Homework Assignment 2

Due **Friday 30-01-04**

Question 1. Let A be a set and $\wp(A)$ the set of all subsets of A. Show that there is no bijection $f : A \longrightarrow \wp(A)$.

Hint: Suppose $f : A \longrightarrow \wp(A)$ is a bijection. Consider the set

$$X = \{y \in A | y \notin f(y)\};$$

show that there is an element x such that $f(x) = X$ (*Why? 1 point*) ...
Case 1. Suppose $x \in f(x)$: then ... (*2 points*)
and this is a contradiction.
Case 2. Suppose $x \notin f(x)$: then ... (*2 points*)
and this is also a contradiction.
In both cases we get a contradiction, there our assumption that there is a bijection between A and $\wp(A)$ is refuted, as required.

Question 2. Let A be a finite set of cardinality n. By induction on n, show that $\wp(A)$ the set of all subsets of A has cardinality 2^n.

OK folks, let's agree on the following:
it's NOT enough to show an INSTANCE of the problem (such as the case of $A = \{1, 2, 3\}$), *and it is NOT enough to write down some calculations for the inductive step. You must write down something like the following piece of English prose.*

• To prove the result by induction on n, we prove the *base case*:

if A has cardinality 0, then $\wp(A)$ has cardinality $2^0 = 1$.

Indeed, if A has cardinality 0 then A is empty, and therefore ... (*1 point*).

• Next we prove the *inductive step*: we assume that the result is true for k (*inductive hypothesis*)

if A has cardinality k, then $\wp(A)$ has cardinality 2^k,

and we prove that it is true for $k + 1$:

if A has cardinality $k + 1$, then $\wp(A)$ has cardinality 2^{k+1}.

If $A = \{a_1, \ldots, a_k, a_{k+1}\}$, then let $B = \{a_1, \ldots, a_k\}$ so that $A = B \cup \{a_{k+1}\}$. It is clear that the set $\wp(A)$ of all subsets of A can be partitioned in two sets, those which are subsets of B and those which are not.

Now if $X = \{b_1, \ldots, b_i, a_{k+1}\}$ is a subset of A but not of B, then

$$Y = \ldots \qquad (\textit{1 point})$$

is a subset of B and the assignment

$$X \mapsto Y$$

is a bijection from $\wp(A) \setminus \wp(B)$ to $\wp(B)$. (*Why? 1 point*) ....

Now we use the *inductive hypothesis*:
(i) the cardinality of $\wp(B)$ is ... because ...
(ii) the cardinality of $\wp(A) \setminus \wp(B)$ is ... because ...
(iii) therefore the cardinality of $\wp(A)$ is $\ldots + \ldots = \ldots$ (*2 points*)
and this concludes the inductive step.
Having proved the base case and the inductive step, we may conclude that

for all n, if A has cardinality n, then $\wp(A)$ has cardinality 2^n,

as required.

Question 3. Prove Euclid's theorem: There are infinitely many prime numbers.
Hint: Suppose $p_1, \ldots, p_k$ were all the prime numbers and consider the number $p_1 \cdot \ldots \cdot p_k + 1$. Can it be divided by some p_i?
For further details, go to the library and read Cohn, *Algebra*, vol. 1, Chapter 2. (*5 points*)