Categorical Proof Theory of Classical Propositional Calculus

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Abstract

We investigate semantics for classical proof based on the sequent calculus. We show that the propositional connectives are not quite well-behaved from a traditional categorical perspective, and give a more refined, but necessarily complex, analysis of how connectives may be characterised abstractly. Finally we explain the consequences of insisting on more familiar categorical behaviour.

Key words: classical logic, proof theory, category theory

1 Introduction

In this paper we describe the shape of a semantics for classical proof in accord with Gentzen's sequent calculus. For constructive proof we have the familiar correspondence between deductions in minimal logic and terms of a typed lambda calculus. Deductions in minimal logic (as in most constructive systems) reduce to a unique normal form, and around 1970 Per Martin-Löf (see [18]) suggested using equality of normal forms as the identity criterion for proof objects in his constructive Type Theories: normal forms serve as the semantics of proof. But $\beta\eta$-normal forms for typed lambda calculus give maps in a free cartesian closed category; so we get a whole range of categorical models of constructive proof. This is the circle of connections surrounding the Curry-Howard isomorphism. We seek analogues of these ideas for classical proof. There are a number of immediate problems.

The established term languages for classical proofs are either incompatible with the symmetries apparent in the sequent calculus (Parigot [16]) or in reconciling themselves to that symmetry at least make evaluation deterministic (cf Danos et al [5,21]). Either way the ideas, which derive from analyses of continuations in programming (Griffin [9], Murthy [15]) can be thought of as reducing classical proof to constructive proof via a double negation translation. (A categorical semantics is described in Selinger [23].) There are term calculi associated directly with the sequent calculus (Urban [25]) but it is not clear how to formulate mathematically appealing criteria for identity of such terms. What we do here suggests many commutative conversions for Urban's terms, but the matter is not straightforward. Also since reductions of classical proofs in sequent calculus form are highly non-deterministic, normal forms do not readily provide a criterion for identity of such proofs.

There are problems at the level of semantics. There are more or less degenerate models giving invariants of proofs ([7] and [12]) and we know how to construct some more general models. But all that is parasitic on experience with Linear Logic. We lack convincing examples of models sensitive to the issues on which we focus here. The connection with established work on polarised logic, modelling both call-by-name and call-by-value reduction strategies ([23], [26], [10]), is also problematic. Even if one considers a system (as in [5]) that mixes the two and considers all the normal forms reachable from representations in it of a proof, one still does not exhaust all normal forms to which a proof in the sequent calculus can reduce (see for example [24, Page 127]). Moreover, there is no easy way to extract models for our system from categorical models in the style of Selinger.

The project on which we report here was motivated by Urban's strong normalisation result ([25] and [24]) for a formulation of classical proof. In [11], one of us then outlined a proposal for a semantics. Unfortunately, the axioms of [11] entail full naturality of logical operations contrary to the clear intentions of the paper. Here we make that good and analyse the issue. Since then, another of us suggested in [19] basing analysis of classical proof on a simple (box-free) notion of proof net. Such systems have implicit naturalities built in so this is in contrast with [11]. In [6] Führmann and Pym analyse Robinson's proposal further. They give categorical combinators, add η-equalities to the implicit naturalities and succeed in axiomatizing reduction. The interaction between the equalities and reduction presents computational difficulties, so this is a substantial achievement. The proof net model is better dynamically than feared, and suggests a notion of model of classical proof simpler than that analysed here. We give an exact account of the relation between the two, and show in what sense the Führmann-Pym equalities identify proofs which differ on a sequent calculus reading.

The question of what are the sensible criteria for identity of proofs is a delicate one. The referee rightly stressed that this is true also of constructive proofs, the difference between the classical and constructive case being that in the latter we have a robust semantic notion which is generally agreed on. We do not expect that in the classical case. At the very least different systems of proof can be expected to lead to different semantics. A compelling example is the recent work of Lamarche and Strassburger [14].

2 Modelling classical proofs

2.1 *Sequent Calculus and Polycategories*

It is a familiar idea that what the sequent calculus provides is not a collection of ideal proofs-in-themselves, but something more like instructions for building proofs. With this in mind we formulate design criteria for our semantics.

(1) Associativity. Cut should be an associative operation on proofs.
(2) Identities. We require that there be a canonical axiom (identity proof) $A \vdash A$ for all A, and that it should act as an identity under cut.
(3) de Morgan Duality. We take a strict duality on propositions and proofs.

Of these the first two seem compelling while the third could be regarded as a matter of convenience. While we have not written out the details, it is our impression that the basics of our analysis would not need to change if we did not take full de Morgan duality.

2.1.1 *Polycategories*

The general category-like structure which encapsulates the first two criteria is Szabo's notion of a polycategory (Szabo [22]). Rather than being definitive, in the way that the notion of an ordinary category is definitive, there are any number of variants adapted to particular contexts (recent treatments include [4] and [2]).

Definition 2.1. A *symmetric polycategory* (henceforth just polycategory) $\mathcal{P}$ consists of

- A collection $\mathrm{ob}\mathcal{P}$ of objects of $\mathcal{P}$; and for each pair of finite sequences Γ and Δ of objects, a collection $\mathcal{P}(\Gamma;\Delta)$ of (poly)maps from Γ to Δ.
- For each re-ordering of the sequence Γ to produce the sequence Γ', an isomorphism from $\mathcal{P}(\Gamma';\Delta)$ to $\mathcal{P}(\Gamma;\Delta)$, functorial in its action, and dually for Δ.
- An identity $\mathrm{id}_A \in \mathcal{P}(A;A)$ for each object A; and a composition

$$\mathcal{P}(\Gamma;\Delta,A) \times \mathcal{P}(A,\Pi;\Sigma) \to \mathcal{P}(\Gamma,\Pi;\Delta,\Sigma)\,.$$

  for each Γ, Δ, A, Π, Σ, coherent with re-ordering.

This data should satisfy identity and associativity laws, which we do not give here.

One thinks of $\mathcal{P}(\Gamma;\Delta)$ as the collection of abstract proofs of $\Gamma \vdash \Delta$. We write a polymap $f \in \mathcal{P}(\Gamma;\Delta)$ as $f : \Gamma \to \Delta$. We picture it as a box

$$\Gamma - \boxed{f} - \Delta$$

with input wires Γ and output wires Δ. We have explicit identities id_A. Composition corresponds to cut: in particular maps are plugged together at a single object, not an entire sequence. We adopt a lazy algebraic notation for composition. For $f : \Gamma \to \Delta, A$

and $g : A, \Pi \to \Sigma$ we write the composite in the diagrammatic order as $f; g : \Gamma, \Pi \to \Delta, \Sigma$. We do not introduce a formal notation for composing many polymaps, but note that such compositions are determined by trees. However it is useful to have a little home-spun notation for simple cases. We write

$$\{f, g\}; \{h, k\}$$

to indicate compositions involving the four multimaps f, g, h and k, where the f and g come before the h and k. For example f and g might plug into h and g also into k. (There are essentially four distinct cases.) It will always be possible to determine what we mean from the context.

2.1.2 $*$-polycategories

Our third design criterion amounts to the simplifying decision to treat negation implicitly. In proof theoretic terms that is to take a formulation with an involutory negation

$$(-)^* : p \to p^* \,,\; p^* \to p$$

on atomic formulae, and extend it to all formulae by setting

$$\begin{array}{cc} \top^* = \bot & \bot^* = \top \,, \\ (A \wedge B)^* = B^* \vee A^* & (A \vee B)^* = B^* \wedge A^* \,, \end{array}$$

that is, more or less, by de Morgan duality. The cyclic choice of order may be familiar from non-commutative linear logic (Ruet [20]). It is not strictly necessary here, but serves as there to preserve a strict duality at the level of proofs. Exact duality permits a purely one-sided sequent calculus as in Girard [8], but we prefer to keep both sides in play at the semantic level. Abstractly we get a *$*$-polycategory*.

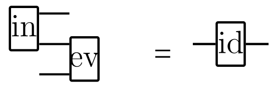
Definition 2.2. A *symmetric $*$-polycategory* (henceforth just *$*$-polycategory*) $\mathcal{P}$ consists of a polycategory $\mathcal{P}$ equipped with an involutory negation $(-)^*$ on objects together with for each Γ, Δ, A, an isomorphism $\mathcal{P}(\Gamma; \Delta, A) \cong \mathcal{P}(A^*, \Gamma; \Delta)$ coherent with re-ordering and composition.

With this in place one should not take the talk of input and output above too literally: according to the $*$-polycategorical perspective an input wire of kind A is effectively an output wire of type A^*. We shall not need to pay much attention to the $(-)^*$ operation which takes polymaps $\Gamma \to \Delta, B$ to polymaps $B^*, \Gamma \to \Delta$. However we shall need notation for variants of the identity $\mathrm{id}_A : A \to A$. We write these as

$$\mathrm{in}_A : - \to A^* \,, A \quad \text{and} \quad \mathrm{ev}_A : A, A^* \to - \,.$$

These can be pictured as follows.

We note that the operation taking a polymap $f : \Gamma \to \Delta, B$ to $f^* : B^*, \Gamma \to \Delta$ say is implemented by composition: one has $f^* = f; \mathrm{in}$. Similarly for the operation taking $g : A, \Gamma \to \Delta$ to $g^* : \Gamma \to \Delta, A^*$, one has $g^* = \mathrm{ev}; g$. In particular we have equations of the form $\mathrm{in}; \mathrm{ev} = \mathrm{id}$ as in the following picture.[1]



The notion of a $*$-polycategory satisfies our design criteria and so gives a first step towards a definition of a model for classical proof. It describes a notion of proof with associative cut, identities and strict duality, but without logical operations and without structural rules. For classical logic we need to add the propositional connectives and the structural rules of weakening and contraction. We treat these two in turn.

2.2 *Logical rules*

We consider how rules of inference for the classical connectives should be treated. We first describe the operations together with the properties (naturality, commutative conversions) which we regard as implicit; and then we consider which proof diagrams should further be identified as a result of meaning preserving reductions.

2.2.1 *Logical operations*

As logical operators we consider only $\top$, $\wedge$, and their de Morgan duals, $\bot$, $\vee$. Negation is defined implicitly by de Morgan duality, and other logical operators in terms of those given.

We recall the rules for $\wedge$ and $\top$ in sequent calculus form.

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge\text{-L} \qquad \frac{\Gamma \vdash \Delta, C \quad \Pi \vdash \Lambda, D}{\Gamma, \Pi \vdash \Delta, \Lambda, C \wedge D}\ \wedge\text{-R} \qquad \frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\ \top\text{-L} \qquad \frac{}{\vdash \top}\ \top\text{-R}\ .$$

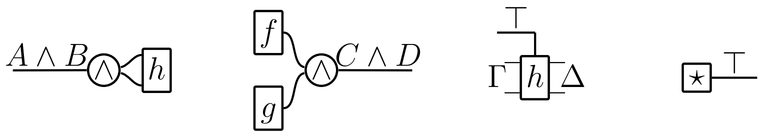
We recast these rules in terms of $*$-polycategories. So we require operations

$$\mathcal{P}(A, B, \Gamma; \Delta) \longrightarrow \mathcal{P}(A \wedge B, \Gamma; \Delta)\ :\ h \to \overline{h}\ ,$$
$$\mathcal{P}(\Gamma; \Delta, C) \times \mathcal{P}(\Pi; \Lambda, D) \longrightarrow \mathcal{P}(\Gamma, \Pi; \Delta, \Lambda, C \wedge D)\ :\ (f, g) \to f \cdot g\ ,$$
$$\mathcal{P}(\Gamma; \Delta) \longrightarrow \mathcal{P}(\top, \Gamma; \Delta)\ :\ h \to h^+\ ,$$
$$\star \in \mathcal{P}(\ ; \top)\ ,$$

encapsulating the $\wedge$-L, $\wedge$-R, $\top$-L and $\top$-R rules. This imprecise notation will serve for

[1] To avoid misunderstanding we stress that there is no composition of the form $\mathrm{ev}; \mathrm{id}$. There is nothing to plug into.

this paper. We can picture the rules thus.



A notion of duality is built into the notion of $*$-polycategory. So given what we have said about the operations $\top$ and $\wedge$, there is no need for substantial discussion of the de Morgan duals $\perp$ and $\vee$. We may as well overload the notation and take operations

$$\mathcal{P}(\Gamma;\Delta,C,D)\longrightarrow\mathcal{P}(\Gamma;\Delta,C\vee D)\ :\ h\to\overline{h}$$

$$\mathcal{P}(A,\Gamma;\Delta)\times\mathcal{P}(B,\Pi;\Lambda)\longrightarrow\mathcal{P}(A\vee B,\Gamma,\Pi;\Delta,\Lambda)\ :\ (f,g)\to f\cdot g$$

$$\mathcal{P}(\Gamma;\Delta)\longrightarrow\mathcal{P}(\Gamma;\Delta,\perp)\ :\ h\to h^{+}$$

$$\star\in\mathcal{P}(\perp;\ )$$

each being the dual of the corresponding operation above.

2.2.2 Naturality

Composition in a $*$-polycategory corresponds to Cut, so the general naturality conditions implicit in proof nets are clear. Two involve a local operation on just one proof, and are compelling. In our imprecise notation, these are as follows.

- Naturality for $\wedge$-L. Suppose $h : A,B \to E$. Then for $w : E \to E'$ we have the naturality condition
$$\overline{h};w=\overline{h;w}\ .$$
- Naturality for $\top$-L. Suppose $h : A \to B$. Then for $v : B \to B'$ we have the naturality condition
$$h^{+};v=(h;v)^{+}\ .$$

(We omit irrelevant contexts.) By duality that gives us naturality as follows

$$w;\overline{h}=\overline{w;h}\quad\text{and}\quad v;h^{+}=(v;h)^{+}\ ,$$

in the right rules for $\vee$ and $\perp$. We adopt these naturality equations.

On the other hand we shall argue against adopting the following condition.

- Naturality for $\wedge$-R. Suppose $f : A \to C$, $g : B \to D$. Then for $u : A' \to A$ and $v : B' \to B$ we have the naturality equation
$$\{u,v\};(f\cdot g)=(u;f)\cdot(v;g)$$
where on the right we have the obvious composition of $f\cdot g : A,B\to C\wedge D$ with u and v.

(Note that there is no context in $\top$-R and so no corresponding naturality.) The problem which we will come to in 4.3 is that taken together with contraction and weakening this naturality equation identifies proofs with essentially different collections of normal forms.

However there are cases where that cannot happen; and it does seem reasonable to allow some maps u and v to slip harmlessly past the imagined box around $(f \cdot g)$. After all we inevitably have

$$\{\mathrm{id}, \mathrm{id}\}; (f \cdot g) = f \cdot g = (\mathrm{id}; f) \cdot (\mathrm{id}; g) \ .$$

So we adopt a restricted form of an idea from [11]. We call maps u, v for which both the $\wedge$ equations

$$u; (f \cdot g) = (u; f) \cdot g \ , \ v; (f \cdot g) = (f) \cdot (v; g) \ , \text{ and so } \{u, v\}; (f \cdot g) = (u; f) \cdot (v; g)$$

and the dual equations for $\vee$ hold *linear*. (This definition does make sense!) We have the following.

Additional assumption *Linear maps are closed under the logical operations introduced above.*

In view of the other naturalities, the essential assumption is that $\star$ is linear and that linear maps are closed under $- \cdot -$.

2.2.3 Commutation: logical rules

The polycategorical perspective supports equalities arising from the commuting conversions in sequent calculus. We sketch, again using our imprecise notation, the basic phenomena for the binary operators.

First given proofs

$$f : \Gamma_1 \to \Delta_1, A, B \ , \quad g : \Gamma_2 \to \Delta_2, C \ , \quad h : \Gamma_3 \to \Delta_3, D \ ,$$

we have (perhaps modulo exchange) an equality of the form

$$(f \cdot g) \cdot h = (f \cdot h) \cdot g : \Gamma \to \Delta, A \wedge C, B \wedge D$$

(with Γ, Δ, the sum of the Γ_i and Δ_i respectively). Of course there are other versions obtained by duality

Secondly given proofs

$$f : A, B, \Gamma_1 \to \Delta_1, C \ , \quad g : \Gamma_2 \to \Delta_2, D \ ,$$

we have an equality of form

$$\overline{f} \cdot g = \overline{f \cdot g} : A \wedge B, \Gamma \to \Delta, C \wedge D$$

(with Γ, Δ, the sum of the Γ_i and Δ_i respectively). As before there are variants by duality.

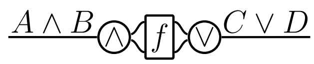
Finally from a proof

$$f : A, B, \Gamma \to \Delta, C, D \ ,$$

we can apply the operation $\overline{(\ )}$ in two different orders getting an equality of the form

$$\overline{\overline{f}} = \overline{\overline{f}} : A \wedge B, \Gamma \to \Delta, C \vee D \ .$$

There are variants by duality. The picture is as follows.



So far we have only considered the binary operators. There are many similar examples involving also the rules for $\top$ which we merely list.

$$f^+ \cdot g = (f \cdot g)^+ \,, \qquad \overline{f}^+ = \overline{f^+} \,, \qquad f^{++} = f^{++} \,.$$

(The final equation reflects the two different orders of applying rules to obtain a proof of $\top, \Gamma \vdash \Delta, \bot$.) We are happy to adopt all these equalities.

2.2.4 Reduction

Most of our equalities on proofs keep track of inessential rewritings, but in itself that is dull. The critical equalities take account of meaning preserving reductions. We take these to arise from logical cuts.

Suppose that $f : A \vdash C$, $g : B \vdash D$ and $k : C, D \vdash E$ are proofs. (Again we suppress further contexts.) We can form the proof

$$\dfrac{\dfrac{A \vdash^f C \quad B \vdash^g D}{A, B \vdash C \wedge D} \quad \dfrac{C, D \vdash^k E}{C \wedge D \vdash E}}{A, B \vdash E} \ \mathrm{CUT}$$

which reduces to

$$\dfrac{A \vdash^f C \quad B \vdash^g D \quad C, D \vdash^k E}{A, B \vdash E} \ \mathrm{CUTs}$$

where by associativity we write the two Cuts together. This gives a simple equation for our polycategory:

$$(f \cdot g); \overline{k} = \{f, g\}; k \ .$$

Similarly suppose that $f : A \vdash B$ is a proof. We can form the proof

$$\dfrac{\vdash^{\star} \top \quad \dfrac{A \vdash^f B}{\top, A \vdash B}}{A \vdash B} \ \mathrm{CUT}$$

and this reduces outright to

$$A \vdash^f B \ .$$

This gives another equation in our polycategory:

$$\star; f^{+} = f \ .$$

These equations (and their duals) constitute *the reduction principle for logical cuts*. For us the reduction of logical cuts is meaning preserving.

2.3 *Structural Rules*

2.3.1 *Implementation*

The structural rule of Exchange is implicit in our notion of symmetric $*$-polycategory, but we need to consider Weakening and Contraction.

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{W-L} \quad , \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ \text{W-R} \quad , \quad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{C-L} \quad , \quad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ \text{C-R} \ .$$

Naturalities implicit in proof nets in tandem with our reduction principle for logical cuts suggest a nice way to represent these in our $*$-polycategory.

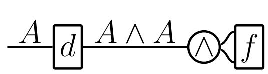
We treat contraction first. For all A, 'generic' instances of contraction give maps $d : A \to A \wedge A$ and $m : A \vee A \to A$ arising from the proofs

$$\frac{\dfrac{A \vdash A \quad A \vdash A}{A, A \vdash A \wedge A}\ \wedge\text{-R}}{A \vdash A \wedge A}\ \text{C-L} \quad , \quad \frac{\dfrac{A \vdash A \quad A \vdash A}{A \vee A \vdash A, A}\ \vee\text{-L}}{A \vee A \vdash A}\ \text{C-R} \ .$$

These are obviously constructed as de Morgan duals, so we assume that they are interchanged by the duality in our $*$-polycategory, that is,

$$(d_A)^* = m_{A^*} \ , \quad (m_A)^* = d_{A^*} \ .$$

It is consonant with earlier assumptions to suppose that we can implement the C-L rule by composition with its 'generic' instance d: that is, we form $\overline{f} : A \wedge A, \Gamma \vdash \Delta$ and then compose with d to give $d; \overline{f} : A, \Gamma \vdash \Delta$ as in the following picture.



Dually $m : A \vee A \to A$ implements contraction on the right: contracting $g : B \to D, D$ on the right is $\overline{g}; m$.

Similarly we have a way to implement weakening. In our polycategory we should have maps $t : A \to \top$ and $u : \bot \to A$ arising from the proofs

$$\frac{\dfrac{}{\vdash \top}\ \top\text{-R}}{A \vdash \top}\ \text{W-L} \quad , \quad \frac{\dfrac{}{\bot \vdash}\ \bot\text{-L}}{\bot \vdash A}\ \text{W-R} \ .$$

Again these are de Morgan duals and should be interchanged by duality:

$$(t_A)^* = u_{A^*} \ , \quad (u_A)^* = t_{A^*} \ .$$

Now suppose that we have a proof $f : \Gamma \vdash \Delta$, and we wish to weaken on the left. We form $f^+ : \top, \Gamma \vdash \Delta$ and compose with t to give $t; f^+ : A, \Gamma \vdash \Delta$. Thus t can be used to implement weakening on the left. Dually u can be used to implement weakening on the right: in that case g is weakened to $g^+; u$.

2.3.2 *Commuting conversions*

Implementing rules by composition with generic instances takes care of naturality issues; and some commuting conversions are an immediate consequence of the associativity of composition in a polycategory. However there are more such.

We expect C-L to enjoy the same commuting possibilities as $\wedge$-L. This requires equations of the form

$$(d; f) \cdot g = d; (f \cdot g) \ , \quad d; \overline{f} = \overline{d; f} \ , \quad d; f^+ = (d; f)^+ \ .$$

(In the second equation, the typing should give a commuting conversion in $d; \overline{f}$, not a logical cut.) Similar considerations for W-L and $\top$-L give the equations

$$(t; f) \cdot g = t; (f \cdot g) \ , \quad t; \overline{f} = \overline{t; f} \ , \quad t; f^+ = (t; f)^+ \ .$$

(In the last equation the typing should give a commuting conversion in $t; f^+$, not a logical cut.) We take all these.

2.3.3 *Correctness equations*

There are further issues to consider arising from the decision to implement the structural rules. We implement contraction via composition with $d : A \to A \wedge A$ and $m : A \vee A \to A$. But d and m are themselves produced by contractions on proofs $\mathrm{id}_A \cdot \mathrm{id}_A : A, A \to A \wedge A$ and $\mathrm{id}_A \cdot \mathrm{id}_A : A \vee A \to A, A$ respectively. So we need to make these agree. This gives us equations:

$$d; \overline{(\mathrm{id}_A \cdot \mathrm{id}_A)} = d \ , \quad \overline{(\mathrm{id}_A \cdot \mathrm{id}_A)}; m = m \ .$$

Similarly, we implement weakening via composition with $t : A \to \top$ and $u : \bot \to A$. But again these are themselves produced by weakening proofs $\star : (-) \to \top$ and $\star : \bot \to (-)$ respectively. Making these agree gives us equations

$$t; \star^+ = t \ , \quad \star^+; u = u \ .$$

There is a further delicate point which we mention here. Given $\Gamma \vdash^f \Delta$ there are two distinct ways to introduce $\top$ on the left:

$$\frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\ \top\text{-L} \ , \qquad \frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\ \text{W-L} \ .$$

In our notation these are f^+ and $t; f^+$ respectively: they are not taken as equal. This decision arises from an austere view of cut reductions where a last rule is structural. In this paper we make no equality assumptions in such circumstances.

2.3.4 Structural congruence

In the interests of simplicity, we subject the structural rules to structural congruence in a sense popular in concurrency theory.

Consider the process of Weakening only immediately to Contract:

$$\cfrac{\cfrac{A, \Gamma \vdash \Delta}{A, A, \Gamma \vdash \Delta}}{A, \Gamma \vdash \Delta} \; .$$

That seems as pointless a detour as a logical Cut, and we allow it to be deleted. Given the analysis above we can express this by the equation:

$$d; \overline{(t; f^{+})} = f \; .$$

Similarly it seems willful to distinguish between the various ways in which a series of contractions may be performed. This provides the seemingly pointless equation

$$d; (\overline{d; \overline{f}}) = d; (\overline{d; \overline{f}}) \; ,$$

which properly indexed is a version of associativity. Finally there is an issue relating contraction to exchange: one can exchange before contracting two copies of A. One may as well identify the proofs. Write $(-)^s$ to indicate a use of symmetry. Then modulo elimination of logical cuts we can express this by

$$d; \overline{\mathrm{id}_A \cdot \mathrm{id}_A}^{s} = d : A \longrightarrow A \wedge A \; .$$

Thus structural congruence gives us identity, associativity and commutativity conditions. We assume these in the interests of mathematical elegance.

3 Categorical formulation

In section 2 we surveyed all the structure on a $*$-polycategory needed to model classical proofs, and we gave the equations which we think should hold. This gives us a genuine though unwieldy notion of model. We shall not spell it out. Instead we shall extract from the $*$-polycategorical formulation structure on its underlying category giving an equivalent notion of categorical model.

Before we get down to work, we note that the involutary negation $(-)^*$ extends to maps as we have (for example) natural isomorphisms

$$\mathcal{C}(A; B) \cong \mathcal{C}(-; A^*, B) \cong \mathcal{C}(B^*; A^*) \, .$$

It is easy to see that

Proposition 3.1. *The operation $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ is a strict functorial self-duality on our category $\mathcal{C}$.*

The duality more or less halves the work which we now have to do. Whenever we have structure we shall have its dual.

3.1 Categorical Preliminaries

We start by introducing some preliminary notions. We consider categories $\mathcal{C}$ equipped with a special class of $\mathcal{C}_{\mathrm{id}}$ of idempotents, which we shall call *linear idempotents*. In our application these will be idempotents (maps e with $e;e = e$) which are linear in the sense of 2.2.2. For the moment we need assume nothing beyond the obvious requirement that every identity is in the class. We call such data a *guarded category*. [2]

Definition 3.2. *A guarded functor* $F : \mathcal{C} \to \mathcal{D}$ between guarded categories consists of the usual data for a functor such that F maps linear idempotents to linear idempotents; and whenever e and e' are linear idempotents, then

$$F(e); F(f); F(g); F(e') = F(e); F(f;g); F(e')$$

We say that a guarded functor F is *domain absorbing* when $F(e); F(f) = F(e;f)$ for linear idempotents e; it is *codomain absorbing* when $F(f); F(e) = F(f;e)$ for linear idempotents e.

We should interpret this in the case $\mathcal{C}$ is the trivial one object category 1 with its only choice of linear idempotents. A guarded functor $D : 1 \to \mathcal{D}$ is a choice of object $D \in \mathcal{D}$ and linear idempotent $e_D : D \to D$. We call this a *guarded object*.

We also need some notion of 2-cell between guarded functors

Definition 3.3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be guarded functors. A *guarded transformation* or simply *transformation* consists of data $\alpha_A : FA \to GA$ satisfying

$$F(\mathrm{id}_A); F(u); \alpha_B = \alpha_A; G(u); G(\mathrm{id}_B)$$

for all $u : A \to B$ in $\mathcal{C}$.

We do not spell out here the consequences of these definitions, but note the following.

Theorem 3.4. *Guarded categories, guarded functors and transformations form a 2-category, the* guarded 2-category.

The only subtle point is the composition of 2-cells along a 0-cell, where one needs to compose additionally with maps of the form $GF(\mathrm{id})$. We shall not need that here. The composition of 2-cells along a 1-cell by contrast is straightforward, and we shall need terminology suggested by it.

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[2] The terminology is intended to suggest a focus on good behaviour once we compose with the idempotents or guards. There is no stronger connections with other uses of "guarded" in logic or computer science.

Definition 3.5. Suppose that $\alpha : F \to G$ and $\beta : G \to F$ are (guarded natural) transformations. α and β are *mutually inverse* (*α inverse to β*) just when $\alpha_A;\beta_A = F(\mathrm{id}_A)$ and $\beta_A;\alpha_A = G(\mathrm{id}_A)$.

This amounts to taking inverses of 2-cells in the guarded 2-category.

3.2 Logical operators

3.2.1 Extension to maps

Clearly $\mathcal{C}$ must be equipped on objects with the structure, *true*, *and*, *false*, *or*, *not* of classical logic: we write this structure as 1, $\wedge$, 0, $\vee$. There is a compelling way to extend the propositional operators to maps. Given proofs $A \vdash^f B$ and $C \vdash^g D$, there is a canonical proof $A \wedge C \vdash^{f \wedge g} B \wedge D$ given by the following

$$\dfrac{\dfrac{A \vdash B \quad C \vdash D}{A, C \vdash B \wedge D}}{A \wedge C \vdash B \wedge D}$$

Similarly we have $f \vee g$ a proof of $A \vee C \vdash B \vee D$. So in terms of our algebraic notation we should define

$$f \wedge g = \overline{(f \cdot g)} \quad , \quad f \vee g = \overline{(f \cdot g)}\,.$$

Thus $\mathcal{C}$ is equipped with operations $\wedge$ and $\vee$ on maps. It turns out that they are not functorial, but in a suitable sense guarded functorial. To make sense of that we need a collection of linear idempotents. We identify that class as follows.

We first note a useful computation in our $*$-polycategories for classical logic. We give just the version for conjunction as that for disjunction is dual to it.

Proposition 3.6. *Suppose that $f : A \to C$, $g : B \to D$, $h : C \to E$ and $k : D \to F$ are maps. Then $(f \wedge g);(h \wedge k) = \overline{\{f, g\};(h \cdot k)}$.*

Using also the Additional Assumption of 2.2.2 we deduce at once the following.

Proposition 3.7. *If $e_A : A \to A$ and $e_B : B \to B$ are linear and idempotent, then so are $e_A \wedge e_B$ and $e_A \vee e_B$.*

We now associate with our categorical model $\mathcal{C}$ a class of linear idempotents. We simply close the collection of identity maps under the logical operations. (We make clear what that means in case of $\top$ and $\perp$.) We introduce some notation for the *canonical linear idempotents* which we have identified. We write

$$e_{A,B} = e_{A \wedge B} = \mathrm{id}_A \wedge \mathrm{id}_B\ , \qquad\qquad e_{A,B} = e_{A \vee B} = \mathrm{id}_A \vee \mathrm{id}_B\ .$$

We also take a nullary version of these, setting

$$e_\top = \star^+ \qquad\qquad e_\perp = \star^+$$

with the obvious interpretation in each case.

Theorem 3.8. *(i) $\top$ with $e_\top$ and dually $\bot$ with $e_\bot$ are guarded objects.*
(ii) The operator $\wedge : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a domain absorbing guarded functor, while dually $\vee : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a codomain absorbing guarded functor.

3.2.2 Coherence

When we come to reconstruct a $*$-polycategory from our category we need to observe some relations between our canonical linear idempotents. We illustrate the point here. Concentrating on conjunction we have on the one hand the idempotent

$$e_{A,B\wedge C} = \mathrm{id}_A \wedge \mathrm{id}_{B\wedge C} : A \wedge (B \wedge C) \longrightarrow A \wedge (B \wedge C)$$

and on the other

$$e_{A,B,C} = \mathrm{id}_A \wedge (\mathrm{id}_B \wedge \mathrm{id}_C) : A \wedge (B \wedge C) \longrightarrow A \wedge (B \wedge C)\ .$$

Intuitively the second decomposes things more than the first, and this is reflected in the fact that the second absorbs the first in the sense that

$$e_{A,B,C}; e_{A,B\wedge C} = e_{A,B,C} \quad \text{and} \quad e_{A,B\wedge C}; e_{A,B,C} = e_{A,B,C}\ .$$

The first calculation depends on the linearity of $\mathrm{id}_B \cdot \mathrm{id}_C$ from the Additional Assumption of 2.2.2.

Generally the situation is as follows. Given propositions A_i we have many bracketings to give a conjunction $\bigwedge A_i$. Given one such we have a variety of idempotents depending on how deeply we 'analyse the bracketings'. The shallowest analysis yields $\mathrm{id}_{\bigwedge A_i}$, the deepest $e_{\bigwedge A_i} = \bigwedge \mathrm{id}_{A_i}$. The coherence of these idempotents is the following fact.

Proposition 3.9. *Suppose in the given situation that e_1 is an idempotent corresponding to a deeper analysis than e_2. Then $e_1; e_2 = e_1 = e_2; e_1$.*

We note the nullary version of the proposition: $e_\top; \mathrm{id}_\top = e_\top = \mathrm{id}_\top; e_\top$.

3.3 Structure

3.3.1 Units and associators

Our logical operations are only guarded functorial, but they do come equipped with structure familiar in the case of tensor products. We concentrate on the case of $\top$ and $\wedge$; the case of $\bot$ and $\vee$ follows by duality.

First we can define maps

$$l = \star \cdot \mathrm{id}_A : A \to \top \wedge A \qquad \tilde{l} = \overline{\mathrm{id}_A^+} : \top \wedge A \to A$$
$$r = \mathrm{id}_A \cdot \star : A \to A \wedge \top \qquad \tilde{r} = \overline{(\mathrm{id}_A^+)^s} : A \wedge \top \to A$$

where the superscript s indicates a tacit use of exchange. We also have associativity maps defined as follows

$$a = \overline{\overline{(\mathrm{id}_A \cdot \mathrm{id}_B) \cdot \mathrm{id}_C}} : A \wedge (B \wedge C) \longrightarrow (A \wedge B) \wedge C \ ,$$
$$\tilde{a} = \overline{\overline{\mathrm{id}_A \cdot (\mathrm{id}_B \cdot \mathrm{id}_C)}} : (A \wedge B) \wedge C \longrightarrow A \wedge (B \wedge C) \ .$$

(There is only one sensible way to read those definitions!) We note at once that all these structural maps are linear.

By direct computation we show the following.

Theorem 3.10. *The pairs of maps l and $\tilde{l}$, r and $\tilde{r}$, a and $\tilde{a}$, are in each case mutually inverse guarded transformations.*

Note that the equations given by our definitions are not quite the familiar ones. For example since $\wedge$ is domain absorbing we do have

$$(f \wedge g) \wedge h; \tilde{a} = \tilde{a}; f \wedge (g \wedge h); e_{A' \wedge (B' \wedge C')}$$

but we only have the more familiar

$$(f \wedge g) \wedge h; \tilde{a} = \tilde{a}; f \wedge (g \wedge h)$$

when f, g and h are linear.

Perhaps surprisingly, it is automatic that our associativities satisfy the Mac Lane pentagon condition and the usual unit conditions on the nose. The diagrams are familiar and we do not exhibit them here.

Theorem 3.11. *The Mac Lane pentagon and unit conditions*

$$a_{A,B,C \wedge D}; a_{A \wedge B,C,D} = \mathrm{id}_A \wedge a_{B,C,D}; a_{A,B \wedge C,D}; a_{A,B,C} \wedge \mathrm{id}_D$$

$$a_{A,I,C}; r_A \wedge \mathrm{id}_B = \mathrm{id}_A \wedge l_B$$

both hold.

Of course many other version of the diagrams (e.g. involving $\tilde{a}$, $\tilde{l}$, $\tilde{r}$) hold. However the information contained in the coherence diagrams is quite subtle. One needs to bear in mind that e.g. $a_{A,B \wedge C,D}$ is not guarded natural in B and C. Let us say that a *mixed path* in the pentagon is one which involves both a and $\tilde{a}$. Many but by no means all mixed paths are equal. For example, the two maps

$$a : A \wedge (B \wedge (C \wedge D)) \longrightarrow (A \wedge B) \wedge (C \wedge D)$$

and

$$\mathrm{id}_A \wedge a; a; a \wedge \mathrm{id}_D; \tilde{a} : A \wedge (B \wedge (C \wedge D)) \longrightarrow (A \wedge B) \wedge (C \wedge D)$$

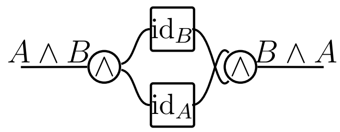
are not equal. (There are some similar issues for the triangle diagrams.)

3.3.2 *Symmetry*

We have maps induced by the symmetry of our $*$-polycategory. We use a superscript $(\ )^s$ to indicate a use of a symmetry in $\mathcal{P}$, and define a twist map

$$c = \overline{(\mathrm{id}_B \cdot \mathrm{id}_A)^s} : A \wedge B \longrightarrow B \wedge A \ .$$

The picture is as follows.



We note at once that this further structural map is linear. We then compute.

Proposition 3.12. $c_{A,B}; c_{B,A} = e_{A \wedge B} : A \wedge B \longrightarrow A \wedge B$*, that is,* c *is a transformation inverse to itself in the guarded sense.*

Finally we look at coherence.

Theorem 3.13. *The Mac Lane hexagon and unit conditions*

$$a_{A,B,C}; cA \wedge B, C; a_{C,A,B} = \mathrm{id}_A \wedge c_{B,C}; a_{A,C,B}; c_{A,C} \wedge \mathrm{id}_B \qquad c_{\top,A}; r_A = l_A$$

both hold.

We note a nuance. A symmetry of the form $c_{A,(B \wedge C)} : A \wedge (B \wedge C) \rightarrow (B \wedge C) \wedge A$ cannot be defined in the usual way from associativities and symmetries $c_{A,B}$ and $c_{A,C}$. Rather one has an equation of the form

$$c_{A,(B \wedge C)}; e_{(B \wedge C) \wedge A} = a_{A,B,C}; c_{A,B} \wedge \mathrm{id}_C; \tilde{a}_{B,A,C}; \mathrm{id}_B \wedge c_{A,C}; a_{B,A,C} \ .$$

Thus the usual definition holds in the guarded sense. However this is quite enough to establish the following.

Theorem 3.14. *The symmetry* c *satisfies the standard braid identities.*

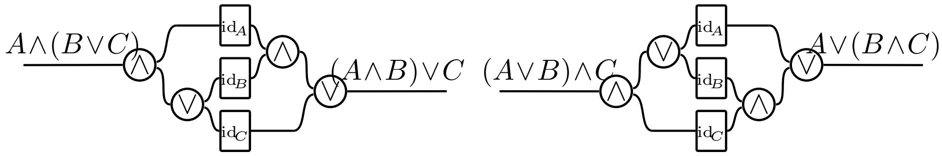
$$c_{A,B} \wedge \mathrm{id}_C; \mathrm{id}_B \wedge c_{A,C}; c_{B,C} \wedge \mathrm{id}_A = \mathrm{id}_A \wedge c_{B,C}; c_{A,C} \wedge \mathrm{id}_B; \mathrm{id}_C \wedge c_{A,B} \ .$$

3.3.3 *Linear distributivity*

So far we have the operations $\top$, $\wedge$ and $\bot$, $\vee$, which are dual. We need something like the usual connection between them from Linear Logic to capture general polycategorical composition. We define

$$w = \overline{\overline{\mathrm{id}_A \cdot (\mathrm{id}_B \cdot \mathrm{id}_C)}} = \overline{\overline{(\mathrm{id}_A \cdot \mathrm{id}_B) \cdot \mathrm{id}_C}} : A \wedge (B \vee C) \longrightarrow (A \wedge B) \vee C$$
$$\tilde{w} = \overline{\overline{(\mathrm{id}_A \cdot \mathrm{id}_B) \cdot \mathrm{id}_C}} = \overline{\overline{\mathrm{id}_A \cdot (\mathrm{id}_B \cdot \mathrm{id}_C)}} : (A \vee B) \wedge C \longrightarrow A \vee (B \wedge C)$$

where the many commuting conversions are indicated in the following pictures.



Note that these maps are linear.

There are two distinct kinds of symmetry at play here. On the one hand we have the following.

Proposition 3.15. *w and $\tilde{w}$ are self-dual: that is, we have*

$$(w_{A,B,C})^* = w_{C^*,B^*,A^*} \quad \text{and} \quad (\tilde{w}_{A,B,C})^* = \tilde{w}_{C^*,B^*,A^*} .$$

Essentially this follows from the de Morgan duality of the proof rules. On the other hand we have the following.

Proposition 3.16. *w and $\tilde{w}$ are interderivable using the symmetry: that is, we have the following equation and its dual.*

$$w_{A,B,C} = c_{A,B\vee C}; c_{B,C} \wedge \mathrm{id}_A; \tilde{w}_{C,B,A}; \mathrm{id}_C \vee c_{A,B}; c_{C,A\wedge B}$$

Many other relations between w and $\tilde{w}$ are consequences of these equations and the idempotency of the symmetry c.

The basic result is as follows.

Theorem 3.17. *The linear distributivities are guarded transformations.*

There are a considerable number of coherence diagrams for weak distributivities. They are clearly laid out in [2] and we do not have space to repeat them here.

Theorem 3.18. *The coherence diagrams for weak distributivities hold.*

The only place where this bears interpretation is in the case of 'Unit Coherence' where one finds canonical idempotents (identities in the 2-category of guarded functors).

3.3.4 Duality

A $*$-polycategory supports polymaps $\mathrm{in}_A : - \to A^*, A$ and $\mathrm{ev}_A : A, A^* \to -$ which enable us to define something like a unit

$$\eta^A_B = \overline{\mathrm{in}_A \cdot \mathrm{id}_B} : B \to A^* \vee (A \wedge B) ,$$

and something like a counit

$$\varepsilon^A_B = \overline{\mathrm{ev}_A \cdot \mathrm{id}_B} : A \wedge (A^* \vee B) \to B .$$

One expects that the unit and counit are interchanged by the self-duality, though our conventions on duality require mediating symmetries. (With the other choice of convention, the problem emerges elsewhere!)

Proposition 3.19. *The η and ε are dual in the sense that the equations* $c_{A^*,B^*} \wedge \mathrm{id}_A; (\eta^A_B)^* = c_{A^* \vee B^*,A}; \varepsilon^A_{B^*}$ *and* $(\varepsilon^A_B)^*; c_{A^*,B^* \wedge A^*} = \eta^A_{B^*}; \mathrm{id}_{A^*} \vee c_{B^*,A}$ *hold.*

Finally we get triangle identities in a guarded sense.

Theorem 3.20. *We have* $\mathrm{id}_A \wedge \eta^A_B; \varepsilon^A_{A \wedge B} = e_{A \wedge B}$ *and* $\eta^A_{A^* \vee B}; \mathrm{id}_{A^*} \vee \varepsilon^A_B = e_{A^* \vee B}$.

This essentially gives an adjunction in the guarded 2-category.

3.3.5 Algebras and coalgebras

We consider now the structural maps

$$d : A \to A \wedge A \ t : A \to \top \qquad m : A \vee A \to A \ u : \bot \to A$$

The de Morgan duality of the proof rules shows that these structures are dual to one another:

$$d^*_A = m_{A^*}\,, \quad m^*_A = d_{A^*}\,, \quad t^*_A = u_{A^*}\,, \quad u^*_A = t_{A^*}\,.$$

In familiar category theoretic settings maps of these kinds are usually associated with product and coproduct structure; but here we do not even have guarded naturality. But the correctness equations of 2.3.3 give at once the following relation to canonical linear idempotents.

Proposition 3.21. *The maps d, t are codomain absorbing while m and u are domain absorbing in the sense that following equations hold.*

$$d; e_{A \wedge A} = d\,, \quad t; e_\top = t \quad \text{and} \quad e_{A \vee A}; m = m\,, \quad e_\bot; u = u\,.$$

Moreover some structure holds on the nose.

Proposition 3.22. *The structure (A, t_A, d_A) forms a commutative comonoid, while the structure (A, m_A, u_A) forms a commutative monoid.*

We list the equations involved in the comonoid case.

$$d; t \wedge \mathrm{id}_A; \tilde{r} = \mathrm{id}_A \qquad d; \mathrm{id}_A \wedge t; \tilde{l} = \mathrm{id}_A$$
$$d; \mathrm{id}_A \wedge d; a = d; d \wedge \mathrm{id}_A \qquad d; d \wedge \mathrm{id}_A; \tilde{a} = d; \mathrm{id}_A \wedge d$$
$$d; c = d\,.$$

We are now in a position to explain a notion of categorical model for classical proof. In the definition one should think of the hom-sets $\mathcal{C}(A; B)$ as being the collection of classical proofs of $A \vdash B$. Proofs of more complex sequents are coded indirectly in the model.

Definition 3.23. A *(static) model for classical (propositional) proofs* consists of the following data satisfying the given axioms.

- A guarded category $\mathcal{C}$ equipped with a (strictly) involutive self-duality $(-)^*$.
- Guarded objects $\top$ and $\bot$ of $\mathcal{C}$ and guarded functors $\wedge$, $\vee$ (respectively domain and codomain absorbing) satisfying the usual de Morgan laws with respect to the duality. Linear maps are maps u, v such that

$$u \wedge v; f \wedge g = (u; f) \wedge (v; g) \quad \textit{and} \quad f \vee g; u \vee v = (f; u) \vee (g; v) .$$

- Linear mutually inverse guarded transformations for $\top$ and $\wedge$

$$\begin{array}{cc} l : A \to \top \wedge A, & \tilde{l} : \top \wedge A \to A, \\ r : A \to A \wedge \top & \tilde{r} : A \wedge \top \to A \\ a : A \wedge (B \wedge C) \to (A \wedge B) \wedge C & \tilde{a} : (A \wedge B) \wedge C \to A \wedge (B \wedge C) \end{array}$$

  of the left and right unit laws and associativity satisfying the usual pentagon and triangle laws .
  By duality we have also the same structure for $\bot$ and $\vee$.
- A linear self-inverse guarded transformation

$$c : A \wedge B \to B \wedge A$$

  giving a symmetry for $\wedge$, and satisfying the usual hexagon condition.
  By duality we have also the same structure for $\vee$.
- Linear guarded transformations

$$w : A \wedge (B \vee C) \to (A \wedge B) \vee C, \quad \tilde{w} : (A \vee B) \wedge C \to A \vee (B \wedge C)$$

  interchanged by duality, interdefinable using the symmetry and satisfying standard coherence conditions for distributivities.
- Linear and mutually dual guarded transformations $\eta^A_B : B \to A^* \vee (A \wedge B)$ and $\varepsilon^A_B : A \wedge (A^* \vee B) \to B$ satisfying the triangle identities.
- An association to all objects A of maps $d : A \to A \wedge A$, $t : A \to 1$, and their duals $m : A \vee A \to A$ and $u : \bot \to A$ in $\mathcal{C}$, codomain and dually domain absorbing, and giving to each object A the structure of a commutative comonoid with respect to $\wedge$ and the structure of a commutative monoid with respect to $\vee$.

This definition may seem substantially more complex than analogues for linear logic; but that may well be more a matter of lack of familiarity. Much of the definition is concerned to say that one has a $*$-autonomous category, modulo issues of canonical idempotents.

We now explain how, given a model $\mathcal{C}$ of classical proof in the sense just described, we can construct a $*$-polycategory $\mathcal{C}$ modelling classical proof in the sense analyzed earlier. Splitting idempotents is a basic tool in category theory, familiar in particular from the theory of Morita equivalence. Here we could use it for a novel purpose: splitting some canonical idempotents provides objects representing polysets of objects on either sides of polymaps. This means that we recover the sets of polymaps $\mathcal{C}(\Gamma, \Delta)$. We explain the point in a simple case. We have canonical polymaps

$$i_{A\wedge B} = \mathrm{id}_A \cdot \mathrm{id}_B : A, B \to A \wedge B \quad \text{and} \quad i_{C\vee D} = \mathrm{id}_C \cdot \mathrm{id}_D : C \vee D \to C, D\ .$$

Since $i_{A\wedge B}; \overline{\overline{f}}; i_{C\vee D} = \{\mathrm{id}_A, \mathrm{id}_B\}; f; \{\mathrm{id}_C, \mathrm{id}_D\} = f$, we can regard $\mathcal{C}(A, B; C, D)$ as arising by splitting the idempotent

$$g \to \overline{\overline{i_{A\wedge B}; g; i_{C\vee D}}} = e_{A\wedge B}; g; e_{C\vee D}$$

on $\mathcal{C}(A \wedge B; C \vee D)$.

So in outline the construction of the polycategorical model is as follows. We make a choice of bracketings of both Γ and Δ. This gives us hom-sets $\mathcal{C}(\bigwedge \Gamma, \bigvee \Delta)$ and canonical idempotents $e_{\bigwedge \Gamma}$ and $e_{\bigvee \Delta}$. We can then take $\mathcal{C}(\Gamma; \Delta)$ to consist of the $f \in \mathcal{C}(\bigwedge \Gamma, \bigvee \Delta)$ such that $e_{\bigwedge \Gamma}; f; e_{\bigvee \Delta} = f$. Finally we have a series of fiddly but routine tasks.

(1) We show that $\mathcal{C}(\Gamma; \Delta)$ is essentially independent of the bracketing chosen. This follows from the coherence of the canonical linear idempotents.
(2) We show how to define composition on the sets of polymaps. This combines point (1) with heavy use of the linear distributivities. And we show that the result is indeed a $*$-polycategory.
(3) We define the logical operations on the collections of polymaps and derive the many equations. This is pretty much routine.

4 Explanation and comparison

4.1 Representable polycategories

We recall the relationship between $*$-polycategories and $*$-autonomous categories (see [2] or [11] for example). Take the obvious 2-categories $*\,\mathbf{Poly}$ of $*$-polycategories and $*\mathbf{Aut}$ of $*$-autonomous categories: all 2-cells are invertible so we are in the groupoid enriched setting. Any $*$-autonomous category determines a $*$-polycategory, with the linear tensor and par representing polymaps; so one sees that there is a groupoid enriched forgetful functor $SPoly : *\mathbf{Aut} \to *\mathbf{Poly}$. On the other hand one can freely construct a $*$-autonomous category generated by a $*$-polycategory, subject to obvious identifications. This gives a groupoid enriched functor $SAut : *\mathbf{Poly} \to *\mathbf{Aut}$ and a groupoid enriched adjunction $SAut \dashv SPoly$. The basic conservativity result proved by direct syntactic considerations in [2] (though see [11] for an indication of a semantic proof) is as follows.

Theorem 4.1. *In the groupoid enriched adjunction $SAut \dashv SPoly$, the unit*

$$\mathcal{P} \to SPolySAut(\mathcal{P})$$

is full and faithful for any $*$-polycategory $\mathcal{P}$.

When does a $*$-polycategory $\mathcal{P}$ arise from a $*$-autonomous category, that is when is it in the essential image of $SPoly$? This occurs just when there are maps

$$i_{A,B} : A, B \to A \wedge B \;\; i_\top : - \to \top \qquad i_{C,D} : C \vee D \to C, D \;\; i_\perp : \perp \to -$$

composition with which induces isomorphisms

$$\mathcal{P}(A \wedge B, \Gamma; \Delta) \cong \mathcal{P}(A, B, \Gamma; \Delta) \qquad \mathcal{P}(\top, \Gamma; \Delta) \cong \mathcal{P}(\Gamma; \Delta)\,,$$
$$\mathcal{P}(\Gamma; \Delta, C \vee D) \cong \mathcal{P}(\Gamma; \Delta, C, D) \qquad \mathcal{P}(\Gamma; \Delta\, \perp) \cong \mathcal{P}(\Gamma; \Delta)\,.$$

In particular for any Γ, Δ we have isomorphisms $\mathcal{C}(\Gamma; \Delta) \cong \mathcal{C}(\bigwedge \Gamma; \bigvee \Delta)$ where we write $\bigwedge \Gamma$ and $\bigvee \Delta$ for a conjunction and disjunction according to some bracketings. In these circumstances we say that $i_{A,B}$, $i_\top$, $i_{C,D}$ and $i_\perp$ *provide a representation of polymaps*, or more loosely that $\wedge$, $\top$, $\vee$, $\perp$ *represent polymaps*.

4.2 Representability and functoriality

Consider now a $*$-polycategorical model $\mathcal{C}$ for classical proof: it comes equipped with structure

$$i_{A,B} = i_{A \wedge B}\,, \quad i_\top = \star\,, \quad i_{C,D} = i_{C \vee C}\,, \quad i_\perp = \star$$

(using earlier notation) potentially providing a representation of polymaps.

From our outline of the reconstruction of the $*$-polycategory, we see that we have representability just when the canonical linear idempotents

$$e_{A \wedge B} = \overline{i_{A,B}}\,, \quad e_\top = (i_\top)^+\,, \quad e_{C \vee D} = \overline{i_{C,D}}\,, \quad e_\perp = (i_\perp)^+$$

are in fact identities. By duality, we only need half of this so representability is equivalent to the conditions

$$e_\top = \mathrm{id}_\top \quad \text{and} \quad \mathrm{id}_A \wedge \mathrm{id}_B = \mathrm{id}_{A \wedge B}\,.$$

Next note that, as $\wedge$ is guarded domain absorbing, we have

$$f \wedge g; h \wedge k; \mathrm{id}_E \wedge \mathrm{id}_F = (f; h) \wedge (h; k); \mathrm{id}_E \wedge \mathrm{id}_F$$

so that $\mathrm{id}_E \wedge \mathrm{id}_F = \mathrm{id}_{E\wedge F}$ gives

$$f \wedge g; h \wedge k = (f; h) \wedge (h; k)$$

which is functoriality of $\wedge$. One should regard $e_\top = \mathrm{id}_\top$ as functoriality of $\top$. Then one can summarise the discussion in the following.

Theorem 4.2. *Let $\mathcal{C}$ be a model for classical proof. Then the following are equivalent.*

(1) The identity conditions $\mathrm{id}_A \wedge \mathrm{id}_B = \mathrm{id}_{A\wedge B}$ and $e_\top = \mathrm{id}_\top$.
(2) Full functoriality of $\wedge$, and $\top$.
(3) Representability of polymaps by $\wedge$, $\top$ and $\vee$, $\bot$.

This makes clear the oversight in [11]. There linear maps were assumed to form a $*$-autonomous category; but that gives $\mathrm{id}_A \wedge \mathrm{id}_B = \mathrm{id}_{A\wedge B}$ and so functoriality of the logical operators. Note also that the condition $f \wedge g; h \wedge k = (f; h) \wedge (h; k)$ follows from that naturality of the $\wedge$-R rule which we did not adopt. However that condition is weaker than full functoriality. It is easy to find models in which it holds but $\mathrm{id}_A \wedge \mathrm{id}_B = \mathrm{id}_{A\wedge B}$ fails.

4.3 Why functoriality should fail

As we shall see the assumption of representability provides a substantial simplification of the notion of categorical model. So it is time to explain why we do not adopt it.

First we argue against the tempting naturality of $\wedge$-R

$$\{u, v\}; (f \cdot g) = (u; f) \cdot (v; g)\,.$$

Consider first $\{m, \mathrm{id}_B\}; (\mathrm{id}_A, \mathrm{id}_B)$. Composing with id_B does nothing so this is equal to $m; (\mathrm{id}_A, \mathrm{id}_B)$, which is represented by the proof

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad A \vdash A}{A \vee A \vdash A, A}}{A \vee A \vdash A} \qquad \dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{A \vee A, B \vdash A \wedge B} \tag{1}$$

There are two distinct ways to eliminate the Cut. One results in the normal form

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad A \vdash A}{A \vee A \vdash A, A}}{A \vee A \vdash A} \qquad B \vdash B}{A \vee A, B \vdash A \wedge B} \tag{2}$$

and the other in the normal form

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B} \qquad \dfrac{A \vdash A \quad B \vdash B}{A, B \vdash A \wedge B}}{A \vee A, B, B \vdash A \wedge B, A \wedge B}}{A \vee A, B \vdash A \wedge B} \tag{3}$$

Now consider $(m; \mathrm{id}_A) \cdot (\mathrm{id}_B; \mathrm{id}_B)$. This is clearly equal to $m \cdot \mathrm{id}_B$ which is represented by the first of the above two normal forms. There is no way to get at the second (though that is the normal form in which m has done its intended job of copying). Now we take the view that failure to have the same normal forms (even modulo obvious rewritings) is a clear sign of non-identity. We conclude that the naturality equation

$$\{m, \mathrm{id}_B\}; (\mathrm{id}_A \cdot \mathrm{id}_B) = (m; \mathrm{id}_A) \cdot (\mathrm{id}_B; \mathrm{id}_B)$$

is not faithful to the notion of proof encapsulated in the sequent calculus.

We explain the significance of this for the functoriality of $\wedge$. Consider $\mathrm{id}_A \wedge \mathrm{id}_B$. Note that

$$(m \wedge \mathrm{id}_B); (\mathrm{id}_A \wedge \mathrm{id}_B) = \overline{\{m, \mathrm{id}_B\}; (\mathrm{id}_A \cdot \mathrm{id}_B)} \quad \text{and} \quad m \wedge \mathrm{id}_B = \overline{m \cdot \mathrm{id}_B}\ .$$

Now we just argued that we should not have

$$\{m, \mathrm{id}_B\}; (\mathrm{id}_A \cdot \mathrm{id}_B) = m \cdot \mathrm{id}_B\ .$$

But as $i_{A \wedge B}; \overline{h} = h$, the operation $\overline{(\ )}$ is injective. So we cannot have the equation

$$(m \wedge \mathrm{id}_B); (\mathrm{id}_A \wedge \mathrm{id}_B) = m \wedge \mathrm{id}_B\ .$$

The general point seems to be this. If we cut a classical proof even with such simple proofs as given by our canonical linear idempotents, then we can, in general, obtain additional normal forms that were not available from the classical proof on its own.

4.4 Führmann-Pym Axioms

We observed already that a $*$-polycategory in which the polymaps are represented by $\wedge$ and $\vee$ is in effect a $*$-autonomous category. If one has a model for classical proof of this kind the structure simplifies drastically.

Theorem 4.3. *To give a model of classical proof in which $\wedge$, $\top$ and $\vee$, $\bot$ represent polymaps is to give the following data.*

- *A $*$-autonomous category $\mathbb{C}$ (with a strict duality): tensor is $\wedge$ and par $\vee$.*
- *The equipment on each object A of $\mathbb{C}$ of the structure of a commutative comonoid with respect to tensor (and so dually the structure of a commutative monoid with respect to par).*

This is the equality component of the structure proposed in Führmann and Pym [6]. (It is not the only simple possibility. We have recently seen work [14] of Lamarche and Strassburger which leads to an even more restrictive notion.)

There are a number of further connections between the Führmann-Pym notion and the one described in this paper. One simple thought is as follows. Suppose that $\mathcal{C}$ is a model for classical logic in the general sense, freely generated by some category of objects and maps. (This makes sense by Kelly-Power [13].) We can inductively define idempotents e_A on objects A of $\mathcal{C}$: we set $e_A = \mathrm{id}_A$ for atomic objects (which includes the duals A^*) and then set $e_{A\wedge B} = e_A \wedge e_B$ and $e_{A\vee B} = e_A \vee e_B$. (Implicitly we have taken $e_\top$ and $e_\bot$ as we found them.) Then we can define a quotient $\hat{\mathcal{C}}$ of $\mathcal{C}$ with

$$\hat{\mathcal{C}}(A, B) = \{f \in \mathcal{C}(A, B) \,|\, e_A; f; e_B = f\} \ .$$

The quotient functor is given by

$$\mathcal{C}(A, B) \longrightarrow \hat{\mathcal{C}}(A, B) \ : \ f \to e_A; f; e_B \ .$$

Now it is easy to see that $\hat{\mathcal{C}}$ has on the nose the structure which $\mathcal{C}$ has up to idempotents.

Theorem 4.4. *If $\mathcal{C}$ is a model for classical proof freely generated by a category, then $\hat{\mathcal{C}}$ is a model in the Führmann-Pym sense.*

4.5 Semantic possibilities

We hope to write more fully about models in further papers, so for now we survey the possibilities. We distinguish between the following.

- Degenerate models: that is categorical models based on compact closed categories (and so ignoring the difference between $\wedge$ and $\vee$). We think of these as abstract interpretations, allowing one in particular to associate a variety of invariants to proofs. Preliminary observations are in [12], [7].
- Categorical models: that is models satisfying the Führmann-Pym equality axioms [6]. We know some examples of these, and have a little theory, but there is more to do.
- General models: that is, models which are equivalent to polycategories which do not arise from $*$-autonomous categories. We know almost nothing about these.

5 Provisional Conclusions

5.1 Guiding Principles

The notion of model for classical proof theory which we have developed has unfamiliar features. Hence it seems worth reflecting on the principles which have informed our analysis.

Reduction principle for logical cuts. For us this is the remnant of the Martin-Löf criterion (see Prawitz [18]) for identity of proofs. At least some part of normalisation preserves meaning: we ask that simple detours should not matter. This is an essential component of our analysis, without which we would not have interesting equalities between (representations of) proofs.

Structural congruence. This an idea taken from concurrency theory. We follow that culture in taking the structural rules of Weakening and Contraction to behave well with respect to themselves: so we end up with commutative comonoid structure for $\top$, $\wedge$ and commutative monoid structure for $\perp$, $\vee$. However not a great deal rides on this choice. We note that the optical graphs of Carbone [3] provide free models for a notion of abstract interpretation in which this choice is not made.

Computation of values. We take something from ideas of non-determinism: a classical proof has a non-deterministic choice as to the normal forms to which it reduces. We take account of all plausible commuting conversions and the like, with a view to having some good representation of proofs. For these we hope that it is plausible that if proofs are equal then they should have the same normal forms. Where we have evidence of distinct normal forms we have taken it to be evidence that the proofs are distinct. Though we need to say more about equality to make the claim precise, we believe that our analysis is consistent with this principle in the following sense.

Proposition 5.1. *If two proofs are equal then they reduce to the same* <u>*collection*</u> *of normal forms.*

5.2 Further issues

Normal forms and meaning. We consider the question whether our general principle in the last proposition should be an equivalence: does having the same set (or maybe multiset) of normal forms entail equality of proofs? At the moment we would argue against that.

MIX. There is something right about the idea that proofs in classical logic involve some kind of non-determinism: the computation or reduction process is in principle non-deterministic. But we do not for example have primitives for non-deterministic choice. In particular in view of [1] we should investigate an approach to the idea of non-deterministic choice in proofs using the MIX rule.

Idempotents. While it is not clear whether our formulation of semantics for classical proof is robust, its use of canonical idempotents would bear further investigation. We have not space to describe here the consequence of splitting idempotents in a model for classical proof in our sense.

Linearity. In this paper we have used a notion of linearity which has mitigated to some extent the general failure of functoriality of the logical operations. We have not troubled with natural refinements (linearity in the domain or codomain). In a properly algebraic formulation we would expect to follow Power [17] and take this explicitly as part of the structure. Before doing that we should probably decide just how much use to make of it. In [11] where already an explicit notion of linearity is proposed, the idea was that linear maps would also be maps of the commutative coalgebra and commutative algebra structure. It seems that to make good sense of that one must forbid some superficially natural ways to reduce Cuts. (For example we would allow to reduce proof (1) in Section 4.3 to only (2) but not (3).)

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