding.

1.4. Inductive Proof Structures.

A sequent derivation is inductively built from axioms by applications of the rules of inference. In a similar way, we can inductively generate the proof structures that correspond to linear derivations. More precisely, a proof structure for $\mathbf{MLL}^{-}$ is said *inductively generated* (an IPS) if it results from a finite number of applications of the following steps:

(0) An axiom

$$\overline{A \quad A^{\perp}}$$

is an IPS;

(1) If $\mathcal{S}$ and $\mathcal{S}'$ are IPS's with the multisets Γ, A and $A^{\perp}, \Delta$, respectively, as conclusions, and moreover $\mathcal{S} \cap \mathcal{S}' = \emptyset$, then

$$\begin{array}{cccc} & \mathcal{S} & \mathcal{S}' & \\ \Gamma, & \underline{A \quad} & \underline{A^{\perp}} & \Delta \end{array}$$

is an IPS with the multiset Γ, Δ as conclusions;

(2) If $\mathcal{S}$ and $\mathcal{S}'$ are IPS's with the multisets Γ, A and B, Δ, respectively, as conclusions, and moreover $\mathcal{S} \cap \mathcal{S}' = \emptyset$, then

$$\begin{array}{ccc} & \mathcal{S} \quad\quad \mathcal{S}' & \\ \Gamma, & \dfrac{A \quad B}{A \otimes B} & \Delta \end{array}$$

is an IPS with the multiset $\Gamma, A \otimes B, \Delta$ as conclusions;

(3) If $\mathcal{S}$ is an IPS with the multiset Γ, A, B as conclusions, then

$$\begin{array}{cc} & \mathcal{S} \\ \Gamma, & \dfrac{A \quad B}{A \sqcup B} \end{array}$$

is an IPS with the multiset $\Gamma, A \sqcup B$ as conclusions.

(4) If $\mathcal{S}$ is an IPS with the multiset $\Gamma, A(a)$ as conclusions, and a does not occur in Γ, then

$$\begin{array}{cc} & \mathcal{S} \\ \Gamma, & \dfrac{A(a)}{\bigwedge x.A} \end{array}$$

is an IPS with the multiset $\Gamma, \bigwedge x.A$ as conclusions.

(5) If $\mathcal{S}$ is an IPS with the multiset $\Gamma, A(t)$ as conclusions, then

$$\begin{array}{cc} & \mathcal{S} \\ \Gamma, & \dfrac{A(t)}{\bigvee x.A} \end{array}$$

is an IPS with the multiset $\Gamma, \bigvee x.A$ as conclusions.

The above definition is extended to the noncommutative case $\mathbf{NCMLL}^-$ by considering the conclusions and the cut formulas of an IPS as a set with ciclic ordering, defined arbitrarily at steps (1) and in the obvious way at each step (2) - (5). For instance, in case (2) let $\Gamma, A = A_0, \ldots, A_{n-1}$, be the conclusions and the cut formulas of $\mathcal{S}$; let $B, \Delta = B_0, \ldots, B_{m-1}$ be the conclusions and the cut formulas of $\mathcal{S}'$, where the indices represent the ordering of the conclusions *mod n* and *mod m*, respectively; then the conclusions and the cut formulas $\Gamma, A \otimes B, \Delta$ of the resulting IPS are $C_0, \ldots, C_{n-1}, \ldots, C_{n+m-2}$, where the indices represent the order *mod* $n + m - 1$.

Remarks. (*i*) The following is clear (by a glance at section 1.2). There is an obvious map $(\ )^*$ that transforms any sequent derivation $\mathcal{D}$ in (first order) **MLL** or **NCMLL** into a proof structure $\mathcal{S}$ for the same fragment. If $\mathcal{S}$ is an IPS, then from the inductive generation of $\mathcal{S}$ we immediately obtain a derivation $\mathcal{D}$ such that $\mathcal{S} = (\mathcal{D})^*$.

(*ii*) We refer the reader to Girard [1989] for explanations of the meaning of logical symbols in relation to their introduction rules. To motivate the formal choices it is important to keep in mind the following slogans. In step (0) the axiom $\overline{A\ A^\perp}$ asserts the existence of a *relevant connection* (a "wire") between the occurrences A and $A^\perp$. In step (3) the $\sqcup$ link can be regarded as an introduction of a linear implication with conclusion $A^\perp \multimap B$ or $A \circ\!\!- B^\perp$. Step (1) and (2) connect two separate structures in different ways: in step (2), the connective $\otimes$ joins the occurrences A and B as separate non-communicating entities; in step (1) the linkage of the cut formulas A and $A^\perp$ creates the possibility of communication (a "plug" between wires). Similarly, thinking in terms of "formulas-as-types", step (3) is a "symmetric lambda abstraction" and step (1) is lambda application.

(*iii*) A Cut link may be regarded as a $\otimes$-link

$$\frac{X \quad X^\perp}{X \otimes X^\perp}$$

with the understanding that $X \otimes X^\perp$ is not a formula occurrence but rather a ghost. It will be shown that from a *geometric* point of view, and *statically* (i.e., without consideration of cut elimination) Cut and $\otimes$ links are basically the same. In other words, the expression $X \otimes X^\perp$ can be regarded either as one of the conclusions of the deductive process or as a temporary inconsistency to be removed. More precisely, in a *propositional multiplicative* proof structure, with respect to its *internal* connections, there is no difference between a ghost and a real formula occurrrence. The difference matters (i) when we add quantifiers and additives or (ii) in the noncommutative case or (iii) when we ask, from the *outside*, what are the conclusions of (the deductive process represented by) a given structure.

1.5. The notion of a box.

In Girard [1987] there are *boxes* in correspondence with following axioms and inference rules of sequent calculus for linear logic: (1) the $\top$-axiom

$$\vdash \top, X_1, \ldots, X_n,$$

(2) the one-premise rule

$$\bot : \frac{\vdash \Gamma}{\vdash \Gamma, \bot} \qquad \forall : \frac{\vdash \Gamma, A(a)}{\vdash \Gamma, \forall x.A} \qquad \mathbf{!} : \frac{\vdash ?C_1, \ldots, ?C_n, A}{\vdash ?C_1, \ldots, ?C_n, !A}$$

(as usual, in the case of the $\forall$ rule a does not occur in Γ) and (3) the two-premise rule

$$\sqcap : \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \sqcap B}.$$

In correspondence with each one-premise rule there is a box of the form

$$\begin{array}{c} \boxed{\begin{array}{c} \mathcal{S}' \\ \\ \Gamma \;\; Y \end{array}} \\ \mathcal{S}^o \end{array}$$

and in correspondence with the $\sqcap$ rule we have

$$\begin{array}{c} \boxed{\begin{array}{cc} \mathcal{S}' & \mathcal{S}'' \\ \\ \Gamma & A \sqcap B \end{array}} \\ \mathcal{S}^o \end{array}$$

where the indicated multisets of formulas belong to a 'new kind of link': we may regard Γ, Y [or $\Gamma, A \sqcap B$] as *non-logical axioms* of $\mathcal{S}^o$ and we write sometimes $\overline{\Gamma, Y}$ [or $\overline{\Gamma, A \sqcap B}$].

We have the following *global restriction*:

(§) In all the above boxes, the lowermost formula occurrencess of $\mathcal{S}'$ [and of $\mathcal{S}''$] must correspond to the non-logical axiom as follows (cf. the related inference of the calculus of sequents):

$$\bot: \begin{array}{c} \boxed{\begin{array}{c} \mathcal{S}' \\ \\ \Xi, \; \Gamma \end{array}} \\ \Gamma, \bot \end{array} \qquad \forall: \begin{array}{c} \boxed{\begin{array}{c} \mathcal{S}' \\ \\ \Xi, \; \Gamma, \; A(a) \end{array}} \\ \Gamma, \forall x.A \end{array} \qquad !: \begin{array}{c} \boxed{\begin{array}{c} \mathcal{S}' \\ \\ \Xi, \; \Gamma, \; A \end{array}} \\ \Gamma, !A \end{array} \qquad \sqcap: \begin{array}{c} \boxed{\begin{array}{cc} \mathcal{S}' & \mathcal{S}'' \\ \\ \Xi, \; \Gamma, \; A & \Theta, \Gamma, B \end{array}} \\ \Gamma, A \sqcap B \end{array}$$

Here Ξ and Θ are multisets of ghosts.[2]

[2] To test condition (§) we need maps between the occurrences of Γ in the conclusions of $\mathcal{S}'$, $\mathcal{S}''$ and in the non-logical axiom. There may be lots of such maps. A 'non-constructive' operation is implicit here (which would be obvious if we considered infinitary multisets).

The additional *global restrictions* hold:

(§§) In the $\forall$ box, a does not occur in Γ (but it may occur in Ξ).

(§§§) In the **!** box, Γ has the form $?C_1, \ldots, ?C_n$, for some n.

Finally, there is an obvious *global restriction* implicit in the above notation:

(§§§§) In all the above boxes, $\mathcal{S}'$, $\mathcal{S}''$ and $\mathcal{S}^o$ are pairwise disjoint sets.

A *proof structure with boxes* is defined like a proof structure, with the addition the new kind of link represented by the box. We may assume that the conditions (§)-(§§§) are checked by a *flagging function* φ, leaving the details to the reader. We will refer to Conditions (§)-(§§§§) as the *Box Condition, version 1.*

Also, *Inductive Proof Structures with Boxes* (IPSB) for the full system of linear logic can be defined by adding to the clauses for IPS the clause

(6) If $\mathcal{S}'$ is an IPSB [if $\mathcal{S}'$ and $\mathcal{S}''$ are IPSBs and $\mathcal{S}' \cap \mathcal{S}'' = \emptyset$], then so is the configuration $\mathcal{S}$ consisting of $\mathcal{S}'$ [of $\mathcal{S}'$ and $\mathcal{S}''$] inside a box and of a new non-logical axiom outside the box, provided that the conclusions of $\mathcal{S}'$ [of $\mathcal{S}'$ and $\mathcal{S}''$] and the non-logical axiom satisfy conditions (§), (§§) and (§§§); the conclusions of $\mathcal{S}$ are the occurrences of the non-logical axiom.

A proof structure $\mathcal{S}$ with boxes and with conclusion Γ satisfying the box condition can also be regarded as a collection $\mathbf{C}$ of proof structures with non-logical axioms and without boxes (i.e., proof structures for **MLL** with $\oplus$ and **?** links with non-logical axioms), with the following properties:

(i) There is an $\mathcal{S}_0$ in $\mathbf{C}$ having Γ as conclusions.

(ii) There is a function φ (*flagging function*) with the following property: if $\mathcal{S}'$ in $\mathbf{C}$ is different from $\mathcal{S}_0$ then φ associates to the multiset of conclusions of $\mathcal{S}'$ with one non-logical axiom occurring in some $\mathcal{S}''$, in accordance with (§)-(§§§). Every non-logical axiom occurring in some $\mathcal{S}' \in \mathbf{C}$ is associated by φ to the multisets of conclusions of one or two structures in $\mathbf{C}$.

Let $\mathcal{S}_1 \triangleleft^0 \mathcal{S}_2$ be the relation defined on $\mathbf{C}$ which holds if and only if the conclusions of $\mathcal{S}_1$ correspond to a non-logical axiom of $\mathcal{S}_2$ by φ; let $\triangleleft_t$ be the transitive closure $\triangleleft^0$. The following is a global restriction equivalent to the box condition, version 1:

Box condition, version 2: $\mathbf{C}$ satisfies (i) and (ii), the structures in $\mathbf{C}$ are pairwise disjoint and $\triangleleft_t$ is a strict order.

We can define proof structures for **NCMALL** by using the box condition, version 2: we require that each proof structure in $\mathbf{C}$ has a cyclic ordering 1 (defined as in section 1.3) and that if $\mathcal{S}_1 \triangleleft^0 \mathcal{S}_2$, then the orderings of $\mathcal{S}_1$ be compatible with that of $\mathcal{S}_2$ in an obvious sense. We leave it to the reader to fill in the details.

2. Part I. Proof Nets for Multiplicative Linear Logic.

In this section we extract from Girard [1987] and [1987a] some conditions that characterize multiplicative proof nets in terms of their subnets (or empires), pointing at the connections with Ketonen and Weyhrauch [1984] (cf. also Danos and Regnier [1989]). Working with the notion of empire makes the generalization to the additives easier.

2.1. Empires.

Given a Proof-Structure $\mathcal{S}$ for $\mathbf{MLL}^-$ and $\mathbf{NCMLL}^-$ and a formula occurrence $A \in \mathcal{S}$, we define $e(A)$, the *empire of* A *in* $\mathcal{S}$, as the smallest set of formula occurrences closed under the following conditions. (Remember that $X_i \neq A$ means that X_i and A are different formula *occurrences*.)

(i) $A \in e(A)$;

(ii) if $\frac{X_1 \quad X_2}{Y}$ [resp. $\frac{X_0}{Y}$] is a link in $\mathcal{S}$ and $Y \in e(A)$, then $X_1, X_2 \in e(A)$, $[X_0 \in e(A)]$ ($\uparrow$-*step*);

(iii) if $\overline{X_1, \ldots, X_p}$ is an axiom in $\mathcal{S}$ and $X_i \in e(A)$ for some i, then for all j, $X_j \in e(A)$ ($\rightarrow$-*step*);

(iv) if $\frac{X_1 \quad X_2}{X_1 \otimes X_2}$ is a link in $\mathcal{S}$, and for $i = 1$ or 2 $X_i \neq A$ and $X_i \in e(A)$, then $X_1 \otimes X_2 \in e(A)$ [resp. if $\frac{X}{Y}$ is a unary link in $\mathcal{S}$, $X \neq A$ and $X \in e(A)$, then $Y \in e(A)$] ($\downarrow$-*step*);

(v) if $\frac{X_1 \quad X_2}{X_1 \sqcup X_2}$ is a link in $\mathcal{S}$, $X_1 \neq A \neq X_2$ and $X_1, \in e(A)$, $X_2 \in e(A)$, then $X_1 \sqcup X_2 \in e(A)$ ($\Downarrow$-*step*).

A ghost or formula occurrence $D \in e(A)$ which is *not* a premise of a link with conclusion in $e(A)$ is called a *door* of $e(A)$. It is clear that such a D which is not a ghost must be one of the following:

- A itself (*main door*);
- A conclusion of $\mathcal{S}$ (*open door*);
- A premise of a link $\frac{C \quad D}{C \sqcup D}$ or $\frac{D \quad C}{D \sqcup C}$ (*closed door*).

Remark. We choose the terminology 'open', 'closed' doors since there is an analogy (to be justified below) between a closed door in a proof structure and the conclusion of an implication introduction in Gentzen-Prawitz's natural deduction system; similarly, there is a correspondence between an open door and an open assumption.

A computation verifying that $B \in e(A)$ in $\mathcal{S}$ consists of sequences of steps (ii)-(v), starting from A and reaching B. We can represent a computation as a tree $\tau : [A] \rightarrow B$, with root in B and leaves in A, where a branching occurs only in correspondence with a step ($\Downarrow$). A tree $\tau : [A] \rightarrow B$ is said *normal* if no sequence in it reaches twice the same formula occurrence. Every (finite) tree $\tau : [A] \to B$ is reduced to a normal tree $\tau' : [A] \to B$ in finitely many steps.[3]

We write

$X \emptyset Y$ for $X \notin e(Y)$ and $Y \notin e(X)$;

$X @ Y$ for $X \in e(Y)$ and $Y \notin e(X)$;

$X \diamondsuit Y$ for $X \in e(Y)$ and $Y \in e(X)$.

Let $\frac{X(a)}{\bigwedge x.X}$ be any link in $\mathcal{S}$ and let b be any eigenvariable. We write

$a <^0 b$ if b occurs *both inside and outside* $e(X(a))$

and we let $<_t$ be the transitive closure of the relation $<^0$ (over the eigenvariables of $\mathcal{S}$).

2.2. Proof Nets.

Let $\mathbf{S}$ be a proof structure for $\mathbf{MLL}^-$ or $\mathbf{NCMLL}^-$. We consider the following requirements on $\mathcal{S}$.

(0) (*Noncommutative case only*) If $\overline{X, \ldots, X'}$ and $\overline{Y, \ldots, Y'}$ are any pair of distinct axiom links, or $\underline{X \quad X'}$ and $\underline{Y \quad Y'}$ are any pair of disjoint Cut links, then in the induced ordering we cannot have $X\,\mathrm{l}\ \ Y\,\mathrm{l}\ \ X'\,\mathrm{l}\ \ Y'$ or $X\,\mathrm{l}\ \ Y'\,\mathrm{l}\ \ X'\,\mathrm{l}\ \ Y$ (the *planarity* condition);

(1) For every link $\frac{X \quad Y}{X \otimes Y}$ in $\mathcal{S}$, $X \emptyset Y$ (the [no] *vicious circle* condition);

(2) For every link $\frac{X \quad Y}{X \sqcup Y}$ in $\mathcal{S}$, and for every X, Y that are conclusions of $\mathcal{S}$ or ghosts, $X \diamondsuit Y$ (the *connectedness* condition);

(3) The ordering $<_t$ is strict (the *parameters* condition).

A proof structure in [first order] $\mathbf{MLL}^-$ is a *proof net* if it satisfies the vicious circle and connectedness [and parameters] conditions. A proof net for $\mathbf{NCMLL}^-$ must also satisfy the planarity condition.

Remarks. (i) In linear logic the *vicious circle* condition is the main requirement of *propositional consistency* and the *connectedness* condition is the requirement of *relevance.*

(ii) The 'naturality' of the above conditions, as well as the connection between the parameters condition and the usual restrictions on the eigenvariables, can be evaluated as soon as the role of empires is understood. The main fact about empires to keep in mind is the following Theorem, whose proof can be obtained using the techniques below.

2.2.1. Subnet Theorem. *Let $\mathcal{S}$ be a proof net for* $\mathbf{MLL}^-$ *or* $\mathbf{NCMLL}^-$ *and let A be any formula occurrence in $\mathcal{S}$. Then $e(A)$ is a substructure of $\mathcal{S}$ which is a proof net.*

2.2.2. Theorem. *Every IPS in* $\mathbf{MLL}^-$ *or* $\mathbf{NCMLL}^-$ *is a proof net.*

[3] Such trees are particularly simple instances of inductively presented systems considered in Feferman [1982].

The proof is by induction on the definition of IPS. For the vicious circle condition see the Remark below. The parameters condition is immediate if we make sure that different eigenvariables are used at each application of (4) (*pure parameters property*). The details are left to the reader. ∎

Remark. Consider step (2) in the definition of IPS, section 1.4. Clearly, the requirement that $\mathcal{S} \cap \mathcal{S}' = \emptyset$ implies $e(A) \cap e(B) = \emptyset$ in the resulting IPS, thus $A \emptyset B$. Conversely, in a proof net $A \emptyset B$ implies $e(A) \cap e(B) = \emptyset$, by the Tiling Lemma below (section 2.5). By the Subnet Theorem, $e(A)$ and $e(B)$ are IPS.

Moreover, Step (2) corresponds to the operation of substitution of a derivation for an open assumption in natural deduction. If $e(X)$ has no open door in $\mathcal{S}'$ then $e(X)$ remains unchanged after an application of (2). Hence consideration of the doors of an empire provides adequate information about dependencies in a subderivation, in particular about the effects of the process of substitution on a subderivation.

2.3. Chains; Properties of Empires.

Let $\mathcal{S}$ be a proof structure for $\mathbf{MLL}^-$ or $\mathbf{NCMLL}^-$. Suppose X, Y and $A \otimes B$ are formula occurrences in $\mathcal{S}$, such that $X \not\prec A$ and $Y \not\prec B$. We write $X - (A \otimes B) - Y$ if axioms connect X with A and B with Y, i.e., if there are axioms $\overline{P', \ldots, P}$ and $\overline{Q, \ldots, Q'}$ such that $P' \prec X$, $P \prec A$, $Q \prec B$, $Q' \prec Y$. A *chain* is a configuration of the form

$$X - (A_0 \otimes B_0) - (A_1 \otimes B_1) - \ldots - (A_n \otimes B_n) - Y.$$

A chain is *pure* if $A_i \otimes B_i$ is a different formula occurrence from $A_j \otimes B_j$, for $i \neq j$. A pure chain where Y is $A_0 \otimes B_0$ is called a *cycle* (this include the case of a conjunction $A_0 \otimes B_0$, where A_0 and B_0 are connected by an axiom). We say that a proof structure satisfies the *acyclity condition* if it contains no cycle.

2.3.1. Quasi-Transitivity Lemma. *Assume $A \in e(B)$ and $B \in e(C)$, and moreover suppose that $\tau : [B] \to A$ is a computation tree in which no sequence contains a step of the form $\downarrow \frac{C}{X}$ or $\Downarrow \frac{C}{X}$ for any X. Then $A \in e(C)$.* ∎

2.3.2. Corollary. *If $B@C$, then $e(B) \subset e(C)$. Hence @ is a strict partial order.* ∎

2.3.3. Proposition. *Suppose $B \not\prec A$ and let α be any sequence in a normal $\tau : [A] \to B$;*
(1) *if the last step in α has the form ($\uparrow$), then there exists $C_0 \otimes C_1$ such that α ends with*

$$\downarrow \frac{C_i}{C_0 \otimes C_1}, \uparrow \frac{C_{1-i}}{C_0 \otimes C_1}, \quad \text{only } (\uparrow) \text{ steps;}$$

(2) *if the last step of α has the form ($\downarrow$) or ($\Downarrow$), then there exists an axiom $\overline{X_1, \ldots, X_p}$ such that α ends with*

$$\uparrow \frac{X_i}{Y}, \quad \text{only } (\downarrow) \text{ or } (\Downarrow) \text{ steps.}$$ ∎

2.3.4. Corollary. (i) *Every α in $\tau : [A] \to B$ is a chain.*

(ii) *The* acyclicity *condition implies the* vicious circle *condition.* ■

Remarks. (i) The converse is not true, i.e., a chain need not belong to the empire of its first member. However, if the connectedness condition holds, then for every chain $\mathcal{C}$ there is an element $A_i \otimes B_i$ such that for every $A_j \otimes B_j \in \mathcal{C}$ with $j \neq i$, we have $A_j \otimes B_j \in e(A_i) \cup e(B_i)$. Using the Tiling Lemma below, we can show that if $\mathcal{C}$ is cyclic, then the vicious circle fails for $A_i \otimes B_i$. That is, the *vicious circle* and *connectedness* condition imply the *acyclicicty* condition.

(ii) In general, absence of cycles is a stronger condition, of conceptual interest since it guarantees consistency for systems in which the connectedness condition is not required in general. Such systems are variants of *direct logic.* Consider the language of multiplicative linear logic without the constants $\perp$ and **1** and let **DL**, **DL**$^+$ be the logics formalized by the extension of sequent calculus for **MLL** with the rule of Mingle or Weakening, respectively (see section 1.2). We have the following result below, [4] which provides an alternative proof of the sequentialization theorem for **MLL**.

Call a non-empty set $\mathcal{P}$ of axiom links in a proof structure with conclusions Γ a *path for* Γ if it satisfies the following *relevance condition on conjunctions.* We write $\mathcal{P} \mapsto X$ if for some $\overline{P, P^\perp} \in \mathcal{P}$, we have $P \prec X$ or $P^\perp \prec X$; then the relevance condition on $A \circ B$ requires that $\mathcal{P} \mapsto A$ if and only if $\mathcal{P} \mapsto B$.

A path $\mathcal{P}$ for Γ is *minimal* if no proper subset is a path for Γ. It is easy to show that minimality of $\mathcal{P}$ is equivalent to the connectedness condition (for those disjunctions and pairs of conclusions for which the relevance condition holds).

Theorem. $\vdash \Gamma$ *is provable without Cut in the calculus of sequents for* **DL** [**DL**$^+$] *if and only if there is a proof structure $\mathcal{S}$ with conclusions Γ such that the axiom links are a* [*minimal*] *path for Γ and no chain is a cycle in $\mathcal{S}$.*

2.4. Properties of consistent Proof Structures.

2.4.1. Proposition. *Let $\mathcal{S}$ be a proof structure for* **MLL**$^-$ *or* **NCMLL**$^-$ *satisfying the vicious circle condition and let $\tau : [A] \to B$ be normal.*

(i) *if B is $C_0 \otimes C_1$ and no $\alpha \in \tau : [A] \to B$ ends with a step $\downarrow \frac{C_i}{C_0 \otimes C_1}$, then $C_i @ A$;*

(ii) *If $B \prec A$, then $B@A$.*

(iii) *if all $\alpha \in \tau : [A] \to B$ end with a step of the form $(\uparrow)$, then $B@A$.*

Proof. (i) Clearly $C_i \in e(A)$. Also, $C_{1-i} \in e(A)$ and by the assumption of the case from $\tau : [A] \to B$ we can build a normal tree $\tau : [A] \to C_{1-i}$ which does not contain any

[4] Bellin and Ketonen `[1989?]` present a proof of the sequentialization theorem for **DL**$^+$. Proofs of the sequentialization theorem for **DL** and **DL**$^+$ can be found in Bellin `[1990]`.

step $\downarrow \frac{C_i}{C_0 \otimes C_1}$. Now if $A \in e(C_i)$ then $C_{1-i} \in e(C_i)$, by the Quasi-Transitivity Lemma (2.3.1.),and this is a 'vicious circle'. (ii) We assume that $A \in e(B)$, in order to find a contradiction. We claim that there is a conjunction $D_0 \otimes D_1 \in e(B)$ and a normal tree $\tau : [B] \to D_0 \otimes D_1$ with the property that $B \prec D_j$ and every β in $\tau : [B] \to D_0 \otimes D_1$ ends with the step $\downarrow \frac{D_{1-j}}{D_0 \otimes D_1}$. It follows from our claim that $\downarrow \frac{D_j}{D_0 \otimes D_1}$ never occurs in $\tau : [B] \to D_0 \otimes D_1$. Consider the subtree $\tau : [B] \to D_{j-1}$; since obviously $B \in e(D_j)$, we can apply the Quasi-Transitivity Lemma to obtain $D_{1-j} \in D_j$, which is a 'vicious circle'.

To prove the claim, let $\tau : [B] \to A$ be normal. If some $\alpha \in \tau : [B] \to A$ ends with a step $(\uparrow)$, then the desired $D_0 \otimes D_1$ is given by Proposition 2.3.3. in the last section. Otherwise, starting from the root A of $\tau : [B] \to A$ proceed as follows. If A is $X_0 \sqcup X_1$, then a branching $\Downarrow \frac{X_0 \quad X_1}{X_0 \sqcup X_1}$ occurs, and we choose the subtree $\tau' : [B] \to X_j$ such that $B \prec X_j$. Otherwise, A is $X_0 \otimes X_1$ and every sequence in the tree under consideration ends with a step $\downarrow \frac{X_k}{X_0 \otimes X_1}$. If $B \not\prec X_k$, then we must have $B \prec X_{1-k}$, and we are done. If $B \prec X_k$, then consider the subtree $\tau' : [B] \to X_k$. Since there are only finitely many Y such that $B \prec Y$ in $\mathcal{S}$, failure to encounter the desired $D_0 \otimes D_1$ would mean that we end with a step $\downarrow \frac{Y_i}{Z}$ or $\Downarrow \frac{Y_0 \quad Y_1}{Y_0 \sqcup Y_1}$ where Y_i is B *itself*. But this is absurd, since such a step is not admissible in a computation of $e(B)$.

Part (iii) easily follows from (i), (ii) and proposition 2.3.3. ∎

2.5. Properties of Proof Nets.

2.5.1. Proposition. *Let $\mathcal{S}$ be a proof net for* **MLL**$^-$ *or* **NCMLL**$^-$*. The following are equivalent*:

(i) $B@A$;

(ii) *for some door D of $e(B)$, there is a normal $\tau : [A] \to D$ such that every $\alpha \in \tau : [A] \to D$ ends with a step* $(\uparrow)$;

(iii) *there is a normal $\tau : [A] \to B$ such that every $\alpha \in \tau : [A] \to B$ ends with a step* $(\uparrow)$.

Proof. $(iii) \Rightarrow (i)$ is part (iii) of the proposition in the last section and $(i) \Rightarrow (ii)$ is trivial. $(ii) \Rightarrow (iii)$. If D is the main door of $e(B)$, then we are finished. Otherwise, the last step in every sequence in $\tau : [A] \to D$ crosses a link $\frac{C \quad D}{C \sqcup D}$ by a step $(\uparrow)$. By changing the last step of every sequence we obtain a tree $\tau : [A] \to C$. By part (iii) in Proposition 2.4.1. $e(C) \subset e(A)$. By the connectedness condition $C \Diamond D$.

Next consider a normal tree $\tau : [C] \to D$ and $\alpha \in \tau : [C] \to D$. Since $D \in e(C) \cap e(B)$ and $C \notin e(B)$, $\alpha \in \tau : [C] \to D$ must enter $e(B)$ through some door D' by a step $(\uparrow)$. If D' is the main door, i.e., D is B, then the subtree $\tau : [C] \to B$ cannot contain steps $\downarrow \frac{A}{Z}$ or $\Downarrow \frac{A}{Z}$, since $e(C) \subset e(A)$; combining with $\tau : [A] \to C$ we obtain a tree $\tau : [A] \to B$. (Notice that the computation represented by $\tau : [A] \to B$ takes place outside $e(B)$, except for the last step. If $\tau : [A] \to C$ is not normal, simplification certainly yields a normal $\tau' : [A] \to C$ with the desired property.) Otherwise, D' is a closed door of $e(B)$ and the last step of α has the form $\uparrow \frac{D'}{C' \sqcup D'}$. We obtain trees $\tau : [C] \to C'$ and $\tau : [A] \to C'$; the computations they represent take place entirely outside $e(B)$. Again we have $C'\Diamond D'$, by connectedness.

Proceeding in this way, we obtain a sequences of doors D, D', D'',... and a sequence C, C', C'',... such that

$$\ldots \subsetneq e(C'') \subsetneq e(C') \subsetneq e(C) \subsetneq e(A) \subset \mathcal{S}.$$

This implies that C, C', C'', are distinct formula occurrences in $\mathcal{S}$; thus $C \sqcup D$, $C' \sqcup D'$, $C'' \sqcup D''$,... are also distinct. But $\mathcal{S}$ is finite, thus we cannot have an infinite discending sequence of proper inclusions. Therefore we eventually reach a D^* which is the main door of $e(B)$, by a computation taking place outside $e(B)$, as required. ∎

2.5.2. Tiling Lemma. (cf. Girard [1987a], II.1, Proposition 3) *Let $\mathcal{S}$ be a proof net for* **MLL**$^-$ *or* **NCMLL**$^-$. *Let A, $B \in \mathcal{S}$ be such that $A\emptyset B$. Then $e(A) \cap e(B) = \emptyset$.*

Proof. Suppose $X \in e(A) \cap e(B)$. We show that $A \notin e(B)$ implies $B \in e(A)$. Consider any sequence in a normal $\tau : [A] \to X$: since $A \notin e(B)$, α must enter $e(B)$ through some door D by a step $(\uparrow)$. Consider the subtree $\tau : [A] \to D$ and apply proposition 2.5.1. in the direction $(ii) \Rightarrow (i)$. ∎

2.5.3. Sequentialization Theorem. (cf. Girard [1987a], II.1, Remark 2.) *Every proof net in* **MLL**$^-$ *and in* **NCMLL**$^-$ *is an IPS.*

Proof. (**MLL**$^-$) By induction on the number of formulas in $\mathcal{S}$. If $\mathcal{S}$ is an axiom there is nothing to prove, and if the conclusions of $\mathcal{S}$ contain a disjunction $A \sqcup B$, then the result is immediate from the inductive hypothesis. Let Γ be the set of conclusions and of ghosts of $\mathcal{S}$ and suppose every formula occurrence in Γ is the conclusion of a $\otimes$ link. We need to find a link $\frac{A \quad B}{A\otimes B}$ with the property that if $\mathcal{S}'$ results from $\mathcal{S}$ by removing only the link in question, then $\mathcal{S}$ is partitioned in two subnets $\mathcal{S}_1 = e(A)$ and $\mathcal{S}_2 = e(B)$.

Suppose for some $A \otimes B \in \Gamma$ all the doors of $e(A)$ and of $e(B)$ are open or ghosts. Then for every $C \in \Gamma \setminus \{A \otimes B\}$, either $C \in e(A)$ or $C \in e(B)$. To see this, let $C \in \Gamma$ be outside $e(A) \cup e(B)$. There cannot be any axiom connecting C with any formula in $e(A)$ or of $e(B)$, since otherwise some $D \prec C$ would be a door of either $e(A)$ or of $e(B)$. This implies that $A \otimes B$ and C belong to substructures of $\mathcal{S}$ which are disconnected, i.e., we cannot have $A \otimes B \Diamond C$, a contradiction.

We show that it is enough to find an element $A \otimes B$ of Γ such that $e(A)$ or $e(B)$ is maximal with respect to inclusion. If $A_0 \otimes B_0 \in \Gamma$ is such that B_0 has a side door D, then $D \prec A_1 \otimes B_1 \in \Gamma$, say $D \prec A_1$. Since $D \in e(B_0) \cap e(A_1)$ and $A_1 \notin e(B_0)$, the Tiling Lemma yields $B_0 \in e(A_1)$; hence $e(B_0) \subset e(A_1)$. We also obtain $e(A_0) \subset e(A_1)$ as follows. $B_0 \in e(A_1)$ implies $A_0 \otimes B_0 \in e(A_1)$, hence $A_0 \in e(A_1)$. Now $A_1 \in e(A_0)$ would imply $D \in e(A_0) \cup e(B_0)$, thus by the Tiling Lemma $A_0 \in e(B_0)$ or $B_0 \in e(A_0)$, a 'vicious circle'. Hence $e(A_0) \cup e(B_0) \subsetneq e(A_1)$ i.e., neither $e(A_0)$ nor $e(B_0)$ can be maximal with respect to inclusion.

To extend the above argument to the case of **NCMLL**$^-$, we also need to show that in the non-trivial case there are cyclic orderings 1_1 and 1_2 with the following property. Let

Δ, A and B, Λ be the lowermost formula occurrences in $e(A)$ and $e(B)$, ordered according to l_1 and l_2, respectively: then $\Delta, A \otimes B, \Lambda$ is Γ in the given ordering l .

By applying a cyclic permutation to Γ, if necessary, we may assume that Γ in the given ordering is

$$C_1, \dots, C_{i-1}, A \otimes B, C_{i+1}, \dots, C_n$$

where C_1 is in $e(A)$, and $\Delta_1 = C_{i+1}, \dots, C_n$ are in $e(B)$. Let Δ_0 be the set of C_j such that C_j is in $e(B)$ and $j < i$; suppose Δ_0 is non-empty. By connectedness there must be an axiom $\overline{P, \dots, P'}$ that connects some formula in Δ_0 with a formula in Δ_1, say $P \prec \Delta_0$. Therefore we can partition Λ into Λ_0 and Λ_1, according to their position relatively to P in the given ordering l , say $C_1 \in \Lambda_0$ and $A \in \Lambda_1$. Again by connectedness we must have an axiom $\overline{Q, \cdots, Q'}$ connecting Λ_0 and Λ_1, say $Q \prec \Lambda_0$. Hence

$$Q \, l \;\; P \, l \;\; Q' \, l \;\; P'$$

violates the planarity condition. Hence Δ_0 is empty, and the ordering l_1 and l_2 are easily obtained. ∎

By applying the Tiling Lemma and similar techniques we can prove the following useful facts.

2.5.2. Door Lemma. *Let $\mathcal{S}$ be a proof net for* **MLL**$^-$ *or* **NCMLL**$^-$*, let A occur in $\mathcal{S}$ and let D, D' be any two doors of $e(A)$. Then $e(A) \subset e(D)$ and $D \Diamond D'$.* ∎

2.5.3. Shared Empires Lemma. (cf. Girard [1987a], II.2) *Let $\mathcal{S}$ be a proof net for* **MLL**$^-$ *or* **NCMLL**$^-$ *and let $A \Diamond B$ in $\mathcal{S}$. $e(A) \cap e(B)$ is the set of all formula occurrences X in $\mathcal{S}$ such that $X \in e(A)$ and $X \in e(B)$ can be computed without exiting $e(A) \cap e(B)$.* ∎

The Subnet Lemma (section 5) is an immediate consequence of the Shared Empires Lemma.

2.6. Extension to First Order Proof Nets.

The notions of proof structure with non-logical axioms and of substructure are defined in section 1.3.

Let $\mathcal{S}'$ be a proper substructure of $\mathcal{S}$, where the inclusion map strongly preserves the axioms, and let Δ be the conclusions of $\mathcal{S}'$. The *complementary substructure* $\overline{\mathcal{S}}$ *of* $\mathcal{S}'$ *in* $\mathcal{S}$ is the set $\mathcal{S} \setminus \mathcal{S}'$, satisfying the same links as $\mathcal{S}$ (relatively to $\mathcal{S} \setminus \mathcal{S}'$) and in addition the axiom link $\overline{\Delta}$. (Notice that strictly speaking $\overline{\mathcal{S}}$ is not a substructure of $\mathcal{S}$.)

To sequent derivations with non-logical axioms there correspond the notions of *IPS with non-logical axioms* and of *proof net with non-logical axioms*. The results in the last sections, in particular the Tiling Lemma and Sequentialization Theorem, still hold for proof nets with non-logical axioms. Moreover we have the following

2.6.1. Lemma. *If A and B both belong to $\mathcal{S}'$ [to $\overline{\mathcal{S}}$], then $B \in e(A)$ in $\mathcal{S}'$ [in $\overline{\mathcal{S}}$] if and only if $B \in e(A)$ in $\mathcal{S}$*

Proof. Use the Shared Empires Lemma in the case of $\mathcal{S}'$, and direct consideration of a computation tree $\tau : [A] \to B$ in the case of $\overline{\mathcal{S}}$. ∎

It follows that $\mathcal{S}$ *is a proof net if and only if* $\mathcal{S}'$ [*and* $\overline{\mathcal{S}}$] *are proof nets* [*with non-logical axiom* Δ.]

To prove the Sequentialization Theorem in the first order case, we reduce to the propositional case by induction on the number of eigenvariables as follows. Let $\mathcal{S}$ be a proof net for first order $\mathbf{MLL}^-$ with conclusions Γ, where all formulas in Γ are closed. Notice that *eigenvariables may still occur in the ghosts* of $\mathcal{S}$, although no eigenvariable occurs in the *conclusions* of $\mathcal{S}$. Choose an eigenvariable a maximal with respect to the ordering $<_t$, and let $\mathcal{S}_A$ be $e(A(a))$, where a corresponds to a link $\frac{A(a)}{\bigwedge x.A}$. Let $\mathcal{S}'$ be $e(\bigwedge x.A)$, which coincides with

$$\begin{array}{c}\mathcal{S}_A\\ A(a)\\ \hline \bigwedge x.A\end{array}$$

Let $\overline{\mathcal{S}}$ be the complementary substructure of $\mathcal{S}'$ in $\mathcal{S}$. In $\mathcal{S}_A$ the eigenvariable a can be regarded as a constant without affecting correctness and no eigenvariable occurs in any conclusion, apart from the occurrences of a in $A(a)$, since a is maximal w.r.t. $<_t$. Finally, a does not occur in $\overline{\mathcal{S}}$.

We want to apply the induction hypothesis to $\mathcal{S}_A$ and to $\overline{\mathcal{S}}$. We claim that the ordering $<_t$ is strict in $\mathcal{S}_A$ and in $\overline{\mathcal{S}}$ is strict since it is strict in $\mathcal{S}$. In fact, the impossiblity of $b <_t c <^0 b$ in $\overline{\mathcal{S}}$ follows from Lemma 2.6.1. and in $\mathcal{S}_A$ from the following

2.6.2. Proposition. *Let* $\mathcal{S}$ *be a proof net for first order* $\mathbf{MLL}^-$ *and let* a, b *be eigenvariables, where* a *is associated with* $\frac{A(a)}{\bigwedge x.A}$. *If* $b = a$ *or* $b <_t a$, *then* b *occurs only inside* $e(A(a))$. ∎

2.7. Extension to the full Multiplicative Fragment.

To extend the above results to the fragments **MLL** or **NCMLL** containing axioms and rules for $\mathbf{1}$ and $\bot$, we use the extended definition of proof structure with $\mathbf{1}$ axioms and $\bot$ links, as in section 1.3. We extend the definition of empire by adding to conditions (iii) and (iv) the cases

(iii) if $\frac{Y}{X_0 \quad X_1}$ is a $\bot$-link in $\mathcal{S}$ and $X_i \in e(A)$, $X_{1-i} \in e(A)$ ($\to$-*step*);

(vi) if $\frac{X}{Y \quad \bot}$ is a $\bot$ link in $\mathcal{S}$ $X \neq A$ and $X \in e(A)$, then $Y \in e(A)$ ($\downarrow$-*step*).

All the previous results still hold in this context.

Remarks. (i) Given a $\bot$ box

$$\begin{array}{c}\boxed{\begin{array}{c}\mathcal{S}'\\ \Gamma\end{array}}\\ \Gamma^{-}\quad \perp\end{array}$$

we must choose one formula in Γ to create the $\perp$ link, performing a non-deterministic operation.

(ii) In sequent calculus an application of the $\perp$ rule can always be permuted with the inference immediately above it.[5] Similarly, in a proof net we can 'move $\perp$ links upwards' and still obtain a proof net. Hence we could also generalize the definition of proof structure by introducing axiom links of the form

$$\overline{P, P^{\perp}, \perp, \ldots, \perp},$$

as well as $\overline{\mathbf{1}}$.

3. Part II. Proof Networks for Multiplicative and Additive Linear Logic.

We consider now the elimination of the $\sqcap$ boxes. Our solution will be in two steps: first, we extend the characterization of proof nets in terms of empires to proof structures for **MALL** (or **NCMALL**); second, we develop Girard's notion of *slicing of a box*, we introduce the notion of a *quasi structure* and define a generalized notion of empire over families of quasi structures. A family representing a correct proof (*proof network*) is one in which the *multiplicative conditions* (vicious circle and connectedness conditions) are satisfied by each quasi structure and the *additive conditions* (box and parameters conditions) are satisfied by the whole family according to the generalized notion of empire.

3.1. Empires in Proof Structures for MALL and NCMALL.

We define proof structures for **MALL** as for **MLL**, with in addition the $\top$ axioms

$$\overline{\top, X_1, \ldots, X_n}$$

for any X_i and n, and the following $\sqcap$ and Contraction links

$$\frac{A \quad B}{A \sqcap B} \qquad \frac{C' \quad C''}{C},$$

[5] Notice that if the latter is an $\otimes$ application, there is a non-deterministic choice of one branch (unless we decide that the $\otimes$ rule behaves 'additively' with respect all the formulas of the form $?A$; but this may be regarded as a rather inelegant trick).

where C, C' and C'' are occurrences of the same formula. The Contraction links are to be restricted as follows. Consider a proof structure with a $\sqcap$ box

$$\begin{array}{c} \boxed{\begin{array}{ccc} \mathcal{S}' & \qquad & \mathcal{S}'' \\ \\ \Xi\ \Gamma'\ A & & \Theta\ \Gamma''\ B \end{array}} \\ \Gamma \qquad A \sqcap B \end{array} \qquad (\P)$$

satisfying condition (§) of section 1.5, i.e., such that Γ, Γ' and Γ'' are the same multiset. In the case of **NCMALL**, we are given orderings $\mathcal{S}'$, $\mathcal{S}''$ and of $\overline{\Gamma, A \sqcap B}$ that preserve the correspondence (§). In the case of **MALL** there may be many ways of establishing the correspondence (§): we choose one. Let Γ be $C_1, \ldots, C_n$. Now we can regard the structure in (¶) as consisting of $\mathcal{S}' \bigcup \mathcal{S}'' \bigcup \Gamma \bigcup \{A \sqcap B\}$ with the same links and in addition and of a configuration of the form (¶¶) (a sequence of Contraction links and precisely one $\sqcap$ link)

$$\frac{C'_0 \quad C''_0}{C_0} \quad \ldots \quad \frac{C'_{n-1} \quad C''_{n-1}}{C_{n-1}} \qquad \frac{A \quad B}{A \sqcap B}. \qquad (\P\P)$$

The Contraction links in (¶¶) are justified by their association with the $\sqcap$ link. We define a *flagging function* φ that associates each Contraction link with a $\sqcap$ links and marks some ghosts and some premises of Contraction or $\sqcap$ links as belonging to the 'left' or to the 'right' proof structure in the box. We may simply say that φ *flags* $C'_1, \ldots, C'_n$, A and the ghosts in Ξ *with* A and it *flags* $C''_1, \ldots, C''_n$, B and the ghosts in Θ *with* B. Of course, if C'_i is flagged with A, then C''_i will be flagged with B, and viceversa. A ghost is *external* if it is not flagged with any formula occurrence.

We assume that *every proof structure $\mathcal{S}$ in notation* (¶¶) *is given together with a flagging function φ*. Clearly the difference between notations (¶) and (¶¶) is inessential.

Next we extend the definition of empire (section 2.1) to proof structures in notation (¶¶) by letting clause (v) be

(v) if $\frac{X_1 \quad X_2}{X_1 \sqcup X_2}$ [or $\frac{X_1 \quad X_2}{X_1 \sqcap X_2}$ or a Contraction $\frac{X_1 \quad X_2}{X}$] is a link in $\mathcal{S}$, $X_1 \neq A \neq X_2$, $X_1 \in e(A)$ and $X_2 \in e(A)$, then $X_1 \sqcup X_2$ [or $X_1 \sqcap X_2$ or X, respectively] is in $e(A)$ ($\Downarrow$ - *step*).

We define a proof net for first order **MALL** [or **NCMALL**] as a proof structure satisfying the (1) vicious circle, (2) connectedness, (3) parameters, (4) box [and planarity] condition.

The vicious circle, parameters and planarity conditions are as before. The *connectedness* condition in this context is weakened:

(2) For every link $\frac{X \quad Y}{X \sqcup Y}$ in $\mathcal{S}$, and for every X, Y that are conclusions of $\mathcal{S}$ or *external* ghosts, we have $X \diamondsuit Y$.

We take the following formulation of the box condition:

(4) *The official box condition*: For every link $\frac{A_0 \quad A_1}{A_0 \sqcap A_1}$,

(i) the connectedness condition holds among all occurrences flagged with one of its premises, i.e., for every D, if D is flagged with A_i, then $D \diamondsuit A_i$;

(ii) the no vicious circle condition holds for the $\sqcap$ link, i.e., $A_0 \emptyset A_1$, and

(iii) for every premise A' of a $\sqcap$ link, $A_i \diamondsuit A'$ implies $A' = A_i$.

Remark. Such a formulation of the box condition may seem awkward, like the following Box Lemma. However, the advantage is that only (i) will remain as a condition, when the boxes are sliced.

The definition of inductively generated proof structures is easily extended to **MALL** and **NCMALL**. In the clause allowing the introduction of a link $\frac{A_0 \quad A_1}{A_0 \sqcap A_1}$ we require that the structures containing A_0 and A_1 are disjoint (as the case for $\otimes$ links, clause (2), section 1.4). The details are left to the reader. The verification that every IPS satisfies the box condition is almost immediate, thus we have

3.1.1. Theorem. *Every IPS in* **MALL** *or* **NCMALL** *is a proof net.* ∎

The properties of empires (section 2.3) and of consistent proof structures (section 2.4) remain true. In addition, the main facts about boxes are the following.

3.1.2. Box Lemma. *Let $\mathcal{S}$ be a proof net for* **MALL** *or* **NCMALL** *and let $\frac{A_0 \quad A_1}{A_0 \sqcap A_1}$ be a link in $\mathcal{S}$.*

(i) *If $X \in e(A_i)$, then $e(X) \subset e(A_i)$;*

(ii) *no closed door of $e(A_i)$ is a premise of a $\sqcup$ link;*

(iii) *the doors of A_i are precisely the formula occurrences flagged with A_i together with the ghosts flagged with some A', where $A' @ A_i$;*

(iv) *if $\frac{A' \quad A''}{A' \sqcap A''}$ is a link and $A' @ A_i$, then $A'' @ A_i$;*

(v) $e(A_0) \cap e(A_1) = \emptyset$.

Proof. (i) Let $Y \in e(X)$ and consider any sequence in a normal computation $\tau : [X] \to Y$: if $Y \notin e(A_i)$, then there must be a step of the form $\Downarrow \frac{C \quad D}{E}$, where, say, D is a door of $e(A_i)$. Since $C \notin e(A)$, the subtree $\tau : [X] \to C$ must contain a step $\Downarrow \frac{C' \quad D'}{E'}$, where, e.g., D' is a door of $e(A_i)$. Moreover C' and C must be different formula occurrences, because $\tau : [X] \to Y$ is normal; thus D is different from D' too. By repeating this argument we obtain an infinite sequence D, D'. ... of distinct doors of $e(A_i)$, a contradiction, since $\mathcal{S}$ is finite.

(ii) Let $\frac{C \quad D}{C \sqcup D}$ be a link in $\mathcal{S}$ with $D \in e(A_i)$. By the connectedness condition, $C \diamondsuit D$; by part (i), $C \in e(A_i)$; hence D cannot be a door of A_i.

(iii) If A and A' are premises of two distinct $\sqcap$ links, by the box condition we have only three possibilities, $A @ A'$ or $A \emptyset A'$ or $A' @ A$. By Corollary 2.3.2, @ is a strict partial ordering. Therefore we can argue by induction on @ over the set of premises of $\sqcap$ links.

Fix i, say $i = 0$. The fact that every formula flagged with A_0 is a door of $e(A_0)$ follows from the following remarks. A_0 is a door of $e(A_0)$ by part (ii) of Proposition 2.4.1. A ghost flagged with A_0 is obviously a door. Let X' be flagged with A_0 and occur in a Contraction link $\frac{X' \quad X''}{X}$. Hence X'' is flagged with A_1 and $A_1 \in e(X'')$ by the box condition. If $X \in e(A_0)$, then also $X'' \in e(A_0)$, hence $A_1 \in e(A_0)$ by part (i), contradicting the box condition. Hence $X \notin e(A_0)$, and X'' is a door of $e(A_0)$.

We prove that every door of $e(A_0)$ is flagged with A_0. This is true of the main door A_0. Let D be a door of $e(A_0)$, with $D \neq A_0$. First, D cannot be a conclusion of $\mathcal{S}$ nor an external ghost, since this implies $A_1 \in e(D)$ by connectedness, hence $A_1 \in e(A_0)$ by part (i), contradicting the box condition. Second, let D be a ghost flagged with A'. By the box condition $A' \in e(D)$; by part (i) we have $A' \in e(A_0)$; using again the box condition we obtain either $A' = A_0$ or $A'@A_0$, as required. Third, if D is a closed door of $e(A_0)$, then it must be a premise of a $\sqcap$-link or of Contraction link, say $\frac{D \quad C}{E}$, by part (ii). Now every such premise is flagged, say D is flagged with A' and C is flagged with A'' for some $\frac{A' \quad A''}{A' \sqcap A''}$. As before we obtain $A' \in e(A_0)$. We claim that $A_0 \Diamond A'$: by the box condition the claim implies $A_0 = A'$, as required. If not, the only possibility is $A'@A_0$, hence by the induction hypothesis every door of $e(A')$ is flagged with A'. A door X of $e(A')$ is certainly encountered in any normal computation $\tau : [A_0] \to A'$, which must contain a step $\uparrow \frac{X \quad Y}{Z}$. Since by induction hypothesis X is flagged with A', Y must be flagged with A'', and by the box condition $C \Diamond Y$. But obviously $Y \in e(A_0)$, hence $C \in e(A_0)$, contradicting the fact that D is a door of $e(A_0)$.

(iv) In any α in $\tau : [A_i] \to A'$ consider a step of the form $\uparrow \frac{C \quad D}{E}$, where D is a door of A' and apply parts (i) and (iii).

(v) Let $X \in e(A_0) \cap e(A_1)$. First notice that X can be neither a conclusion of $\mathcal{S}$ nor an external ghost, by the connectedness condition for $\mathcal{S}$ and part (i). Next, X can be neither a ghost flagged with A_i nor a closed door of $e(A_i)$: by parts (iii) and (i) this implies $A_i \in e(A_{1-i})$, a contradiction.

Now suppose that $X \in e(A_0) \cap e(A_1)$ is a ghost flagged with A', where A' is a premise of a $\sqcap$ link and $A'@A_i$. By part (iv) the conclusion of that link also belongs to $e(A_0) \cap e(A_1)$ and we consider the first Y such that $A' \prec Y$ and Y is a door of $e(A_i)$. By the argument of the previous paragraph, this can only be a ghost flagged with some A''. Proceeding this way, we obtain an infinite sequence of premises of $\sqcap$ links

$$\ldots @A''@A'@A_i$$

which is impossible, since $\mathcal{S}$ is finite.

Finally, if X is not a door of $e(A_0)$ nor of $e(A_1)$, consider the first $D \in e(A_0) \sqcap e(A_1)$ such that $X \prec D$ and D is a door of $e(A_i)$ and we apply the previous argument to D. ∎

It is now immediate that the *official* box condition implies *version 1* of the same condition, if in (2) we take $\mathcal{S}' = e(A)$, $\mathcal{S}'' = e(B)$ and let

$$\mathcal{S}^o = \mathcal{S} \setminus \big(e(A) \cup e(B)\big)$$

with the additional non-logical axiom $\overline{\Gamma, A \sqcap B}$. It is also easy to see that *version 2* of the box condition follows from the *official* box condition: by continuing the above decomposition until all $\sqcap$ links are eliminated, we obtain a class $\mathbf{C}$ of multiplicative proof structures (with non-logical axioms and $\otimes$ links). The relation $\lhd_t$ is obtained from @ applied to premises of $\sqcap$ links.

3.1.3. Proposition. *Let $\mathcal{S}$ be a proof net for* **MALL** *or* **NCMALL** *and let* **C** *be the collection of proof nets for* **MLL** *or* **NCMLL** (with non-logical axioms and $\otimes$ links) *resulting by decomposing all the $\sqcap$ links in $\mathcal{S}$. For every $\mathcal{S}_0 \in$* **C**, *if A and B both belong to $\mathcal{S}_0$, then $B \in e(A)$ in $\mathcal{S}$ if and only if $B \in e(A)$ in $\mathcal{S}_0$.*

Proof. By direct inspection of normal trees $\tau : [A] \to B$ in $\mathcal{S}_0$ or in $\mathcal{S}$. ∎

3.1.4. Sequentialization Theorem. *If $\mathcal{S}$ is a proof net for* **MALL** *and* **NCMALL**, *then $\mathcal{S}$ is an IPS.*

Proof. By induction on $\lhd_t$, using the above proposition and the Sequentialization Theorem for **MLL** or **NCMLL** (extended with $\oplus$ links and non-logical axioms). ∎

3.2. Quasi Structures and Structures.

A *Quasi Structure* $\mathcal{Q}$ is defined as a proof structure for **MALL** or **NCMALL** except that (a) there are no Contraction links and (b) $\sqcap$ links have only one premise:

$$\frac{X}{X \sqcap Y} \qquad \frac{Y}{X \sqcap Y}.$$

Maps preserving links, axioms and conclusions are defined for Quasi Structures as in the case of proof structures.

We want to define the category **Struct** of Structures, whose objects are either proof structures for **MALL** or Quasi Structures and whose maps are embeddings (and similarly for the definition of the category of **NCStrct** of noncommutative structures). We need to define an appropriate notion of embedding of a Quasi Structure in a proof structure.

A *segment* σ in $\mathcal{S}$ is a sequence of occurrences $C_0,\ldots,C_n$ of the same formula with the property that

(1) C_0 is not the conclusion and C_n is not the premise of a Contraction link;

(2) if $0 \leq i < n$, then C_{i+1} the conclusion of a Contraction link with premise C_i.

Given a segment $\sigma = C_0, \ldots, C_m$, we say (with some abuse of terminology) that σ is a premise [conclusion] of a link if and only if C_m is a premise [C_0 is a conclusion] of it. We use the notation

$$\frac{}{\sigma_1 \ldots, \sigma_k} \qquad \frac{\sigma_1}{\sigma} \qquad \frac{\sigma_1 \quad \sigma_2}{\sigma}$$

with the obvious meaning. Given a map $m : \mathcal{Q} \to \mathcal{S}$, where $\mathcal{Q}$ is a Quasi Structure and $\mathcal{S}$ a proof structure, we say that *m preserves an axiom or a $\circ$ link*

$$\frac{}{A_1 \ldots, A_n,} \qquad \frac{A_1 \quad A_2}{B} \quad \text{or} \quad \frac{A_0}{B}$$

if there are segments σ_i in $\mathcal{S}$ and an axiom or a $\circ$ link such that

$$\frac{}{\sigma_1 \ldots, \sigma_k,} \qquad \frac{\sigma_1 \quad \sigma_2}{\sigma} \quad \text{or} \quad \frac{\sigma_1}{\sigma}$$

and moreover $m(A_i) \in \sigma_i$ and $m(B) \in \sigma$. Here of course [the premise of] a *unary* $\sqcap$ links is mapped to [a formula occurrence in a segment which is a premise of] a *binary* $\sqcap$ link. Finally, we extend the notion of *preserving the conclusions* in a similar way.

Let $\mathbf{Struct}^*$ be the category whose objects $(\mathcal{O}, A)$ are structures $\mathcal{O}$ with a selected conclusion A and whose morphisms are embeddings preserving links and selected conclusion. Consider the assignments

Times $$((\mathcal{O}_1, A), (\mathcal{O}_2, B)) \mapsto (\mathcal{O}_1 \;{\scriptstyle \frac{A \quad B}{A \otimes B}}\; \mathcal{O}_2, A \otimes B)$$

Plus $$(\mathcal{O}, A) \mapsto (\mathcal{O}_{\frac{A}{A \oplus B}}, A \oplus B) \qquad (\mathcal{O}, A) \mapsto (\mathcal{O}_{\frac{B}{B \oplus A}}, B \oplus A)$$

Exists $$(\mathcal{O}, A(t)) \mapsto (\mathcal{O}_{\frac{A(t)}{\exists x.A}}, \exists x.A)$$

Fact. The assignment *Times* determines a bifunctor on $\mathbf{Struct}^*$. Given B or t, the assigments *Plus* or *Exists* determine functors on $\mathbf{Struct}^*$. ∎

3.3. Slicings of Proof Nets for MALL and NCMALL.

Let $\iota : \mathcal{O} \to \mathcal{O}'$ be an injective mapping such that X and $\iota(X)$ are occurrences of the same formula, for all $X \in \mathcal{O}$. Suppose first the both $\mathcal{O}$ and $\mathcal{O}'$ are either proof structures or quasi structures. We say that ι is a *quasi-embedding* if

(1) ι preserves all links different from axioms;

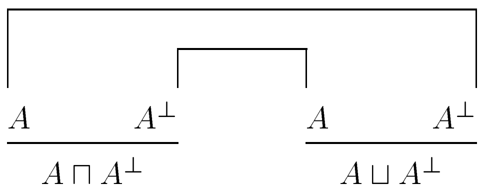
(2) ι preserves axioms in the weak sense that it maps an axiom

$$\overline{X_1, \ldots, X_n} \qquad \text{to} \qquad \overline{\iota(X_1), \ldots, \iota(X_n), Y_1, \ldots, Y_p};$$

(3) whenever $\iota(X_i)$ and $\iota(X_j)$ occur in an axiom $\overline{Y, \ldots, \iota(X_i), \ldots, \iota(X_j)}$ of $\mathcal{O}'$, then also X_i, X_j occur in one axiom of $\mathcal{O}$.

We extend the notion of quasi-embedding to the case $\iota : \mathcal{Q} \to \mathcal{S}$ by considering segments as above.

Example. Consider the following proof-structure $\mathcal{S}$.



There are quasi-embeddings of the following Quasi Structures $\mathcal{Q}_1$, $\mathcal{Q}_2$ into $\mathcal{S}$. Here $\overline{A}$ and $\overline{A^\perp}$ are non-logical axioms.

$$\mathcal{Q}_1: \qquad \frac{A^\perp}{A \sqcap A^\perp} \qquad \frac{A \quad \overline{A^\perp}}{A \sqcup A^\perp} \qquad\qquad \mathcal{Q}_2: \qquad \frac{A}{A \sqcap A^\perp} \qquad \frac{\overline{A} \quad A^\perp}{A \sqcup A^\perp}$$

Let $\mathcal{Q}_3$ be

$$\frac{\overline{A}}{A \sqcap A^\perp} \quad \frac{\overline{A} \quad \overline{A^\perp}}{A \sqcup A^\perp}$$

Then there is no quasi-embedding of $\mathcal{Q}_3$ in $\mathcal{S}$ nor between $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$.

Define *Slicing*$(\mathcal{S})$ to be the set of all quasi-embeddings $\iota : \mathcal{Q} \to \mathcal{S}$ preserving conclusions. We let

$$\mathit{Fam}(\mathcal{S}) \ = \ \{\mathcal{Q} : \text{ for some } \iota \text{ in } \mathit{Slicing}(\mathcal{S}),\ \iota : \mathcal{Q} \to \mathcal{S}\}.$$

Details for the noncommutative case are left to the reader.

3.4. Proof Networks for MALL and NCMALL.

Given a proof structure $\mathcal{S}$ for **MALL** or **NCMALL**, let $\mathcal{F} = \mathit{Fam}(\mathcal{S})$. Define an equivalence relation $\mathcal{R}$ on $\bigcup \mathcal{F}$ by letting $A' \ \mathcal{R}\ A''$ if and only if there are ι', ι'' in *Slicing*$(\mathcal{S})$ such that $\iota'(A') = \iota''(A'')$. If $A = \iota'(A')$, we let $[A]$ be the corresponding equivalence class. of formula occurrences. If $A' \in \mathcal{Q} \cap [A]$, then A' is unique, and in this case we sometimes write $A^{\mathcal{Q}}$ for A'.

For any $\mathcal{Q} \in \mathcal{F}$ the definition of empire of a formula occurrence is as in section 4. Let $\mathcal{F} = \mathit{Fam}(\mathcal{S})$ as before. The *empire* $e[A]$ of an *equivalence class* $[A]$ is a set of equivalence classes, defined as follows: $[B]{\in}e[A]$ in $\mathcal{F}$ if and only if for every $\mathcal{Q}{\in}\mathcal{F}$ such that $[B] \cap \mathcal{Q} \neq \emptyset$, we have $[A] \cap \mathcal{Q} \neq \emptyset$ and $B^{\mathcal{Q}}{\in}e(A^{\mathcal{Q}})$ in $\mathcal{Q}$.

In this context the box condition is as follows. Given $\mathcal{S}$ with flagging function φ, let and $\mathcal{F} = \mathit{Fam}(\mathcal{S})$.

Slicing Box Condition: (1) for every $D, D' \in \mathcal{S}$, if D and D' are flagged with the same formula, then $[D] \Diamond [D']$ in $\mathcal{F}$;

(2) every $\iota \in$ *Slicing*$(\mathcal{S}$ is an embedding.

A family $\mathcal{F} = \mathit{Fam}(\mathcal{S})$ is a *proof network* if every $\mathcal{Q} \in \mathcal{F}$ satisfies the vicious circle and connectedness condition and $\mathcal{F}$ satisfies the slicing box condition.

3.4.1. Proposition. *Let $\mathcal{S}$ be a proof structure for* **MALL** *or* **NCMALL**. *Let $\mathcal{F}$ be* $\mathrm{Fam}(\mathcal{S})$.

(i) *If* $\frac{X_0}{Y}$ $\left[\text{ or } \frac{X_0 \quad X_1}{Y}\right]$ *is a* $\circ$*-link in $\mathcal{S}$ different from a $\sqcap$ or Contraction link, then for each $\mathcal{Q} \in \mathcal{F}$ either* $[X_0] \cap \mathcal{Q} = [X_1] \cap \mathcal{Q} = \emptyset = [Y] \cap \mathcal{Q}$ *or* $\frac{X_0^{\mathcal{Q}}}{Y^{\mathcal{Q}}}$ $\left[\text{ or } \frac{X_0^{\mathcal{Q}} \quad X_1^{\mathcal{Q}}}{Y^{\mathcal{Q}}}\right]$.

(ii) *Conversely, every link in $\mathcal{Q}$ different from a $\sqcap$ link can be written as* $\frac{X^{\mathcal{Q}}}{Y^{\mathcal{Q}}}$ *or* $\frac{X_0^{\mathcal{Q}} \quad X_1^{\mathcal{Q}}}{Y^{\mathcal{Q}}}$ *in correspondence with some link of the same type in $\mathcal{S}$.*

(iii) *For every link* $\frac{X_0 \quad X_1}{X_0 \sqcap X_1}$ *in $\mathcal{S}$, for every $\mathcal{Q} \in \mathcal{F}$ and for some $i = 0, 1$, either* $[X_i] \cap \mathcal{Q} = \emptyset = [Y] \cap \mathcal{Q}$ *or* $\frac{X_i^{\mathcal{Q}}}{(X_0 \sqcap X_1)^{\mathcal{Q}}}$. *If $Z_0 \preceq X_0$ and $Z_1 \preceq X_1$, then $[Z_0] \cap [Z_1] = \emptyset$.*

(iv) *Conversely, every $\sqcap$ link in $\mathcal{Q}$ can be written as* $\frac{X_i^{\mathcal{Q}}}{(X_0 \sqcap X_1)^{\mathcal{Q}}}$ *in correspondence with some link* $\frac{X_0 \quad X_1}{X_0 \sqcap X_1}$ *in $\mathcal{S}$.*

Suppose $\mathcal{F}$ satisfies the box condition.

(v) *For every axiom* $\overline{X_1, \ldots, X_n} \in \mathcal{S}$ *and every $\mathcal{Q} \in \mathcal{F}$, either $[X_i] \cap \mathcal{Q} = \emptyset$, for all $i \leq n$, or* $\overline{X_1^{\mathcal{Q}}, \ldots, X_n^{\mathcal{Q}}}$ *is an axiom of $\mathcal{Q}$.*

(vi) *Conversely, every axiom of $\mathcal{Q}$ can be written as* $\overline{X_1^{\mathcal{Q}}, \ldots, X_n^{\mathcal{Q}}}$ *in correspondence with some axiom* $\overline{X_1, \ldots, X_n}$ *of $\mathcal{S}$.*

(vii) *For every Contraction link* $\frac{X \quad Y}{Z}$ *in $\mathcal{S}$, $[X] \cup [Y] = [Z]$ and for every $Z_0 \preceq X$ and $Z_1 \preceq Y$, $[Z_0] \cap [Z_1] = \emptyset$.* ∎

3.4.2. Equivalence Theorem. *Let $\mathcal{S}$ be a proof structure for propositional* **MALL** *or* **NCMALL**.

(i) *If* $\mathrm{Fam}(\mathcal{S})$ *satisfies the box condition, then $B \in e(A)$ in $\mathcal{S}$ if and only if $[B] \in e[A]$ in* $\mathrm{Fam}(\mathcal{S})$, *for all A, B*;

(ii) *$\mathcal{S}$ is a proof net if and only if* $\mathrm{Fam}(\mathcal{S})$ *is a proof network.*

Proof. (i) Using the proposition above, given a computation for $B \in e(A)$ in $\mathcal{S}$ we obtain family of computations of $B^{\mathcal{Q}} \in e(A^{\mathcal{Q}})$ for each $B^{\mathcal{Q}}$ in $[B]$, and conversely. (ii) If $\mathcal{S}$ is a proof net, then it is generated inductively and so is $\mathcal{F}$, i.e., each $\mathcal{Q}$ in $\mathcal{F}$ can be generated inductively. By theorem 2.2.1. (extended to quasi structures), each $\mathcal{Q}$ in $\mathcal{F}$ satisfies the vicious circle and connectedness conditions. Also in the inductive generation of $\mathcal{F}$ each axiom of $\mathcal{Q} \in \mathcal{F}$ is defined as a copy of an axiom of $\mathcal{S}$ and the box condition for $\mathcal{F}$ follows from the connectedness condition.

Conversely, if $\mathcal{F}$ is a proof network, then by part (i) we immediately obtain that the vicious circle and connectedness conditions hold for $\mathcal{S}$. To show that the *official box condition* (condition (4), section 3.1) also holds for $\mathcal{S}$, consider that condition (4)(i) is immediate from the *slicing box condition* for $\mathcal{F}$ and part (i). Next notice that for every $\frac{X' \quad X''}{X' \sqcap X''}$ in $\mathcal{S}$ there are $\mathcal{Q}'$ and $\mathcal{Q}''$ in $\mathcal{F}$ such that $[X'] \cap \mathcal{Q}' \neq \emptyset \neq [X''] \cap \mathcal{Q}''$ and $[X''] \cap \mathcal{Q}' = \emptyset = [X'] \cap \mathcal{Q}''$. Therefore condition (4)(ii) is immediate, again by part (i). To prove condition (4)(iii), consider two distinct $\sqcap$ links in $\mathcal{S}$, with conclusions $X = X_0 \sqcap X_1$

and $Y = Y_0 \sqcap Y_1$ and write $X \triangleleft Y$ if $X \preceq Z$, where Z is a formula flagged with Y_i, for $i = 0$ or 1.

On one hand, suppose $X \triangleleft Y$, say $X \prec Z \diamondsuit Y_0$, and consider a $\mathcal{Q} \in \mathcal{F}$ such that $[X_i] \cap \mathcal{Q} \neq \emptyset$. Then $X_i^{\mathcal{Q}} \in e(Z^{\mathcal{Q}})$ and $X_i^{\mathcal{Q}} \in e(Y_0^{\mathcal{Q}})$ (using the slicing box condition) and moreover $[X_{1-i}] \cap \mathcal{Q} = \emptyset$. This proves $[X_i] \in e[Y_0]$ and $[Y_0] \notin e[X_{1-i}]$. Excanging X_i and X_{1-i} and applying part (i), we obtain $X_0 @ Y_i$ and $X_1 @ Y_i$ in $\mathcal{S}$.

On the other hand, if neither $X \triangleleft Y$ nor $Y \triangleleft X$, then for each $i, j = 1, 2$ we can certainly find $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in $\mathcal{F}$ such that

$$[X_i] \cap \mathcal{Q}_1 = \emptyset \neq [Y_j] \cap \mathcal{Q}_1$$

and

$$[X_i] \cap \mathcal{Q}_2 \neq \emptyset = [Y_j] \cap \mathcal{Q}_2$$

by construction of $\mathcal{F}$. Hence in both cases we cannot have $X_i \diamondsuit Y_j$. ∎

Remark. Thus we have the following decision procedure for proof structures for propositional **MALL**: given $\mathcal{S}$, test the multiplicative conditions for every $\mathcal{Q} \in \mathit{Fam}(\mathcal{S})$; next test the slicing box condition, and in particular, consider whether every quasi-embedding $\mathit{Slicing}(\mathcal{S})$ is an embedding.

3.5. Slices.

In section 6, pp. 93-97 of Girard [1987] *slices* are defined for the whole system of linear logic, including exponentials. In particular, a *slice* for **MALL** is like a proof structure for $\mathbf{MLL}^-$, with the addition of $\mathbf{1}$ axioms, unary $\sqcap$ and $\oplus$, $\bigwedge$ and $\bigvee$ links and of sets of formula occurrences that are not conclusions of any link and do not belong to any axiom.

In **MALL** maps $*$, $\star$ and $\bullet$ have been constructed

$$\text{Sequent Derivations} \xrightarrow{*} \text{Proof Nets} \xrightarrow{\star} \text{Proof Networks} \xrightarrow{\bullet} \text{Sets of Slices}$$

that identify equivalent proofs. Notice that such a $*$ is unique in $\mathbf{MLL}^-$, $\star$ is unique in $\mathbf{MALL}^-$ and $\bullet$ is unique in **MALL**.

It would be desirable to give sets $\mathbf{S}$ of slices an independent logical meaning, as much as possible. To do this, one must express in some way the restriction on $\mathbf{S}$ as a whole which is given by the $\sqcap$ boxes, and the additional connections within each slice which is created by the $\perp$ boxes. Assuming this to be done, the problem of irrelevance created by the slicing of $\top$ boxes is easily solved if we assume that each slice satisfies the *relevance condition on conjunctions* (cf. section 2.3.). This amounts to the requirement that we check logical correctedness *only after garbage collection*, i.e., that $\mathbf{S}$ is the slicing of a proof structure which is normal with respect to the following reduction.

• *Zero Commutation.* A proof structure $\mathcal{S}$ is transformed into a proof structure $\mathcal{S}'$ by replacing a configuration of the form

$$\begin{array}{lccl} & & & e(C) \\ \overline{\top, X_1, \ldots, X_m, \quad B} & & C & Y_1, \ldots, Y_n \\ & \multicolumn{2}{c}{\overline{\quad B \otimes C \quad}} & \end{array}$$

where $Y_1, \ldots, Y_n$ are all the doors of $e(C)$, with the configuration

$$\overline{\top, X_1, \ldots, X_m, B \otimes C, Y_1, \ldots, Y_n}.$$

Notice that in a $\top$ box, an occurrence of $\top$ is marked as principal. Clearly, one effect of Zero Commutation on proof structures is to erase axiom link and markings on occurrences of $\top$. Define the *path* $\mathcal{P}$ of a proof structure $\mathcal{S}$ for **MALL** to be the set of axiom links different from a $\top$ axiom and of marked occurrences of $\top$ boxes (cf. section 2.3.). The following is clear:

Proposition. *Let $\mathcal{S}$ and $\mathcal{S}'$ be proof structures for* **MALL***, where $\mathcal{S}'$ results from $\mathcal{S}$ by an application of* Zero Commutation*. If the path $\mathcal{P}$ of $\mathcal{S}$ is non-empty and $\mathcal{S}$ satisfies the vicious circle condition, then so does $\mathcal{S}'$.* ∎

The above notions can be easily transfered to slices. Moreover, in a slice the notion of empire of a formula can be relativized to the given path $\mathcal{P}$: in the case of a link $\frac{X_0 \quad X_1}{X_0 \sqcup X_1}$ where X_i is irrelevant we let $X_0 \sqcup X_1 \in e_{\mathcal{P}}(A)$ if and only if $X_{1-i} \in e_{\mathcal{P}}(A)$. Therefore, if a slice satisfies the relevance condition on conjunctions, then the *multiplicative* consistency conditions are the vicious circle, connectedness conditions, relativized to the path $\mathcal{P}$ and, moreover, the requirement that $\mathcal{P}$ is non-empty. Details are left to the reader.

3.6. Strong Normalization Theorem.

The Strong Normalization Theorem for the system of proof nets (with boxes) for full second order linear logic is proved in Girard [1987], Section 4, pp.60-78. The reader is referred to that paper for a definition of all the *contraction rules* (definitions 4.1.- 4.18.) and for a definition of contractions for slices (definition 6.4.). It is also remarked there that the Church-Rosser Property holds for proof nets only in the fragment **MLL**$^-$, but that it does hold for sets of slices in the full system.

As an illustration, we consider the case of propositional **MALL**$^-$, the additive fragment without propositional constants.

A *Contraction for proof structures* transforms $\mathcal{S}$ into $\mathcal{S}'$ by one of the replacements indicated below.

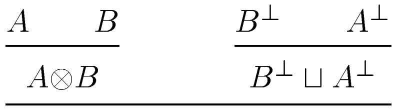
• **Axiom Contraction:** A configuration of the form

$$\begin{array}{lcc} \multicolumn{2}{l}{\overline{A^\perp, \quad A}} & A^\perp \\ & \multicolumn{2}{c}{\overline{\qquad\qquad}} \end{array}$$

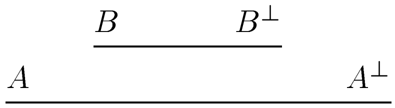
is replaced by

$$A^{\perp}.$$

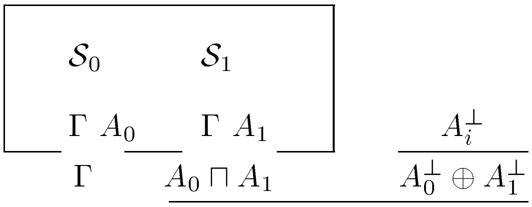
- *Times* **Contraction:** A configuration of the form



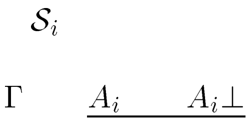
is replaced by



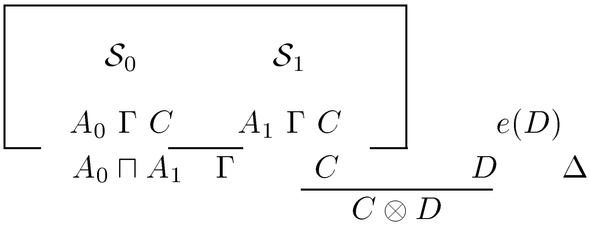
- *With* **Contraction:** A configuration of the form



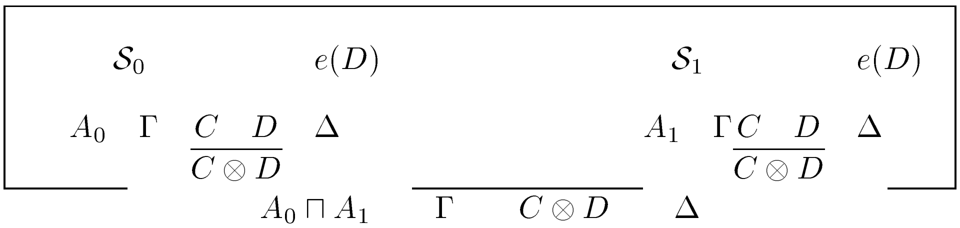
is replaced by



- *With* **Commutation:** A configuration of the form



is replaced by



A *Contraction for Quasi Structures* is defined like for proof structures in the case of an *Axiom* contraction or *Times* contraction. There are no commutations.

• *With* **Contraction:** If a Quasi Structure $\mathcal{Q}$ contains a configuration of the form

$$\underline{\dfrac{A_i}{A_0 \sqcap A_1} \qquad \dfrac{A_i^{\perp}}{A_0^{\perp} \oplus A_1^{\perp}}}$$

then $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by replacing that configuration by

$$\underline{A_i \qquad A_i^{\perp}}.$$

A Quasi Structure $\mathcal{Q}$ containing a configuration

$$\underline{\dfrac{A_i}{A_0 \sqcap A_1} \qquad \dfrac{A_{1-i}^{\perp}}{A_0^{\perp} \oplus A_1^{\perp}}}$$

is deleted (replaced by the empty set of formula occurrences).

3.6.1. Reduction Lemma. *If $\mathcal{S}$ is a proof net for* **MALL**$^-$ *or* **NCMALL**$^-$ *and $\mathcal{S}'$ results from $\mathcal{S}$ by application of one of the above contractions, then $\mathcal{S}'$ is a proof net and* $\bigcup$ Fam $(\mathcal{S}')$ *contains less links than* $\bigcup$ Fam $(\mathcal{S})$; *if $\mathcal{S}'$ results from $\mathcal{S}$ by commutation, then* Fam $(\mathcal{S}')$ = Fam $(\mathcal{S})$.

Proof. In the case of an *Axiom* or *Times* reduction, to prove that the vicious circle condition is preserved by a reduction it is convenient to argue as follows: suppose some $\mathcal{Q}' \in Fam(\mathcal{S}')$ contains a cyclic chain and show that there is a $\mathcal{Q} \in Fam(\mathcal{S})$ that contains a cyclic chain. Further details can be found in Bellin [1990]. ∎

3.6.2. Strong Normalization Theorem. *In* **MALL**$^-$ *or* **NCMALL**$^-$ *every reduction sequence starting from $\mathcal{S}$ terminates, and* Fam$(\mathcal{S})$ *reduces to a unique $\mathcal{F}$.* ∎

4. Part III. Permutability of Inferences in the Sequent Calculus for Linear Logic.

As an application, we give a classification of permutability of inferences in linear sequent calculus. We do not define standard terminology: principal formula, active formulas, ancestors of a formula, etc.

The table below is to be interpreted as follows.

Let $\mathcal{I}_1$, $\mathcal{I}_2$ be inferences such that:

(i) $\mathcal{I}_2$ occurs immediately after $\mathcal{I}_1$;

(ii) the principal formula of $\mathcal{I}_1$ is not active in $\mathcal{I}_2$

(iii) $\mathcal{I}_1$ and $\mathcal{I}_2$ are of the type indicated in the horizontal and vertical entries, respectively, then $\mathcal{I}_2$ can always be permuted above $\mathcal{I}_1$, unless explicitly indicated.

$\mathcal{I}_1$: / $\mathcal{I}_2$:	$\otimes$	$\sqcup$	$\sqcap$	$\oplus$	**!**	Der	Weak	Contr	Cut
$\otimes$					no				
$\sqcup$	no				no				no
$\sqcap$	no			no	no	no	no	no	no
$\oplus$					no				
!	no	no	no	no	no	no			no
Der									
Weak									
Contr	no								no
Cut					no				

Remarks. (i) The fact that Cut can be permuted above any inference other than a **!** rule is part of the proof of the Cut elimination theorem. In the case

!/Cut :

$$\mathcal{I}_2\ \dfrac{\mathcal{I}_3\ \dfrac{\cdots\cdots\cdots}{\vdash \Delta, \mathbf{!}C} \qquad \mathcal{I}_1\ \dfrac{\vdash \mathbf{?}(C^{\perp}), \mathbf{?}\Gamma, A}{\vdash \mathbf{?}(C^{\perp}), \mathbf{?}\Gamma, \mathbf{!}A}}{\vdash \Delta, \mathbf{?}\Gamma, \mathbf{!}A}$$

a permutation of $\mathcal{I}_1$ and $\mathcal{I}_2$ is permissible only if Δ has the form $\mathbf{?}\Lambda$, in particular, if $\mathcal{I}_3$ is itself a **!** rule.

(ii) A Cut $\mathcal{I}_1$ cannot be permuted below another inference $\mathcal{I}_2$ in the following case:

Cut /$\sqcup$:

$$\mathcal{I}_2\ \dfrac{\mathcal{I}_1\ \dfrac{\vdash B, \Delta, C^{\perp} \qquad \vdash C, \Gamma, A}{\vdash B, \Delta, \Gamma, A}}{\vdash A \sqcup B, \Delta, \Gamma}$$

A similar case arises for the pair Cut / Contraction. Cut can be permuted below a **!** rule only if the Cut formula C is of the form **?**D:

Cut /! :

$$\dfrac{\dfrac{\vdash ?\Delta, C^{\perp} \qquad \vdash C, ?\Gamma, A}{\vdash ?\Delta, ?\Gamma, A}\;\mathcal{I}_1}{\vdash ?\Delta, ?\Gamma, !A}\;\mathcal{I}_2$$

(*iii*) A permutation of Cut below a $\sqcap$ rule is permissible only if $\mathcal{I}_3$ is also a Cut with premises $\vdash \Delta, C$ and $\vdash C^{\perp}, \Delta, A$:

Cut/$\sqcap$:

$$\dfrac{\dfrac{\cdots\cdots}{\vdash \Delta, \Gamma, A}\;\mathcal{I}_3 \qquad \dfrac{\vdash \Delta, C^{\perp} \qquad \vdash C, \Gamma, B}{\vdash \Delta, \Gamma, B}\;\mathcal{I}_1}{\vdash \Delta, \Gamma, A \sqcap B}\;\mathcal{I}_2$$

(*iv*) Because of the restrictions on the side formulas of the **!** rule, in a derivation there cannot be a sequence of consecutive inferences $\mathcal{I}_1$, $\mathcal{I}_2$ as above, where $\mathcal{I}_2$ is a **!** rule and where $\mathcal{I}_1$ is one of the $\otimes$, $\sqcup$, $\sqcap$, $\oplus$, **!** rules.

(*v*) The following sequents provide *counterexamples* to the permutability of the indicated inferences: Cut-free proofs of the following sequents must end with a pair of logical inferences in the fixed ordering.

$!/\otimes$	$\vdash P^{\perp}, P \otimes ?(Q^{\perp}), !Q$
$!/\sqcup$	$\vdash ?(P^{\perp}) \sqcup ?(P^{\perp}), !P$
$!/\sqcap$	$\vdash (?(P^{\perp})) \sqcap \top, !P$
$!/\oplus$	$\vdash (?(P^{\perp})) \oplus Q, !P$
$\oplus/\sqcap$	$\vdash P^{\perp} \oplus Q^{\perp}, P \sqcap Q$
Der $/\sqcap$	$\vdash ?(P^{\perp}), !P \sqcap P$

Weak $/\sqcap$ $\qquad\vdash ?(P^\perp), P \sqcap \mathbf{1}$

Contr $/\sqcap$ $\qquad\vdash ?(P^\perp), P \sqcap (P \otimes P)$

$\otimes/\sqcup$ $\qquad\vdash Q^\perp \sqcup P^\perp, P \otimes Q$

$\otimes/$ Contr $\qquad\vdash ?(P^\perp), P \otimes P$

$\otimes/\sqcap$ $\qquad\vdash S^\perp \oplus T^\perp, (T \otimes S^\perp) \oplus R, S \otimes [P^\perp \oplus (R^\perp \otimes Q^\perp)], P \sqcap Q$

With respect to the last counterexample, notice that the permutation of $\mathcal{I}_1$ with $\mathcal{I}_2$ is impossible in the following case

$\otimes/\sqcap$ **:**

$$\dfrac{\mathcal{I}_3\,\dfrac{\cdots\cdots\cdots\cdots}{\vdash \Delta, C \otimes D, \Gamma, A} \qquad \mathcal{I}_1\,\dfrac{\vdash \Delta, C \qquad \vdash D, \Gamma, B}{\vdash \Delta, C \otimes D, \Gamma, B}}{\vdash \Delta, C \otimes D, \Gamma, A \sqcap B}\,\mathcal{I}_2$$

unless $\mathcal{I}_3$ is itself a $\otimes$ rule with premises $\vdash \Delta, C$ and $\vdash D, \Gamma, A$. Now let

$$\Delta = S^\perp \oplus T^\perp, \qquad C = S$$

and let

$$D = [P^\perp \oplus (R^\perp \otimes Q^\perp)], \qquad \Gamma = (T \otimes S^\perp) \oplus R, \qquad B = Q$$

and notice that

$$\nvdash\ (T \otimes S^\perp) \oplus R,\ [P^\perp \oplus (R^\perp \otimes Q^\perp)],\ P.$$

Since the **!** rule cannot be permuted above or below the $\otimes$, $\sqcup$, $\sqcap$ and $\oplus$ rules, the most interesting cases of a Permutability Theorem for linear logic can be stated simply for the fragment without **!** rule.

For such a fragment we can define a notion of proof net as before and the links for **?** require no modification in the definition of empire.

We define inductively what it means for an inference $\mathcal{R}$ in a deduction *to introduce* a formula-occurrence X: either X is the principal formula of $\mathcal{R}$ or X is a side formula and $\mathcal{R}$ introduces the *immediate ancestor* of X.

For simplicity, we consider only derivations in which every application of the $\perp$ *rule occurs above any other rule of inference.* Let $*$ be the mapping sending each derivation to the corresponding proof net, and let $\star$ send each proof net to the corresponding proof network. We write $\mathcal{D}^* = \mathcal{S}$, $\mathcal{S}^\star = \mathcal{F}$. For each formula occurrence A in a derivation $\mathcal{D}$, A^* is a formula occurrence in the proof net $\mathcal{D}^*$ and A^{**} is a class of formulas in the proof network $\mathcal{D}^{**}$. Also, given a $\mathcal{D}$ such that $\mathcal{D}^* = \mathcal{S}$ and B in $\mathcal{S}$, we write B^{-*} for the set of formula occurrences X in $\mathcal{D}$ such that $X^* = B$ and similarly we write $\big(e(B)\big)^{-*}$ for the inverse image of $e(B)$ in $\mathcal{D}$.

4.1. Permutability Theorem. *Let $\mathcal{D}$ be any derivation of Γ in propositional linear logic. For every formula occurrence A in $\mathcal{D}$ there exists a derivation $\mathcal{D}'$ with an inference $\mathcal{R}_+$ introducing an occurrence A' of the same formula as A and with the following properties.*

(0) *$\mathcal{D}'$ is obtained from $\mathcal{D}$ by permuting the order of the inferences.*

(1) *Suppose A is introduced in $\mathcal{D}$ by either the $\otimes$ rule or the $\sqcup$ rule or by a Contraction with active formulas A_1 and A_2; then $\mathcal{R}_+$ is an inference of the same kind and*

(i) $\mathcal{D}^* = (\mathcal{D}')^*$;

(ii) *in $\mathcal{D}'$, $\big(e(A_1^*) \cup e(A_2^*)\big)^{-*}$ is the set of formula occurrences that are introduced above $\mathcal{R}_+$.*

(2) *Suppose A is introduced in $\mathcal{D}$ either by the $\oplus$ rule or by Weakening or Dereliction; then $\mathcal{R}_+$ is an inference of the same kind and*

(i) $\mathcal{D}^* = (\mathcal{D}')^*$;

(ii) *in $\mathcal{D}'$, $\big(e(A^*)\big)^{-*}$ is the set of formula occurrences that are introduced above $\mathcal{R}_+$.*

(3) *Suppose A is introduced in $\mathcal{D}$ by the $\sqcap$ rule; then $\mathcal{R}_+$ is an inference of the same kind and*

(i) $\mathcal{D}^{**} = (\mathcal{D}')^{**}$;

(ii) $(e(A^*))^\star = (e(A'^*))^\star$ *and in $\mathcal{D}'$, $\big(e(A'^*)\big)^{-*}$ is the set of formula occurrences that are introduced above $\mathcal{R}_+$.*

Proof. (1) Let $\mathcal{R}$ be the inference of $\mathcal{D}$ introducing A (we may suppose A is the principal formula of $\mathcal{R}$) and consider the link

$$\frac{B_1 \quad B_2}{B}$$

in $\mathcal{S} = \mathcal{D}^*$, where $B = A^*$, etc. Let k be the number of formula occurrences Z^* in $\mathcal{S}$ that belong to $e(B)$ but such that Z is introduced below $\mathcal{R}$. By induction on $e(B)$ (see section 2.1), we reduce k by permuting the inferences of $\mathcal{D}$. The cases of clauses (i) – (iii) in the definition of empire are trivial.

Clause (iv): for a fixed $i = 1, 2$, $X^* \in e(B_i)$ and $X \neq B_i$ implies $(X \otimes Y)^* \in e(B_i)$. By induction hypothesis we may assume that X is introduced above $\mathcal{R}$. If $\mathcal{R}'$ introduces the principal formula $X \otimes Y$ and occurs below $\mathcal{R}$, there is a passive formula X' in every sequent between $\mathcal{R}$ and $\mathcal{R}'$ such that $(X')^* = X^*$. By the table above, we can certainly permute $\mathcal{R}'$ with the inference $\mathcal{R}''$ immediately above it. In this way it, we do not increase the number of formulas Z^* in $e(B_i)$ such that Z is introduced below $\mathcal{R}$. After a finite number of steps, the inference introducing $X \otimes Y$ is permuted above $\mathcal{R}$ and we have reduced k.

Clause (v): $X^* \in e(B_i), Y^* \in e(B_i)$ and $X^* \neq B_i \neq Y^*$ imply $(X \sqcup Y)^* \in e(B_i)$. If $\mathcal{R}'$ introduces the principal formula $X \sqcup Y$, then by the table above the only problematic case is when the inference immediately above $\mathcal{R}'$ is a $\otimes$ rule. By induction hypothesis we assume that both X and Y are introduced above $\mathcal{R}$. It follows that for each application $\mathcal{R}^{\otimes}$ of the $\otimes$-rule between $\mathcal{R}$ and $\mathcal{R}'$ both X and Y have ancestors only in one of the upper sequents of $\mathcal{R}^{\otimes}$. But in this case it is certainly possible to permute $\mathcal{R}'$ with $\mathcal{R}^{\otimes}$ too.

The cases of clause (v) that deal with the conclusion of a $\sqcap$ link or of a Contraction link flagged with a $\sqcap$ link do not apply here. Indeed, if $\mathcal{R}'$ introduces the principal formula $X \sqcap Y$ and $\mathcal{R}$ occurs above $\mathcal{R}'$, then $B_i^* \in e(X^*)$ or $B_i^* \in e(Y^*)$ and by the Box Lemma, section 3.1, $e(B_i) \subset e(X^*)$ or $e(B_i) \subset e(Y^*)$, hence $(X \sqcap Y)^* \notin e(B_i)$.

Finally, it is easy to see that if $\mathcal{D}_1$ comes from $\mathcal{D}_0$ by a permutation of inferences as before, then $\mathcal{D}_0^* = \mathcal{D}_1^*$. This concludes part (1). Parts (2) is similar. In part (3) we must expand the size of the *with*-box by *with*-commutation, but this does not enlarge the empire of A. Details are omitted. ∎

5. Conclusion.

What has been achieved?

(I) We have provided a unified treatment of multiplicative and additive linear logic, essentially based on the notion of *empire*. We have already stressed the significance of the empire of a formula as a subnet in a proof net. The vicious circle, connectedness, box and parameters conditions depend on the possibility of a simple subdivision (*tiling*) of a structure and correspond to natural properties of proofs in linear logic.

(II) The *slicing* of a proof structure with boxes has the Church Rosser property as most obvious motivation. In the case of proof networks, the result from our slicing (the quasi structures) still maintain enough connections, so that the test of the *multiplicative* condition for correctness can still be done on them.

It may be asked whether proof networks are adequate *per se* to represent proofs, or only in connection with proof nets, as the outcome of the process of *slicing*. One could informally describe a proof network as a family of quasi structures satisfying certain properties. (a) All quasi structures in the family satisfy the conditions for consistency and relevance. In addition, (b) all quasi structures in the family are identified at the conclusions. It must be possible to chose the points of identification in such a way that the can be grouped in patterns of the form $C_1, \ldots, C_n$, $A \sqcap B$. (c) Putting the same flag on all the points of the same group, the set of substructures occurring above flagged points must constitute two subnetworks. To express these conditions precisely is complicated enough; the alternative route, the embedding of the family in a proof structure seems more elegant.

The importance of avoiding *bricolage* in the theory of slicing is related to our understanding of the distinction between $\mathbf{0}$ and $\perp$: practically useful applications of this distinction would certainly encourage further attention to the matter.

It is not our intention to advertise proof networks as the ultimate representation of provability for the systems in question. Ultimately, the significance of formalisms has to be proved in applications. If *slicing* represents ‘the absolute limit for a parallelization of the syntax, i.e., the removal of all boxes but !-ones’ (Girard [1987], p.94), then its value ought to be tested in the field of parallelism.

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