Ackermann's function

Definition. Ackermann's function is recursively defined as follows:

$$\begin{aligned}
\alpha(m,0) &= m+1 && \text{(i)}\\
\alpha(0,n+1) &= \alpha(1,n) && \text{(ii)}\\
\alpha(m+1,n+1) &= \alpha(\alpha(m,n+1),n). && \text{(iii)}
\end{aligned}$$

The Ackermann function is well defined, i.e., we can prove the following lemma:

Lemma 0. *For all $y,x \in \mathbf{N}$, there exists a $z \in \mathbf{N}$ such that $\alpha(x,y) = z$.*

Proof. By a main induction on y and a secondary induction on x.

The Ackermann function is *strictly increasing*:

Lemma 1. *For all $m,n \in \mathbf{N}$, $\alpha(m,n) > m$.*

Proof. By a main induction on n and a secondary induction on m.

In fact, the Ackermann function is *monotonic* in both arguments:

Lemma 2.(i) *For all $y,z,x \in \mathbf{N}$, if $x < z$ then $\alpha(x,y) < \alpha(z,y)$.*

Proof. By a main induction on y and a secondary induction on z.

Lemma 2.(ii) *For all $y,z,x \in \mathbf{N}$, if $y < z$ then $\alpha(x,y) < \alpha(x,z)$.*

Proof. Similar.

The following Lemma is very helpful:
Lemma 3. *For all $m,n \in \mathbf{N}$, $\alpha(m,n+1) \geq \alpha(m+1,n)$.*

Proof. By principal induction on n and secondary induction on m.

Main base case: $\forall m.\alpha(m,1) \geq \alpha(m+1,0)$. We prove it by a secondary induction on m. For $m = 0$ we have

$$\alpha(0,1) = \alpha(1,0)$$

by definition of α, and this proves the *secondary base case.*

Suppose $\alpha(m,1) \geq \alpha(m+1,0)$ (*secondary inductive hypothesis*):

$$\begin{aligned}
\alpha(m+1,1) = \alpha(\alpha(m,1),0) &= \alpha(m,1)+1 && \text{def. of } \alpha;\\
&\geq \alpha(m+1,0)+1 && \text{secondary ind. hyp.}\\
&= m+3 && \text{def. of } \alpha;\\
&= \alpha(m+2,0) && \text{def. of } \alpha.
\end{aligned}$$

This concludes the *secondary inductive step*, and thus the main base case.

Suppose $\forall m.\alpha(m,n+1) \geq \alpha(m+1,n)$ (*main inductive hypothesis*); we want to show that $\forall m.\alpha(m,n+2) \geq \alpha(m+1,n+1)$ (*main inductive step*). To do this, we need a secondary induction on m: the *secondary base case* is proved using the definition of α:

$$\alpha(0,n+2) = \alpha(1,n+1).$$

Now suppose $\alpha(m, n+2) \geq \alpha(m+1, n+1)$ (*secondary inductive hypothesis*). Then

$$\begin{aligned} \alpha(m+1, n+2) &= \alpha(\alpha(m, n+2), n+1) && \text{def. of } \alpha; \\ &\geq \alpha(\alpha(m+1, n+1), n+1 && \text{secondary ind. hyp., Lemma 2(i)} \\ &\geq \alpha(m+2, n+1) \end{aligned}$$

The last step follows from Lemma 2(i), using $\alpha(m+1, n+1) \geq m+2$; this follows from $\alpha(m+1, n+1) > m+1$, which holds by Lemma 2(i). This concludes the *secondary inductive step* and therefore also the main inductive step. The proof of Lemma 3 is finished.

We consider the recursion scheme in the following restricted form:
If the unary function h is primitive recursive, then so is f defined as follows:

$$\begin{aligned} f(0) &= 0 \\ f(n+1) &= h(f(n)) \end{aligned}$$

(Using suitable codings, it can be shown that every primitive recursive function can be defined using only the above scheme.)

Now we can prove the Majorization Lemma:

Majorization Lemma: *For every primitive recursive function $f(x_1, \ldots, x_k)$ there exists an $n \in \mathbf{N}$ such that*

$$f(x_1, \ldots, x_k) < \alpha(max(x_1, \ldots, x_k), n)$$

for all x_1, …, x_k.

The proof is by induction on the definition of a primitive recursive function. There are five cases:

constant functions: let $\mathbf{B}_1 = 0$; then

$$c_0^n(x_1, \ldots, x_n) = 0 < max(x_1, \ldots, x_n) + 1 = \alpha(max(x_1, \ldots, x_n), \mathbf{B}_1).$$

projection functions: let $\mathbf{B}_2 = 0$; then

$$\pi_i^n(x_1, \ldots, x_n) = x_i < max(x_1, \ldots, x_n) + 1 = \alpha(max(x_1, \ldots, x_n), \mathbf{B}_2).$$

successor function: let $\mathbf{B}_3 = 1$; then

$$succ(x) = x + 1 < x + 2 = \alpha(x+1, 0) \leq \alpha(x, \mathbf{B}_3)$$

Composition: let $h(x_1, \ldots, x_m)$ be defined by composition from primitive recursive functions $g(x_1, \ldots, x_k)$ and $f_i(x_1, \ldots, x_m)$ for $i \leq k$. Suppose there is a $\mathbf{D}$ such that for all y_1, …, y_k

$$g(y_1, \ldots, y_k) < \alpha(max(y_1, \ldots, y_k), \mathbf{D})$$

and suppose for each $i \leq k$ there is a $\mathbf{C}_i$ such that for all x_1, …, x_n,

$$f_i(x_1, \ldots, x_m) < \alpha(max(x_1, \ldots, x_m), \mathbf{C}_i).$$

Let $\mathbf{B}_4 = max(\mathbf{C}_1, \ldots, \mathbf{C}_k, \mathbf{D})$. Then

$$\begin{aligned}
\alpha(max(x_1, \ldots, x_m), \mathbf{B}_4 + 2) &\geq \alpha(max(x_1, \ldots, x_m) + 1, \mathbf{B}_4 + 1) && \text{(Lemma 3)} \\
&= \alpha(\alpha(max(x_1, \ldots, x_m), \mathbf{B}_4 + 1), \mathbf{B}_4) && \text{(def. of } \alpha\text{)} \\
&> \alpha(max_{i \leq k}\{\alpha(max(x_1, \ldots, x_m), \mathbf{C}_i)\}, \mathbf{B}_4) && \text{(def. of } \mathbf{B} \text{ and monot.)} \\
&> \alpha(max_{i \leq k}\{f_i(x_1, \ldots, x_m)\}, \mathbf{B}_4) && \text{(by hypothesis and monot.)} \\
&\geq \alpha(max_{i \leq k}\{f_i(x_1, \ldots, x_m)\}, \mathbf{D}) && \text{(def. of } \mathbf{B}_4\text{)} \\
&> g(f_1(x_1, \ldots, x_m), \ldots, f_k(x_1, \ldots, x_m)). && \text{(by hypothesis)}
\end{aligned}$$

Primitive Recursion: Let $f(y)$ be a primitive recursive and suppose h is defined by recursion

$$h(0) = 0 \qquad \text{and} \qquad h(n+1) = f(h(n)).$$

Suppose there exists a $\mathbf{C}$ such that $f(y) < \alpha(y, \mathbf{C})$, for all y. Let $\mathbf{B}_5 = \mathbf{C} + 1$. We prove by induction that $h(x) < \alpha(x, \mathbf{B}_5)$, *for all* x.

Base case:

$$h(0) \ = \ 0 \ < \ 1 \ = \alpha(0,0) \ < \ \alpha(0,1) \ \leq \ \alpha(0, \mathbf{B}_5)$$

Inductive step: Suppose $h(n) < \alpha(n, \mathbf{B}_5)$ (inductive hypothesis).
Then

$$\begin{aligned}
h(n+1) &= f(h(n)) && \\
&< \alpha(h(n), \mathbf{C}) && \text{(by assumption)} \\
&< \alpha(\alpha(n, \mathbf{B}_5), \mathbf{C}) && \text{(by inductive hypothesis and Lemma 2)} \\
&= \alpha(\alpha(n, \mathbf{C}+1), \mathbf{C}) && \text{(by def. of } \mathbf{B}_5\text{)} \\
&= \alpha(n+1, \mathbf{B}_5) && \text{(by def. of } \alpha\text{)}
\end{aligned}$$

END OF PROOF OF THE LEMMA.

Theorem. *The Ackermann function is not primitive recursive.*

Proof. Define $\beta(n) =_{df} \alpha(n,n) + 1$, where α is the Ackermann function. Suppose α is primitive recursive, then also β is primitive recursive. So by the Lemma, there is a k such that $\beta(m) < \alpha(m, k)$ for all m. Therefore

$$\begin{aligned}
\alpha(k,k) + 1 &= \beta(k) && \text{definition of } \beta \\
&< \alpha(k,k) && \text{by the Lemma}
\end{aligned}$$

a contradiction. Therefore α is not primitive recursive.