A pragmatic dialogic interpretation of bi-intuitionism

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Summary. We consider a "polarized" version of bi-intuitionistic logic [9, 7, 10, 11] as a logic of *assertions and hypotheses* and show that it supports a "rich proof theory" and an interesting categorical interpretation, unlike the standard approach of C. Rauszer's *Heyting-Brouwer logic* [48, 49], whose categorical models are all partial orders by Crolard's theorem [19]. We show that P. A. Melliès notion of *chirality* [42, 43] appears as the right mathematical representation of the mirror symmetry between the intuitionistic and co-intuitionistc sides of polarized bi-intuitionism. Philosophically, we extend Dalla Pozza and Garola *pragmatic interpretation of intuitionism* as a logic of assertions [21] to bi-intuitionism as a logic of *assertions and hypotheses.* We focus on the logical role of illocutionary forces and justification conditions in order to provide "intended interpretations" of logic systems that classifies inferential uses in natural language and remain acceptable from an intuitionistic point of view. Although Dalla Pozza and Garola originally provide a constructive interpretation of intuitionism in a classical setting, we claim that some conceptual refinements suffice to make the "pragmatic interpretation" a *bona fide* representation of intuitionism. For co-intuitionism we sketch a *meaning-as-use* interpretation that appears as able to fulfill the requirements of Dummett and Prawitz's justificationist approach. We extend the Brouwer-Heyting-Kolmogorov interpretation of intuitionism by regarding co-intuitionistic formulas as types of the evidence for them. Next, assuming a notion of duality between *assertions* and *hypotheses*, we give a "dialogic interpretation" of *multiplicative linear polarized bi-intuitionistic logic* which can be regarded as a translation into *intuitionistic multiplicative linear logic with products*. Mathematically, the interplay between *evidence for* and *evidence against* assertions and hypotheses is inspired by Chu's construction [8], usually regarded as an abstract form of the "game semantics" for linear logic.

1 Introduction. A mathematical prelude.

The mathematical case study of this paper is a variant of Cecylia Rauzer's *bi-intuitionistic* logic [48, 49] (called *Heyting-Brouwer* logic by Rauszer) and the relations between the two parts that can be identified within it, namely *intuitionistic* logic, on one hand, and *co-intuitionistic* (also known as *anti-intuitionistic* or *dual-lintuitionistic*), on the other hand[1]. Our main goal is to identify, among the mathematical models of bi-intuitionism, those which may be regarded as its *intended interpretations*. The quest for an *intended interpretation* of a formal system often arises when several mathematical structures have been proposed to characterise an informal, perhaps vague notion and furthermore more unfamiliar and vaguer extensions arise by analogy or by opposition: here philosophical analysis may be invoked to assess which formal systems belong to *logic*, in the sense that they do capture actual forms of human reasoning, rather than to pure or applied mathematics.

It is very appropriate to ask such a question about bi-intuitionism: following Rauszer's approach researchers in this area usually define bi-intutionism by extending intuitionistic logic with the connective of subtraction $C \setminus D$, to be read as "*C excludes D*", which in algebraic terms is left adjoint to disjunction in the same way as implication is the right adjoint to conjunction (see the rules in (1) below). This pair of adjunctions establishes a *duality* between the core *minimal fragments* of intuitionism and co-intuitionism, namely, intuitionistic conjunction and implication with a logical constant for validity, on one hand, and co-intuitionistic disjunction and subtraction with invalidity, on the other.

However when bi-intuitionistic logic is defined in this way essential properties of the model theory and proof theory of co-intuitionism and of bi-intuitionism no longer hold. Recently Tristan Crolard ([19, 20]) developed bi-intuitionistic proof theory by adding rules for subtraction to *classical* proof theory and then introduced restrictions to characterize the constructive fragment. A reason for this choice is that if bi-intuitionism is regarded as an extension of intuitionistic logic with the connective of subtraction, then the "intuitionistic status" of bi-intuitionism becomes unclear: it was probably E. G. K. López-Escobar [39] the first to notice that *first order bi-intuitionistic* logic is not a conservative extension of first order intuitionistic logic[2], since in the standard theory of first order bi-intuitionistic logic one can prove the *intuitionistically invalid* formula

$$\forall x(Ax \vee B) \vdash (\forall x.Ax) \vee B.$$

Now it it is well-known that any first order theory containing this formula is complete for the semantics of *constant domains*. Thus first order

[1] We are mostly indebted with Paul-André Melliès for pointing at his work on dialogue chirality and at its relevance to our approach to bi-intuitionism. We are grateful for this insight that does clarify the nature of polarized bi-intuitionistic logic and the issue of its categorical models.

[2] We are grateful to Rodolfo Ertola Biraben for giving us this reference.

bi-intuitionistic logic is an intermediate system between classical and intuitionistic logic (see T. Crolard [19] for a clear and detailed account of this matter). It turns out that when topological and categorical models are taken into account, very serious problems emerge that make Rauszer's bi-intuitionism unsuitable as a framework for developing intuitionistic and co-intuitionistic model-theory and proof-theory.

1.1 Bi-Heyting algebras and Kripke models.

The early model theory of bi-intuitionism, namely, *bi-Heyting algebras* and Kripke-style semantics is due to Cecylia Rauszer [48, 49].

Definition 1. *A Heyting algebra is a bounded lattice* $\mathcal{A} = (A, \vee, \wedge, 0, 1)$ (namely, with *join* and *meet* operations, the least and greatest element)*, and with a binary operation,* Heyting implication ($\rightarrow$)*, which is defined as the right adjoint to meet. A* co-Heyting algebra *is a lattice* $\mathcal{C}$ *such that its opposite* $\mathcal{C}^{op}$ *(reversing the order) is a Heyting algebra.* $\mathcal{C}$ *has structure* $(C, \vee, \wedge, 1, 0)$ *with an operation of* subtraction ($\smallsetminus$) *defined as the left adjoint of join. Thus we have the rules*

$$\begin{array}{ccc} \textit{Heyting algebra} & \qquad & \textit{co-Heyting algebra} \\ \dfrac{c \wedge b \ \leq \ a}{\overline{c \ \leq \ b \rightarrow a}} & & \dfrac{a \ \leq \ b \vee c}{\overline{a \smallsetminus b \ \leq \ c}} \end{array} \tag{1}$$

A bi-Heyting *algebra is a lattice that has both the structure of Heyting and of a co-Heyting algebra.*

Definition 2. (Rauszer's Kripke semantics) *Kripke models for bi-intuitionistic logic have the form* $\mathcal{M} = (W, \leq, \Vdash)$ *where the accessibility relation* $\leq$ *is reflexive and transitive, and the forcing relation "*$\Vdash$*" satisfies the usual conditions for* $\vee$*,* $\wedge$*, 0 and 1 and moreover*

$$\begin{array}{l} w \Vdash A \rightarrow B \ \ \textit{iff} \ \ \forall w' \geq w.w' \Vdash A \ \textit{implies} \ w' \vdash B; \\ w \Vdash B \smallsetminus A \ \ \textit{iff} \ \ \exists w' \leq w.w' \Vdash A \ \textit{and} \ w' \vdash B. \end{array}$$

Such conditions are sometimes explained by saying that implication has to hold in all possible worlds *"in the future of our knowledge"* and subtraction in some world *"in the past of our knowledge"*. In fact Rauszer's Kripke semantics for bi-intuitionistic logic is associated with a *modal translation* into (what is called today) *tensed* **S4**.

1.2 No categorical bi-intuitionistic theory of proofs.

In the corpus of mathematical intuitionism very basic constructions are the Brouwer-Heyting-Kolmogorov interpretation, where formulas are interpreted as types of their proofs, and the Extended Curry-Howard correspondence between the typed λ-calculus, intuitionistic Natural Deduction and Cartesian

Closed Categories, in the interpretation of William Lawvere. Here model theory and proof theory meet at a new level, where also *categorical proof theory* plays an essential role. Indeed categorical proof theory is concerned not only with algorithm to establish the *provability* of formulas in given proof systems, but has also mathematical tools to characterize the *identity of proofs*. To quote the simplest example, the philosophical conjecture by Martin-Löf and Prawitz that Natural Deduction derivations reducing to the same normal form represent the same intuitive proof can be treated axiomatically and refined in terms of the functorial properties and natural equivalences in Cartesian Closed Categories. Such a mathematical study where the notion of a proof can be appropriately characterised in relation to significant aspects of computation may be called a *rich proof theory*.

How are these ideas extended from intuitionism to co-intuitionism and bi-intuitionism? Recent work in co-intuitionistic and bi-intuitionistic proof theory (starting from the notes in appendix to Prawitz [45]) exploits the formal symmetry between intuitionistic conjunction and implication, on one hand, and co-intuitionistic disjunction and subtraction, on the other, in various formalisms, the sequent calculus, as in Czemark [22] and Urbas [55], the display calculus by Goré [33] or natural deduction by Uustalu [56]. Luca Tranchini [54] shows how to turn Prawitz Natural Deduction trees upside down, as it was done also by the first author in [9, 7, 10], who has also developed a computational interpretation and a categorical semantics for co-intuitionistic linear logic [10, 11].

But the most striking fact is a theorem by Tristan Crolard [19]:

Theorem 1. *If a Cartesian Closed Category has also the dual structure of a co-Cartesian Closed category, then it is a partial order.*

Thus for Rauszer's bi-intuitionistic logic we can no-longer have a *categorical theory of proofs*: between two objects there is at most one morphism. The outcome is devastating: there cannot be a "rich proof theory" for Rauszer bi-intuitionism by a simple notion of duality.

In this paper we explore a solution to this problem that has been suggested in [9, 7], namely, *"polarizing"* bi-intuitionistic logic so as to "keep the dual intuitionistic and co-intuitionistic parts separate", but connected by "mixed operators", most notably, negations.

1.3 Co-intuitionistic disjunction is "multiplicative".

A second results by Tristan Crolard [19] shows that intuitionistic dualities are not modelled in the naif way in the category **Set**. The category **Set** is an important model of intuitionism, as the adjunction between *categorical products*, given by cartesian products, and *exponents*, given by sets of functions, models the adjunction between *conjunction* and *implication*. By duality, a categorical model of *co-intuitionism* is based on the adjunction between *categorical co-products* modelling *disjunction* and *co-exponents* modelling *subtraction*. But here there is a main difference between intuitionism and co-intuitionism: Tristan Crolard [19] shows that in the category **Sets** the co-exponent of two non-empty sets does not exist. A proof of Crolard's lemma is given in Appendix 7.1.

The reason for this failure lies in the fact that in **Sets** co-products are given by *disjoint unions*; in logical terms, this corresponds to the fact that a proof of A *or* B is always either a proof of A or a proof of B; intuitionistic disjunction involves a choice between the disjuncts. Following Girard's classification of connectives in linear logic [27], it is the *additive form of intuitionistic disjunction* that makes it an unsuitable candidate as a right adjoint of subtraction.

The solution advocated in [11] is to take *multiplicative disjunction*, namely, J-Y. Girard's *par*, as basic for the co-intuitionistic consequence relation and construct a categorical model of *linear co-intuitionistic logic* in *monoidal categories*, where *co-exponents* modelling *subtraction*, are indeed the left adjoint of *co-products* modelling *par*. Thus we have categorical models of *multiplicative linear* intuitionistic and co-intuitionistic logic and we are left with the problem of extending such models to full bi-intuitionistic logic, rather than its linear part.

1.4 Bi-intuitionistic logic as a chirality.

In our reformulation of bi-intuitionism as a *polarized* system the idea emerges of a logic where the intuitionistic and the co-intuitionistic sides remain separated and form what P-A. Melliès [42, 43] calls a *chirality*, i.e., a mirror symmetry between independently defined structures $(\mathcal{A}, \mathcal{B})$, rather than a pair $(\mathcal{C}, \mathcal{C}^{op})$ where one element is defined as the opposite of the other. More precisely, a *chirality* is an adjunction $L \dashv R$ between monoidal functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$, where $\mathcal{A} = (\mathcal{A}, \wedge, \mathbf{true})$ and $\mathcal{B} = (\mathcal{B}, \vee, \mathbf{false})$, together with a monoidal functor $(\_)^* : \mathcal{A} \to \mathcal{B}^{op}$ that allows to give a "De Morgan representation of implication" in $\mathcal{A}$ through disjunction of $\mathcal{B}$. The notion of chirality applies both to linear bi-intuitionism and to full bi-intuitionism and it appears as the right mathematical framework to develop these logics. We sketch the proof-theoretic treatment corresponding to the categorical notion of chirality (see also the Appendix, Section 7.2), but we shall not do the categorical construction here.

But linear logic and the consideration of the relations between *classical* and *intuitionistic* linear logic give us also the tools of *Chu's construction* [8], a method to produce models of classical multiplicative linear logic from a pair of models of intuitionistic multiplicative linear logic, namely, from a pair $(\mathcal{C}, \mathcal{C}^{op})$ of monoidal closed categories. A simple application of Chu's construction yields also models of bi-intuitionistic linear logic from a pair of monoidal closed categories. Conceptually, this is important because Chu's construction suggests a *dialogue semantics* of bi-intuitionism inspired by an abstract form of the *game semantics* for linear logic. It is also clear that the two sides of the interpretation are exactly mirror images, i.e., form a chirality in an obvious sense.

In the rest of this paper we discuss the conceptual aspects of our *pragmatic interpretation* of bi-intuitionism. Next we give a precise definition of the language of *polarized bi-intuitionistic* and of *linear polarized bi-intuitionistic* logic and of our *dialogue interpretation.* Finally in the Appendix we recall the basic definitions of our "proof theoretic" *Chu's construction* and show how to produce the dialogue interpretations of linear intuitionism and linear co-intuitionism as mirror images, i.e. as a chirality.

2 Philosophical interpretations of co-intuitionism.

An important contribution to a philosophical understanding of co-intuitionism has been given by Yaroslav Shramko [51]. Co-intuitionistic sentences are interpreted as *statements that have not yet been refuted*, thus evoking the status of scientific laws in Popper's epistemology. In this view universal empirical statements can never be *conclusively justified*, but can be refuted by cumulative evidence against them (if not by a single crucial experiment). A clear merit of this approach is to have pointed at *formal epistemology* as a large domain where co-intuitionistic logic can be usefully applied.

Granted that the hypothetical status of empirical laws opens the way for application of co-intuitionism to formal epistemology, a question arises about the interpretation of the co-intuitionistic consequence relation and of inferences in co-intuitionism. We may consider a relation of the following form:

$$H \vdash H_1, \ldots, H_n \tag{2}$$

to be read as

> **H.0:** *"the disjunction of $H_1, \ldots, H_n$ may* justifiably *be taken as a hypothesis given that it is* justified *to take H as a hypothesis.*

Here we follow ideas of D. Prawitz [47] on the explanation of deductive inference and justification of inference rules and assume that a consequence relation should be explained not only in terms of validity in a Kripke-style semantics, namely, by saying that the disjunction of $H_1, \ldots, H_n$ is true in all possible world in which H is true, but also by in terms of the *justification conditions* for the act of making hypotheses, namely, by explaining how the evidence giving sufficient grounds for making the hypothesis H would also give sufficient grounds for taking the disjunction of $H_1, \ldots, H_n$ as a hypothesis.

Thus assuming that we know what "sufficient grounds for making a hypothesis H" are and borrowing the notion of "effective method" from the Brouwer-Heyting-Kolmogorov interpretations of intuitionism, we may give an effective interpretation of (2) as follows:

> **H.1:** *"there is a method F transforming sufficient evidence for regarding H as a* justified hypothesis *into sufficient evidence for regarding the disjunction of $H_1, \ldots, H_n$ as a* justified hypothesis.

Let's assume that meaning of a co-intuitionistic statement H is *"H is a still un-refuted hypothesis"*: this seems to imply that a justification for taking H as a hypothesis is the fact that H has not been refuted. Also this presupposes that we do know what "sufficient grounds for refuting a hypothesis H" are. But denying a hypothesis is *asserting* its falsity and refuting a hypothesis is giving *conclusive grounds* for such a denial, in particular in the case of a mathematical statement a *proof* of the falsity of H. We are back in the well-known environment of the Brouwer-Heyting-Komogorov interpretation; an *effective* interpretation of the relation (2) is as follows:

> **H.2:** *"there is a method F^{op} to transform evidence* refuting *all the hypotheses $H_1, \ldots, H_n$ into evidence* refuting *the hypothesis H.*

So what is the primary notion, that of *sufficient grounds for making a hypothesis H* (*evidence for H*) or that of *sufficient grounds for refuting H* (*evidence against H*)? Or do we need both notions?

We may expect a fundamental objection to taking **H.1** as primitive. Many would say that no matter how "evidence for a hypothesis" is defined, it is the business of empiric sciences and of probability theory, not of logic, to deal with it. Hypothetical reasoning is inferring assertable propositions from the assumption that some propositions are assertable; strictly speaking, logic can only be about the *refutation* of hypotheses, as in the medieval practice of disputation [4].

Mathematical reasoning is mainly assertive and its proofs provide the paradigmatic notion of "conclusive evidence". However, other areas of deductive reasoning, including legal argumentation [32, 18], are about statements for which only non-conclusive degrees of evidence are available. We cannot discuss such applications here. Let us explore co-intuitionism as a logic of hypotheses and take the elementary expressions of our object language to represent types of hypotheses and the interpretation **H.1** of the consequence relation as primitive, as in work by the first author, [9, 7, 10, 11] aiming at a "rich proof theory" for co-intuitionism and bi-intuitionism. One should recognize that such mathematical treatment has focussed on the duality between intuitionism and co-intuitionism in order to designe Gentzen systems, term assignments and categorical proof-theory for co-intuitionism. One should not underestimate the difficulty of taking co-intuitionism "on its own" and **H.1** as primitive: there is only one degree of *conclusive evidence*, but there are uncountably many degrees of partial evidence, according to probability theory. Do we need infinitely valued logics here?[3]

[3] Of course this problem is already there in intuitionism, if we take into account what counts as *evidence against* an assertion, not only the *evidence for* it.

Remark 1. From a mathematical point of view it would seem appropriate, given a hypothesis $\mathcal{H}\, p$ and the evidence we have to justify it, to assign a probability to $\mathcal{H}\, p$ expressing our degree of confidence in its validity. This could be done in a classical probabilistic model, or in a Bayesian setting. In the literature on linear logic we find work by P. Lincoln, J. Mitchell and A. Scedrov [38] with a stochastic interaction semantics modelling proof search in *multiplicative and additive linear logic* **MALL**; in that framework logical connectives are interpreted as probabilistic operators. But to construct a model of co-intuitionistic logic we would need a translation into linear logic *with exponential operators* ? and ! and we do not have a stochastic interpretation of them. How should we interpret the consequence relation in **H.1**, **H.2** and **H.3** in terms of probability functions? Are probabilities assigned according to proof-search algorithms appropriate in our case? We cannot speculate about such questions here.

It is clear to us that a proper treatment of hypotheses both in applied contexts such as legal or medical evidence or formal epistemology and in a purely theoretical context does eventually require a probabilistic framework[4]. However it is also clear that if we regard *making the hypothesis that* p ($\mathcal{H}\, p$) as an illocutionary act in natural language, then the act of *asserting that* p *is true with probability* $Pr(p)$" conveys more information and is justified by much stronger conditions that simply making a hypothesis; nevertheless a *probabilistic modelling* of $\mathcal{H}\, p$ would certainly be adequate in any application to common sense reasoning.

2.1 A meaning-as-use justification of co-intuitionism?

If co-intuitionism is to stand as a logic on its own, representing informal practices of common sense reasoning, the question may be asked whether its inferential principles are compatible with the basic tenets of intuitionistic philosophy: mathematical duality may not suffice to justify such compatibility. A way to answer such a question and dispel doubts about its constructive nature is to give a *meaning-as-use* interpretation of co-intuitionism in the sense of Michael Dummett [25] and Dag Prawitz (in a sequence of papers from [46] to [47]).

Here we recall the main ingredients of such an interpretation. We take Natural Deduction rules of introduction and elimination for subtraction (in sequent-style form) and check that they satisfy the *inversion principle* (see [45]).

$$\smallsetminus\text{-}intro\ \frac{H \vdash \Gamma, C \qquad D \vdash \Delta}{H \vdash \Gamma, C \smallsetminus D, \Delta} \qquad\qquad \smallsetminus\text{-}elim\ \frac{H \vdash \Delta, C \smallsetminus D \qquad C \vdash D, \Upsilon}{H \vdash \Delta, \Upsilon}$$

[4] Carlo Dalla Pozza in private conversation has often pointed out that hypotheses in science are best modelled in a Bayesian framework rather than through purely logical methods.

Notice that in the $\smallsetminus$-*elimination* rule the evidence that D may be derivable from C given by the *right premise* has become *inconsistent with the hypothesis* $C \smallsetminus D$ in the left premise; in the conclusion we drop D and we *set aside* the evidence for the inconsistent alternative. We may think that such evidence is not destroyed, but rather stored somewhere for future use.

If the *left premise* of $\smallsetminus$-*elimination*, deriving the disjunction of $C \smallsetminus D$ with Δ from H, has been obtained by a $\smallsetminus$-*introduction*, then such an occurrence of $C \smallsetminus D$ is a *maximal formula* and the pair of *introduction/elimination* rules can be eliminated: using the *removed evidence* for D derivable from C (*right premise* of the $\smallsetminus$-*elim.*) we can conclude that the disjunction of $\Delta_1, \Delta_2, \Upsilon$ is derivable from H. This is, in a nutshell, the principle of normalization (or *cut-elimination*) for subtraction.

$$
\dfrac{\smallsetminus\text{-I}\;\dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, C \smallsetminus D} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{H \vdash \Gamma, \Delta, \Upsilon}\;\smallsetminus\text{-E}
$$

reduces to

$$
\dfrac{subst\;\dfrac{\begin{matrix} d_1 \\ H \vdash \Gamma, C \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{H \vdash \Gamma, D, \Upsilon} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Gamma, \Delta, \Upsilon}\;subst
$$

Now suppose d_1 and d_3 are simply *assumptions* (in the sequent form of axioms). Then we have the following reduction:

$$
\dfrac{\smallsetminus\text{-I}\;\dfrac{C \vdash C \qquad D \vdash D}{C \vdash D, C \smallsetminus D} \qquad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}}{C \vdash D, \Upsilon}\;\smallsetminus\text{-E} \quad \text{reduces to} \quad \begin{matrix} d_2 \\ C \vdash D, \Upsilon \end{matrix}
$$

In words, if we use the hypothesis that C *excludes* D ($C \smallsetminus D$) to remove possible consequences of C of the form D from consideration, but the hypothesis $C \smallsetminus D$ was itself derived from C by an inference that yields the hypothesis D as a possible consequence, then nothing has been achieved by performing such a pair of operations. We conclude not only that the two derivations have the same deductive consequences but also that in some sense they may be regarded as *the same deductive process*. The latter assertion may be disputed, but the above argument is the core of a *proof theoretic justification* of the introduction and elimination pair for subtraction. Here we assume that the primary operational meaning is given by the *elimination* rule, by which some conclusions are “excluded from consideration”; the *introduction* rule is shown to be in *harmony* with it (in the sense of Dummett [25]). The choice of the elimination rule as primary is also supported by the fact that it is *invertible* while the introduction rule, in its general form, is not.

A similar procedure may be give justification for disjunction.

$$\curlyvee\text{-}intro\ \frac{H \vdash \Gamma, C, D}{H \vdash \Gamma, C \curlyvee D} \qquad \curlyvee\text{-}elim\ \frac{H \vdash \Gamma, C \curlyvee D \qquad C \vdash \Delta \qquad D \vdash \Upsilon}{H \vdash \Gamma, \Delta, \Upsilon}$$

We have the following reduction:

$$\curlyvee\text{-E}\ \frac{\curlyvee\text{-I}\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}}{H \vdash \Upsilon, C \curlyvee D} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}$$

reduces to

$$subst\ \frac{subst\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix} \qquad \begin{matrix} d_2 \\ C \vdash \Gamma \end{matrix}}{H \vdash \Upsilon, \Gamma, D} \qquad \begin{matrix} d_3 \\ D \vdash \Delta \end{matrix}}{H \vdash \Upsilon, \Gamma, \Delta}$$

Again let's suppose that d_2 and d_3 are simply *assumptions*. Then the reduction is as follows:

$$\curlyvee\text{-E}\ \frac{\curlyvee\ \text{I}\ \dfrac{\begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}}{H \vdash \Upsilon, C \curlyvee D} \qquad C \vdash C \qquad D \vdash D}{H \vdash \Upsilon, C, D} \quad \text{reduces to} \quad \begin{matrix} d_1 \\ H \vdash \Upsilon, C, D \end{matrix}$$

In words, the $\curlyvee$-*introduction* rule connects two possible conclusions C and D into one $C \curlyvee D$ and the $\curlyvee$-*elimination* rule uses the resulting connection to join into the same deductive context two separate contexts, one resulting from C and the other from D. But C and D were already in the same context to start with, thus the possibility for joining the context was already there in the preconditions of the $\curlyvee$-*introduction* rule. Thus the two derivations have not only the same deductive consequences but also they may be regarded as the same deductive method. It is completely clear that the two rules are in *harmony*. We have chosen the *introduction* rule as giving the primary meaning also considering that it is invertible, while the *elimination* in its general form is not. It may be possible to argue that the context-joining operation exhibited by the $\curlyvee$-*elimination* rule is defining the operational meaning of co-intuitionistic disjunction.

If we take the $\curlyvee$-*introduction* rule as defining the operational meaning of (multiplicative) disjunction, we can say that it exhibits a possibility of connection between two conclusions given by the fact of being in the same context. It may be objected that the meaning of multiplicative disjunction is already given by the multiple-conclusion context; so the interpretation is in some sense circular. The objection is sensible, but it may only show a feature of such *meaning-as-use* interpretations that does not make them irrelevant. Making an implicit possibility of connection explicit is precisely what the $\curlyvee$-*introduction* rule does.

In this paper we shall not flesh out the *meaning-as-use* interpretation in full detail. However our pragmatic interpretation contributes to a justificationist approach by providing an analysis of the contribution to meaning given by *elementary expressions* in virtue of their *illocutionary force.* This analysis allows us to extend the *meaning-as-use* interpretation beyond intuitionistic logic. It is because we interpret the elementary expressions of co-intuitionistic logic as expressing the illocutionary force of a *hypothesis* that we are allowed to give co-intuitionistic disjunction a hypothetical mood and to justify its logical properties, which are very different from those of the usual *assertive* intuitionistic disjunction. By regarding co-intuitionism as the logic of the justification of *hypotheses*, we can explain and justify the duality between intuitionism and co-intuitionism in terms of common sense reasoning, in so far as the notion of a hypothesis can be seen as dual to that of an assertion.

We focus on the proposal of a semantic for *multiplicative linear* bi-intuitionistic logic, a *pragmatic dialogue interpretation* of co-intuitionism and to bi-intuitionism in which the two views **H.1** and **H.2** are combined; such a dialogue interpretation uses the general notion of a method that is characteristic of *linear intuitionistic logic* but is applied here to transform not only proofs to proofs, but also *non-conclusive evidence* into non-conclusive evidence. In this framework we have a *stricter* interpretation for *linear* co-intuitionistic multiplicative disjunction than that of a "contextual compatibility" evoked above, which is implicit in the form of the co-intuitionistic co-intuitionistic relation. does not rely on a meta-theoretic understanding of such a disjunction. Moreover such an interpretation can be formalized within *multiplicative intuitionistic linear logic with products*, in a way that evokes Chu's construction [8] (sketched in Appendix, Section 8).

3 Pragmatic interpretation of bi-intuitionism.

We develop our interpretation by expanding and reinterpreting Dalla Pozza and Garola's *pragmatic interpretation of intuitionistic logic* [21], which is in accordance with M. Dummett's suggestion that intuitionism is the logic of *assertions* and of their justifications. The main feature of Dalla Pozza and Garola's approach is to take *elementary expressions* of the form $\vdash p$, where Frege's symbol "$\vdash$" represents an (impersonal) *illocutionary force of assertion* and p is a proposition. The grammar of Dalla Pozza and Garola's language $\mathcal{L}^P$ is as follows:

$$A, B \quad := \quad \vdash p \mid \curlyvee \mid \mathbf{u} \mid A \supset B \mid A \cap B \mid A \cup B \tag{3}$$

and (strong) negation "$\sim$" is defined as $\sim A = A \supset \mathbf{u}$. Here $\curlyvee$ is an assertion which is always justified and $\mathbf{u}$ is an always unjustified assertion.

The justification of intuitionistic formulas is given precisely by Brouwer-Heyting-Kolmogorov's interpretation of intuitionistic connectives: the justification of $\vdash p$ is given by *conclusive evidence* for p (e.g., a proof of the mathematical proposition p) and the justification of an *implication* $A \supset B$ is a method that transforms a justification of A into a justification of B. Moreover a justification of a conjunction $A \cap B$ is a pair $\langle j, k \rangle$ where j is a justification of A and k a justification of B; a justification of a disjunction $A_0 \cup A_1$ is a pair $\langle j, 0 \rangle$ where j is a justification of A_0 or $\langle k, 1 \rangle$ where k is a justification of A_1.

To be sure, from an intuitionistic viewpoint the proposition p must be such that conclusive evidence for it can be effectively given: the (informal) proof justifying $\vdash p$ must be intuitionistic. Thus we cannot let p be $q \vee \neg q$ where q is intuitionistically undecidable and claim that $\vdash (p \vee \neg p)$ is justified by a classical proof. Thus Dalla Pozza and Garola assume that in the representation of intuitionism the proposition p must be regarded as atomic. If this is granted, then the expressions of $\mathcal{L}^P$ are *types of justification methods*; in a *propositions as types* framework they are *intuitionistic propositions*.

Having introduced the consideration of illocutionary forces in the elementary expression of logical languages, we can then ask in which sense intuitionistic types are *assertive expressions*: do molecular expressions inherit illocutionary force from their elementary components? is an illocutionary assertive force implicit in the way of presenting their justification? This is an interesting question, which Dalla Pozza and Garola do not give an explicit answer to. It seems clear to us that the molecular expressions of the above language must have an *"assertive mood"*, which sets them apart from other forms of reasoning, say, in a hypothetical or conjectural mood.

Gödel, McKinsey, Tarski and Kripke's modal **S4** interpretation is naturally considered here as a *reflection* of the *pragmatic layer* of the logic for pragmatics into the *semantic layer*, where the image $\Box A'$ of a pragmatic expression A is indeed a proposition of classical modal logic **S4**, and the necessity operator of **S4** is read as an operator of "abstract knowability". Briefly put, the modal meaning of pragmatic assertions is provided by a translation of pragmatic connectives where

$$\begin{array}{rlrl} (\vdash p)^M &= \Box p, & (A \supset B)^M &= \Box(A^M \to B^M), \\ (\curlyvee)^M &= \mathbf{t}, & (\mathbf{u})^M &= \mathbf{f} \\ (A \cap B)^M &= A^M \wedge B^M, & (A \cup B)^M &= A^M \vee B^M, \end{array} \tag{4}$$

where "$\to$", "$\wedge$", "$\vee$" are the classical connectives, $\mathbf{t}$ and $\mathbf{f}$ the truth values. Thus Dalla Pozza and Garola develop a two-layers formal system where the propositions p occurring in an elementary expression $\vdash p$ are interpreted according to classical semantics. Moreover they seem to think that the meaning of the *intensonal* expressions of intuitionistic pragmatics are adequately represented by their *extensional* translations in **S4**. Finally they develop their pragmatic interpretation in a *classical metatheory.* Thus what they obtain is a constructive interpretation of intutionism in a classical framework. This is certainly unacceptable to an intuitionistic philosopher but is fully in the spirit of Dalla Pozza and Garola's pragmatics: broadly speaking, their goal is to show how classical logic, as a theory of truth, can be reconciled with intuitionism, as a theory of justified assertability, by the principle that a *"change of logic is a change of subject matter"*.

We believe that such a classical twist in not essential to the project of an intuitionistic pragmatics and indeed that not much needs to be changed to obtain a *bona fide* representation of intuitionism. Granted that the *"semantic projection"* into **S4** is only an *"extensional abstract interpretation"* of intuitionistic pragmatic expressions and that we must work in an *intuitionistic metatheory*, the *pragmatic interpretation of intuitionistic logic* becomes compatible with intuitionistically acceptable interpretations according to a *justificationist* approach, either in a theory of *meaning-as-use* or in some kind of *game-theoretic semantics*.

3.1 Co-Intuitionistic Logic as a logic of hypotheses

A clear example of how a change of epistemic attitudes, particularly as expressed in the elementary formulas, drastically affects the resulting logic is given by considering the illocutionary force of *hypothesis* as basic. When hypothetical force is given also to co-intuitionistic formulas, the meaning of connectives changes: assuming that we know what counts as a justification of an *elementary hypothesis*, the meaning of *hypothetical disjunction* $C \curlyvee D$ and *hypothetical conjunction* $C \curlywedge D$ are obviously different from their assertive counterparts. So long as C is *justifiably* given the illocutionary force of a hypothesis, it is inevitable to accept that we there may be justified reasons to set aside such a hypothesis, i.e., we still *justifably* entertain a *doubt* ($\frown C$) about C: thus the principle $C \curlyvee \frown C$ is valid. Also no contradiction seem to derive from a simultaneous considerations $C \curlywedge \frown C$ of the hypotheses C and $\sim C$. Notice however that since $\frown C$ is definable as $\mathbf{j} \smallsetminus C$ it is only because of the *hypothetical mood* of subtraction that the law of excluded middle is valid and para-consistency obviously holds: indeed a non-hypothetical reading of $\frown C$ as *"the valid hypothesis excludes C"* would make the law of excluded middle intuitionistically problematic for such connectives.

It is even possible to have *mixed connectives* operating on assertive and hypothetical sentences and building assertive or hypothetical connectives in the framework of *polarized bi-intuitionistic* loigc: this has been done in [9] and completeness of the resulting logic with respect to the classical **S4** translation has been checked. Here we consider only the fragment of such logic that allows us to express the *duality* between intuitionism and co-intuitionism.

Our *co-intuitionistic logic of hypothesis* is built from elementary hypothetical expressions $\mathcal{H}\, p$ and a constant $\curlywedge$ for a hypothesis which is always unjustified, using the connectives *subtraction* $C \smallsetminus D$ (*"C excludes D"*), *hypothetical disjunction* $C \curlyvee D$ and *hypothetical conjunction* $C \curlywedge D$.

$$C, D \quad := \quad \mathcal{H}\,p \mid \curlywedge \mid \mathbf{j} \mid C \smallsetminus D \mid C \curlyvee D \mid C \curlywedge D \tag{5}$$

Here *supplement* (*weak negation*) "$\frown$" is defined as $\frown C = (\mathbf{j} \smallsetminus C)$; also "$\curlywedge$" is an always unjustified hypothesis and "$\mathbf{j}$" an always justified one.

A straightforward extension of the **S4** modal translation to co-intuitionism is as follows:

$$\begin{array}{rlrl} (\mathcal{H}\,p)^M &= \Diamond p, & (C \smallsetminus D)^M &= \Diamond(C^M \wedge \neg D^M), \\ (\curlywedge)^M &= \mathbf{f}, & (\mathbf{j})^M &= \mathbf{t} \\ (C \curlyvee D)^M &= C^M \vee D^M, & (C \curlywedge D)^M &= C^M \wedge D^M, \end{array} \tag{6}$$

where "$\neg$", "$\wedge$", "$\vee$" are the classical connectives, $\mathbf{t}$ and $\mathbf{f}$ the truth values. Here we clearly see that such an extension of Gödel's, McKinsey and Tarski's and Kripke's translation unacceptably collapses assertive and hypothetical constants:

$$(\curlyvee)^M \;=\; \mathbf{t} \;=\; (\mathbf{j})^M \quad \text{and} \quad (\curlywedge)^M \;=\; \mathbf{f} \;=\; (\mathbf{u})^M. \tag{7}$$

But what constitutes a *justification* for a *hypothesis* ($\mathcal{H}\ p$) and how does it differ from a justification of an *assertion* ($\vdash p$)? In the familiar Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic evidence for a mathematical statement p is a proof of it; in the case of non-mathematical assertive statements, we speak of *conclusive evidence* for p. What constitutes *conclusive* or *inconclusive evidence* for p depends on the context and scientific discipline.

Consider for example, the *theory of argumentation* in legal reasoning. Here six *proof-standards* have been identified from an analysis of legal practice: *no evidence at alll*, *scintilla of evidence*, *preponderance of evidence*, *clear and convincing evidence*, *beyond reasonable doubt* and *dialectical validity*, in a linear order of strength [32, 18]. Can such distinctions be taken up in our approach in some way? It seems that a *scintilla of evidence* suffices to justify $\mathcal{H}\ p$, making the hypothesis that p, and that *dialectical evidence* ought to coincide with assertability $\vdash p$, which in our framework is *conclusive evidence*. The other proof-standards are defined through probabilities; this goes beyond our purely logical approach here.

If we assume the notion of "negative evidence" (*evidence against* the truth of a proposition) as basic, in addition to "positive evidence" (*evidence for*), then another logical relation is evident between *scintilla of evidence* and *conclusive evidence*, in addition to the order of strength: *we cannot have at the same time* conclusive evidence for *and a* scintilla of evidence against *the truth of a proposition.* On this basis we can attempt an interpretation of intuitionistic and co-intuitionistic connectives which is reminiscent of *game semantics* and also of Nelson's treatment of *constructive falsity.*

4 The language and sequent calculus of "polarized" bi-intuitionism.

We consider two formal systems for *"polarized" bi-intuitionism* as our *"official languages"*. One, the logic **AH** of *assertions and hypotheses*, is a conservative extension of both intuitionistic and co-intuitionistic logic; the other **MLAH** is the multiplicative fragment of the linear version of **AH**. The language of **AH** is built from elementary expressions of the form "$\vdash p$" for *elementary assertions* or "$\mathcal{H}\, p$" for *elementary hypotheses*; ; moreover we have the sentential constants $\curlyvee$ (*assertive validity*), $\mathbf{u}$ (*assertive absurdity*), $\curlywedge$ (*hypothetical absurdity*) and $\mathbf{j}$ (*hypothetical validity*). We build *intuitionistic assertive* formulas within the assertive intuitionistic side using implication ($\supset$), conjunction ($\cap$) and disjunction ($\cup$); also we build *co-intuitionistic hypothetical* formulas within the hypothetical co-intuitionistic side using subtraction ($C \smallsetminus D$, disjunction ($\curlyvee$) and conjunction ($\curlywedge$). We also have two *defined negations*, the familiar intuitionistic one $\sim_u A$ and the co-intuitionistic one ${}_j\!\frown C$ (also called *supplement*):

$$\sim_u A \;=\; A \supset \mathbf{u} \qquad {}_j\!\frown C \;=\; \mathbf{j} \smallsetminus C.$$

Two negations relate the two sides, a *strong* one ($\sim$) transforming a hypothesis C into an assertion $\sim C$ and a *weak* one ($\frown$) transforming an assertion A into a hypothesis $\frown A$.Through these negations the duality between the intuitionistic and the co-intuitionistic sides is expressed within the language.

Definition 3. *(intuitionistic assertions, co-intuitionistic hypotheses)*

- *assertive intuitionistic formulas:*
  $A, B \;:= \vdash p \mid \curlyvee \mid \mathbf{u} \mid A \supset B \mid A \cap B \mid A \cup B \mid \sim C$
- *hypothetical co-intuitionistic formulas:*
  $C, D \;:= \mathcal{H}\, p \mid \curlywedge \mid \mathbf{j} \mid C \smallsetminus D \mid C \curlyvee D \mid C \curlywedge D \mid \frown A$
- *defined negations:*
  $\sim_u A \;=_{df}\; A \supset \mathbf{u} \quad {}_j\!\frown C \;=_{df}\; \mathbf{j} \smallsetminus C.$

In this paper we shall not consider assertive disjunction $(A \cup B)$ *and hypothetical conjunction* $C \curlywedge D$*.*

4.1 Informal Interpretation.

The language of polarized bi-intuitionism has an informal "intended interpretation" where formulas denote *types of acts of assertion and of hypothesis* and must be given *justification conditions*, namely, epistemic conditions that constitute *evidence for* illocutionary acts of these types. We take the notions of *"conclusive evidence"* and *"scintilla of evidence"* as primitive notions, with the obvious ordering, namely, we assume that conclusive evidence is also a scintilla of evidence, but not conversely. Thus we can define simultaneously what it means for assertive and hypothetical expressions to be justified.

Definition 4. (**a**.*i*) *the assertion* $\vdash p$ *is justified by* conclusive evidence *of the truth of* p*;*
(**a**.*ii*) *the assertion* $\curlyvee$ *is always justified and assertion* **u** *is never justified;*
(**a**.*iii*) $A \supset B$ *is justified by a method transforming conclusive evidence for* A *into conclusive evidence for* B*; evidence for* $\sim C$ *is a method transforming evidence for* C *into a contradiction;*
(**a**.*iv*) $A \cap B$ *is justified by conclusive evidence for both* A *and* B*;* $A \cup B$ *is justified by conclusive evidence either for* A *or for* B*.*

Dually:
(**h**.*i*) *the hypothesis* $\mathcal{H} p$ *is justified by* a scintilla of evidence *of the truth of* p*;*
(**h**.*ii*) *the hypothesis* $\curlywedge$ *is never justified and hypothesis* **j** *is always justified;*
(**h**.*iii*) $C \smallsetminus D$ *is justified by a scintilla of evidence for* C *together with a method showing that evidence for* C *and for* D *are incompatible; evidence for* $\frown A$ *is a justification for disregarding evidence for* A*, i..e, for* doubting *of justifications of* A*;*
(**h**.*iv*) $C \curlyvee D$ *is justified by a scintilla of evidence for* C *or for* D*;* $C \curlywedge D$ *is justified by a scintilla of evidence for both* C *and* D*.*

Remark 2. (*i*) We have four *justification values*, assertive validity ($\curlyvee$) and invalidity (**u**) and hypothetical invalidity ($\curlywedge$) and validity (**j**). We cannot identify assertive and hypothetical validity, nor hypothetical and assertive invalidity; we must think of **u** as an expression $\vdash p$ which is always invalid although p may be sometimes true, and similarly **j** as an expression $\mathcal{H} p$ which is always valid although p may be sometimes false.

(ii) Assertive validity $\curlyvee$ and hypothetical invalidity $\curlywedge$ can be related to $\top$ and **0** of linear logic as they are interpreted categorically as the terminal and the initial object in their respective categories. However there are no obvious reasons for relating **u** with $\perp$ and **j** with **1**.

(*iii*) The meaning of subtraction is a delicate point. The accepted informal interpretation of $A \smallsetminus B$ as "A excludes B" was proposed by I. Urbas [55], in place of "A but not B", as suggested by N. Goodman [31]. [5] Here however "$C \smallsetminus D$" has a hypothetical mood. Suppose we have a method showing incompatibility between any evidence for C and any evidence for D, the hypothetical character of subtraction may come either (a) from the fact that the actual evidence for C may be fairly weak or (b) from the nature of the evidence for incompatibility. In this former case the hypothetical mood for $\mathbf{j} \frown C$ or $\mathbf{j} \frown A$ would depend on the fact that evidence for **j** is weak. No such hypothesis is necessary in the latter case.

4.2 Sequent calculus for Polarized Bi-intuitionism.

The *sequent calculus* **AH-G1** has sequents of one of the forms

[5] We thank the anonymous referee for this reference.

$$\Theta \ ; \ \Rightarrow \ A \ ; \ \Upsilon \quad \text{or} \quad \Theta \ ; \ C \ \Rightarrow \ ; \ \Upsilon$$

where the multiset Θ and A are *assertive* formulas and the multiset Υ and C are hypothetical formulas. We use the abbreviation

$$\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon$$

where *exactly one* of ϵ or ϵ' is non-null. The inference rules of **AH-G1** are in the following Tables 1, 2, 3, 4, 5.

logical axiom: $A \ ; \ \Rightarrow \ A \ ;$	*logical axiom:* $; \ C \ \Rightarrow \ ; \ C$
cut$_1$: $\dfrac{\Theta \ ; \ \Rightarrow \ A \ ; \ \Upsilon \qquad A, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon'}{\Theta, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \Upsilon'}$	*cut*$_2$: $\dfrac{\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, C \qquad \Theta' \ ; \ C \ \Rightarrow \ \Upsilon'}{\Theta, \Theta' \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, \Upsilon'}$

Table 1. Identity Rules

contraction left $\dfrac{A, A, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{A, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$	*contraction right* $\dfrac{\Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, C.C}{\Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, C}$
weakening left $\dfrac{\Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{A, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$	*weakening right* $\dfrac{\Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon, C}$

Table 2. AH-G1 Structural Rules

Remark 3. (i) The core fragment of *intuitionistic logic* consists of the rules for assertive conjunction ($\cap$), implication ($\supset$)and validity ($\curlyvee$), without the *assertive validity axiom:*

$$\varTheta \;;\; \Rightarrow\; \curlyvee \;;\; \varUpsilon$$

$\supset$ *right:*

$$\frac{\varTheta, A_1 \;;\; \Rightarrow\; A_2 \;;\; \varUpsilon}{\varTheta \;;\; \Rightarrow\; A_1 \supset A_2 \;;\; \varUpsilon}$$

$\supset$ *left :*

$$\frac{\varTheta_1;\; \Rightarrow\; A_1 \;;\; \varUpsilon_1 \qquad A_2, \varTheta_2 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon_2}{A_1 \supset A_2, \varTheta_1, \varTheta_2 \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon_1, \varUpsilon_2}$$

$\cap$ *right:*

$$\frac{\varTheta \;;\; \Rightarrow\; A_1 \;;\; \varUpsilon \quad \varTheta \;;\; \Rightarrow\; A_2 \;;\; \varUpsilon}{\varTheta \;;\; \Rightarrow\; A_1 \cap A_2 \;;\; \varUpsilon}$$

$\cap$ *left:*

$$\frac{A_i, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon}{A_0 \cap A_1, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon}$$

for $i = 0, 1$.

assertive absurdity axiom:

$$\mathbf{u}, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon$$

assertive disjunction left

$$\frac{A, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon \qquad B, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon}{A \cup B, \varTheta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; \varUpsilon}$$

assertive disjunction right (two rules)

$$\frac{\varTheta \;;\; \Rightarrow\; A_i \;;\; \varUpsilon}{\varTheta \;;\; \Rightarrow\; A_0 \cup A_1 \;;\; \varUpsilon}$$

for $i = 0, 1$

Table 3. AH-G1 intuitionistic rules

rules for assertive disjunction ($\cup$) and absurdity ($\mathbf{u}$); in this core fragment the symbol "$\mathbf{u}$" in the definition of intuitionistic negation is just a sentential constant without special properties. Dually, the core fragment of *co-intuitionistic logic* has the rules for assertive disjunction ($\curlyvee$), subtraction ($\smallsetminus$) and absurdity ($\curlywedge$), without the rules for hypothetical conjunction ($\curlywedge$) and validity ($\mathbf{j}$), which in the definition of co-intuitionistic negation is just a sentential constant. *We shall consider only the core fragment of intutionistic and co-intuitionistic logic.*

(*ii*) The form of bi-intuitionistic sequents, where only one expression occurs in the focusing area, forces the rules for assertive disjunction and hypothetical conjunction to have *additive form*; thus "$\cup$" has the disjunction property and $\not\vdash A \cup \sim A$; dually, $\curlywedge$ has the *conjunction property* ($C \curlywedge D \vdash$ implies $C \vdash$ or $D \vdash$) and is para-consistent ($C \curlywedge \frown C \not\vdash \curlyvee$).

(*iii*) On the other hand, the rules for assertive conjunction and hypothetical disjunction could be given in the *additive* or in the *multiplicative* form; in presence of the structural rules of the structural rules of weakening and contraction the two formulations are equivalent. For the categorical considerations sketched above, we give *additive* rules for assertive disjunction "$\cap$" and *multiplicative* rules for hypothetical disjunction "$\curlyvee$".

hypothetical absurdity axiom:

$$\Theta \; ; \; \curlywedge \; \Rightarrow \; ; \; \Upsilon$$

$\smallsetminus$ *right:*

$$\frac{\Theta_1 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon_1, C_1 \qquad \Theta_2 \; ; \; C_2 \; \Rightarrow \; ; \; \Upsilon_2}{\Theta_1, \Theta_2 \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon_1, \Upsilon_2, C_1 \smallsetminus C_2}$$

$\smallsetminus$ *left:*

$$\frac{\Theta ; \; C_1 \; \Rightarrow \; ; \; \Upsilon, C_2}{\Theta \; ; \; C_1 \smallsetminus C_2 \; \Rightarrow \; ; \; \Upsilon}$$

$\curlyvee$ *right:*

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C_0, C_1}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C_0 \curlyvee C_1}$$

$\curlyvee$ *left:*

$$\frac{\Theta_1 \; ; \; C_1 \; \Rightarrow \; ; \; \Upsilon_1 \qquad \Theta_2 \; ; \; C_2 \; \Rightarrow \; ; \; \Upsilon_2}{\Theta_1, \Theta_2 \; ; \; C_1 \curlyvee C_2 \; \Rightarrow \; ; \; \Upsilon_1, \Upsilon_2}$$

hypothetical validity axiom:

$$\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \mathbf{j}$$

hypothetical conjunction left (two rules)

$$\frac{\Theta \; ; \; C_i \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; C_0 \curlywedge C_1 \; \Rightarrow \; ; \; \Upsilon}$$

for $i = 0, 1$

hypothetical conjunction right

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C \qquad \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, D}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C \curlywedge D}$$

Table 4. AH-G1 Co-Intuitionistic Rules

$\sim$ *right:*

$$\frac{\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \Rightarrow \sim C \; ; \; \Upsilon}$$

$\sim$ *left:*

$$\frac{\Theta ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C}{\sim C, \Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}$$

$\frown$ *right:*

$$\frac{\Theta, A \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \frown A}$$

$\frown$ *left:*

$$\frac{\Theta ; \; \Rightarrow \; A \; ; \; \Upsilon}{\Theta \; ; \; \frown A \; \Rightarrow \; ; \; \Upsilon}$$

u/**j** *left*	**u**/**j** *right*	$\vdash / \mathcal{H}$ *left*	$\vdash / \mathcal{H}$ *right*
$\mathbf{u} \; ; \; \mathbf{j} \; \Rightarrow \; ;$	$; \; \Rightarrow \; \mathbf{u} \; ; \; \mathbf{j}$	${\vdash}p \; ; \; \mathbf{j} \; \Rightarrow \; ; \; \mathcal{H}p$	${\vdash}p \; ; \; \Rightarrow \; \mathbf{u} \; ; \; \mathcal{H}p$

Table 5. AH-G1 Duality Rules

Proposition 1. *The sequent calculus* **AH-G1** *is sound and complete for the Kripke semantics over preordered frames determined the* **S4** *interpretation in (4) and (6). The rules for cut are admissible in* **AH-G1**.

Let us use the following abbreviations:

$$\boxdot C \; =_{df} \; \sim_u\!\sim C \quad \text{and} \quad \Diamond A \; =_{df} \; {}_j \frown\frown A$$

Proposition 2. *The following sequents are provable in* **AH-G1**.

(i) $\sim\frown A\,;\ \Rightarrow A;$ *and* $A\,;\Rightarrow\ \sim\frown A;$. *Dually,* $;C\Rightarrow\,;\frown\sim C$ *and* $;\frown\sim C\Rightarrow\,;C$.

(ii) $A\ ;\ \Rightarrow \boxdot\dot{\Diamond}A;$ *and* $;\ \dot{\Diamond}\boxdot C\ \Rightarrow\ ;\ C$.

(iii) $M\supset\boxdot C\iff\boxdot\big((_j\frown\dot{\Diamond}M)\curlyvee C\big)$.

Proof. *(ii)* and *(iii)*

$$
\supset\mathrm{R}\,\dfrac{\sim\mathrm{L}\,\dfrac{\smallsetminus\mathrm{R}\,\dfrac{;\Rightarrow\mathbf{u}\,;\,\mathbf{j}\qquad \frown\mathrm{L}\,\dfrac{A\,;\,\Rightarrow\,A\,;}{A\,;\,\frown A\Rightarrow\,;}}{A\,;\,\Rightarrow\,\mathbf{u}\,;\,{}_j\frown\frown A}}{A,\sim_j\frown\frown A\,;\,\Rightarrow\,\mathbf{u}\,;}}{A\,;\,\Rightarrow\,\underbrace{\sim_u\sim\,{}_j\frown\frown A}_{\boxdot\dot{\Diamond}A};}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{;\,C\,\Rightarrow\,;\,C}{\Rightarrow\sim C\,;\,C}\,\sim\mathrm{R}\qquad \mathbf{u}\,;\,\mathbf{j}\,\Rightarrow\,;}{\sim_u\sim C\,;\,\mathbf{j}\,\Rightarrow\,;\,C}\,\supset\mathrm{L}}{;\,\mathbf{j}\,\Rightarrow\,;\,\frown\sim\sim C,C}\,\sim\mathrm{R}}{\underbrace{{}_j\frown\frown\sim_u\sim C}_{\dot{\Diamond}\boxdot C}\,;\,\Rightarrow\,;\,C}\,\smallsetminus\mathrm{L}
$$

$$
\supset\mathrm{R}\,\dfrac{\sim\mathrm{L}\,\dfrac{\curlyvee\mathrm{R}\,\dfrac{\smallsetminus\mathrm{R}\,\dfrac{;\Rightarrow\,\mathbf{u}\,;\,\mathbf{j}\qquad \smallsetminus\mathrm{L}\,\dfrac{\frown\mathrm{R}\,\dfrac{\supset\mathrm{L}\,\dfrac{M;\,\Rightarrow M\,;\qquad \supset\mathrm{L}\,\dfrac{\sim\mathrm{R}\,\dfrac{;\,C\,\Rightarrow\,;C}{;\,\Rightarrow\sim C\,;\,C}\qquad \mathbf{u}\,;\,\mathbf{j}\Rightarrow}{\boxdot C\,;\,\mathbf{j}\,\Rightarrow\,;\,C}}{M,M\supset\boxdot C\,;\,\mathbf{j}\,\Rightarrow\,;\,C}}{M\supset\boxdot C\,;\,\mathbf{j}\,\Rightarrow\,;\,\frown M,C}}{M\supset\boxdot C\,;\,\dot{\Diamond}M\,\Rightarrow\,;\,C}}{M\supset\boxdot C\,;\,\Rightarrow\,\mathbf{u}\,;\,{}_j\frown\dot{\Diamond}M,C}}{M\supset\boxdot C\,;\,\Rightarrow\,\mathbf{u}\,;\,(_j\frown\dot{\Diamond}M)\curlyvee C}}{M\supset\boxdot C,\sim((_j\frown\dot{\Diamond}M)\curlyvee C)\,;\,\Rightarrow\,\mathbf{u}\,;}}{M\supset\boxdot C\,;\,\Rightarrow\boxdot((_j\frown\dot{\Diamond}M)\curlyvee C)\,;}
$$

$$
\supset\mathrm{R}\,\dfrac{\supset\mathrm{R}\,\dfrac{\sim\mathrm{L}\,\dfrac{\supset\mathrm{L}\,\dfrac{\sim\mathrm{R}\,\dfrac{\curlyvee\mathrm{L}\,\dfrac{\smallsetminus\mathrm{L}\,\dfrac{\smallsetminus\mathrm{R}\,\dfrac{;\,\mathbf{j}\,\Rightarrow\,;\,\mathbf{j}\qquad \dfrac{M;\Rightarrow M;}{M;\frown M\Rightarrow\,;}\,\frown\mathrm{L}}{M\,;\,\mathbf{j}\,\Rightarrow\,;\,\dot{\Diamond}M}}{M\,;\,{}_j\frown\dot{\Diamond}M\,\Rightarrow\,;}\qquad ;\,C\,\Rightarrow\,;C}{M\,;\,(_j\frown\dot{\Diamond}M)\curlyvee C\,\Rightarrow\,;\,C}}{M\,;\,\Rightarrow\sim(_j\frown\dot{\Diamond}M\curlyvee C)\,;\,C}\qquad \mathbf{u}\,;\,\Rightarrow\,\mathbf{u};}{\sim_u\sim((_j\frown\dot{\Diamond}M)\curlyvee C),M\,;\,\Rightarrow\mathbf{u}\,;\,C}}{\boxdot((_j\frown\dot{\Diamond}M)\curlyvee C),M,\sim C\,;\,\Rightarrow\mathbf{u};}}{\boxdot((_j\frown\dot{\Diamond}M)\curlyvee C),M\,;\,\Rightarrow\boxdot C;}}{\boxdot((_j\frown\dot{\Diamond}M)\curlyvee C)\,;\,\Rightarrow M\supset\boxdot C;}
$$

The categorical definition of *dialogue chirality* is given in the Appendix, where we also sketch the proof of the following proposition.

Proposition 3. *Polarized bi-intuitionism forms a dialogue chirality.*

5 Linear Bi-Intuitionism and Chu's constuction.

In several occasions J-Y. Girard has indicated that his *linear logic* [27] should not be understood as a logical system on the same level as classical or intuitionistic logic, but rather as a tool to analyse the underlying common ground of those logics from the viewpoints not only of proof theory, but also of category theory, the geometry of interaction, game semantics and ludics[6]. These mathematical theories characterize the "deep structure" underlying proof theoretic analysis of the static structure of proofs and their dynamic behaviour, including criteria of identity of proofs in the process of cut-elimination. There are many possible translations of classical and intuitionistic logic into linear logic that were used in the early 1990s to analyse and regiment the essential non-determinism of classical cut-elimination in Gentzen's sequent calculus **LK** [28, 23]. In these early studies we find the notion of *polarization* of linear logic, later developed in the work by Olivier Laurent [35], and the technique of sequent with *focussing* [3, 29] which are applied here.

Girard's linear logic is classical in the sense that it has an involutory negation $(\ )^{\perp}$ such that $A^{\perp\perp} \equiv A$ for all A; this negation is modelled categorically by the operation $(\ )^{*}$ of $*$-autonomous categories [5]. But *intuitionistic* linear logic has also been extensively studied especially in categorical treatments of linear logic: indeed the intuitionistic system has very natural models in terms of monoidal closed categories. More recently P-A. Melliès [42], starting from abstract forms of game semantics, on one hand, and of tensor logic and braids, on the other, came to the conclusion that linear systems with *non-involutory negation* are structurally "more basic" than classical linear logic: indeed the study of proofs in terms geometry of braids and the mathematical theory of games do provide most general but also very informative characterizations of logical computation.

With respect to our present concern we notice that non-commutative linear logics, in particular Lambek calculi, have significant applications to linguistics; moreover interesting applications of ludics to conversational analysis have proposed that are related to the theory of meaning (see A. Lecomte [37]). But no attempt has been made, as far as we know, to specify in a systematic way the forms of common sense reasoning that are represented in linear logic. The paper [12] rather than providing a *"pragmatic interpretation"* of linear logic, claims that linear logic is suitable for representing schemes where illocutionary force is *unspecified* but can be determined in various ways; an attempt is made there to show that the absence of structural rules of *contraction* (thus of *reuse* of "logical resources") is required to guarantee such property in general.

Here we show that a natural "dialogue interpretation" of *linear* bi-intuitionistic logic, still regarded as a logic of *justifications conditions for and against*

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6 See [30] for recent reflections on the overall contribution of linear logic to contemporary logic.

assertions and hypotheses, has a mathematical counterpart in a variant of *Chu's construction* [5]. Chu provides a way to construct $*$-autonomous categories from monoidal closed categories: here a $*$-autonomous category $\mathcal{A}$, a model of classical linear logic, is built where objects are pairs (C, C^{op}) where C is an object of a monoidal closed category $\mathcal{C}$, and morphisms $(f, g^{op}) : (C, C^{op}) \to (D, D^{op})$, where $f : C \to D$ may be seen as a proof of D from B and $g : D \to C$ is a refutation of C from a refutation of D. Thus Chu's construction has the "dialogic" form of an abstract game semantics. Moreover there is an involutory operation $(C, C^{op})^{\perp} = (C^{op}, C)$, modelling negation of classical linear logic.

Now if we retain the "pragmatic" interpretation of C as an assertion type, then a "proof" of C is *conclusive evidence for* C but a "disproof" of C is just a *scintilla of evidence against* C; conversely, if C is a hypothetical type, then a justification of C is a *scintilla of evidence for* it and a disproof of C is *conclusive evidence* against C. Hence evidence against an assertion amounts to evidence for the dual hypothesis and conversely; but "conclusive evidence" and a "scintilla of evidence" are entities of differt types. It follows that the involutory operation $(X, Y)^{\perp} = (Y, X)$ does not interpret a classical notion of negation but rather the duality between justifications of assertions and of hypotheses, between intuitionism and co-intuitionism. As suggested above we interpret such a duality in terms of the notion of a *chirality*, i.e., of an adjunction $L \dashv R$ between functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$. This notion generalizes the notion of a duality $(\mathcal{C}, \mathcal{C}^{op})$ in a way that allows us to recognize interesting structure within linear bi-intuitionism. It is a fact that our dialogue interpretation resulting from Chu's construction applies to the connectives, in particular, the notion of disjunction must be Girard's *multiplicative linear disjunction* (*par*): thus here we have a reasonable characterization of a form of common sense reasoning represented in linear logic, which is obtained by determining the abstract schemes of linear logic with suitable assignments of illocutionary forces to atoms and of "illocutionary moods" to molecular formulas, in the spirit of the suggestions in the paper [12].

5.1 Language and sequent calculus of linear polarized bi-intuitionism.

Definition 5. *(multiplicative linear bi-intuitionistic language)*

- *assertive linear intuitionistic formulas:*
  $A, B \ := {\vdash} p \mid \mathbf{1} \mid A \multimap B \mid A \otimes B \mid \ \neg\, C$
- *hypothetical linear co-intuitionistic formulas:*
  $C, D \ := \mathcal{H}\, p \mid \bot \mid C - D \mid C \wp D \mid \ \ulcorner\, A$
- *defined negations:*
  $\neg_u\ A \ =_{df}\ A \multimap \mathbf{u} \qquad {}_{j}\!\!\urcorner\ C \ =_{df}\ \mathbf{j} - C$
  where $\mathbf{u}$ *is an assertive constant and* $\mathbf{j}$ *a hypothetical constant.*

The sequent calculus **MLAH-G1** for *multiplicative linear "polarized" bi-intuitionistic logic* has the inference rules in Tables 1, 6, 7 and 8.

1 *left*

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\mathbf{1}, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

assertive validity axiom:

$$;\, \Rightarrow\; \mathbf{1}\,;$$

$\multimap$ *left :*

$$\frac{\Theta_1 ;\, \Rightarrow\; A_1 \,;\, \Upsilon_1 \qquad A_2, \Theta_2 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_2}{A_1 \multimap A_2, \Theta_1, \Theta_2 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_1, \Upsilon_2}$$

$\multimap$ *right:*

$$\frac{\Theta, A_1 \,;\, \Rightarrow\; A_2 \,;\, \Upsilon}{\Theta \,;\, \Rightarrow\; A_1 \multimap A_2 \,;\, \Upsilon}$$

$\otimes$ *left:*

$$\frac{A_0, A_1 \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{A_0 \otimes A_1, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

$\otimes$ *right:*

$$\frac{\Theta_1 \,;\, \Rightarrow\; A_1 \,;\, \Upsilon_1 \qquad \Theta_2 \,;\, \Rightarrow\; A_2 \,;\, \Upsilon_2}{\Theta_1, \Theta_2 \,;\, \Rightarrow\; A_1 \otimes A_2 \,;\, \Upsilon_1, \Upsilon_2}$$

Table 6. MLAH-G1 intuitionistic rules

hypothetical absurdity axiom:

$$;\, \bot \;\Rightarrow\, ;$$

$\bot$ *right*

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \bot}$$

$-$ *left:*

$$\frac{\Theta ;\, C \;\Rightarrow\, ;\, D, \Upsilon}{\Theta \,;\, C - D \;\Rightarrow\, ;\, \Upsilon}$$

$-$ *right:*

$$\frac{\Theta_1 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_1, C \qquad \Theta_2 \,;\, D \;\Rightarrow\, ;\, \Upsilon_2}{\Theta_1, \Theta_2 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon_1, C - D, \Upsilon_2}$$

$\wp$ *left:*

$$\frac{\Theta_0 \,;\, C_0 \;\Rightarrow\, ;\, \Upsilon_0 \qquad \Theta_1 \,;\, C_1 \;\Rightarrow\, ;\, \Upsilon_1}{\Theta_0, \Theta_1 \,;\, C_0 \wp C_1 \Rightarrow\, ;\, \Upsilon_1, \Upsilon_2}$$

$\wp$ *right:*

$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C_0, C_1}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C_0 \wp C_1}$$

Table 7. MLAH-G1 Co-intuitionistic rules

The following proposition is analogue to (but easier than) proposition 3.

Proposition 4. *Multiplicative linear polarized bi-intuitionism forms a dialogue chirality.*

$\urcorner$ *right:*

$$\frac{\Theta \,;\, C \;\Rightarrow\; ;\, \Upsilon}{\Theta \,;\, \Rightarrow \urcorner C \,;\, \Upsilon}$$

$\urcorner$ *left:*

$$\frac{\Theta;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, C}{\urcorner C, \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}$$

$\ulcorner$ *right:*

$$\frac{\Theta, A \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \ulcorner A}$$

$\ulcorner$ *left:*

$$\frac{\Theta;\, \Rightarrow\; A \,;\, \Upsilon}{\Theta \,;\, \ulcorner A \;\Rightarrow\; ;\, \Upsilon}$$

u/j *left*	**u/j** *right*	$\vdash/\mathcal{H}$ *left*	$\vdash/\mathcal{H}$ *right*
$\mathbf{u} \,;\, \mathbf{j} \;\Rightarrow\; ;$	$;\, \Rightarrow\; \mathbf{u} \,;\, \mathbf{j}$	$\vdash p \,;\, \mathbf{j} \;\Rightarrow\; ;\; \mathcal{H} p$	$\vdash p \,;\, \Rightarrow\; \mathbf{u} \,;\; \mathcal{H} p$

Table 8. MLAH-G1 Duality Rules

5.2 Dialogue semantics.

Remember that *evidence for* an elementary assertive expression $\vdash p$ is conclusive evidence that p is true and that *evidence against* $\vdash p$ is a "scintilla of evidence" that p is false. Dually, *evidence for* a hypothetical assertive expression $\mathcal{H} p$ is a "scintilla of evidence" that p is true and that *evidence against* $\vdash p$ is conclusive evidence that p is false. For *linear* co-intuitionistic expressions we have the following interpretation.

Definition 6. (dialogue semantics for linear co-intuitionistic logic)

1. *Evidence for* $\mathcal{H} p$ *is a scintilla of evidence that* p *may be true;*
   - evidence against $\mathcal{H} p$ *is conclusive evidence that* p *is not true.*
2. *Evidence* for a subtraction $C - D$ *is given by evidence* for C *together with evidence* against D*;*
   - evidence against $C - D$ *is a method that transforms evidence* for C *into evidence* for D *and also evidence* against D *into evidence* against C*.*
3. *Evidence* for a disjunction $C \wp D$ *is a method that transforms evidence* against C *into evidence* for D *and also evidence* against D *into evidence* for C*;*
   - evidence against $C \wp D$ *is evidence* against both C *and* D*.*
4. *Evidence* for a conjunction $C \& D$ *is evidence* for C *and also* for D*; evidence* against $C \& D$ *is conclusive evidence* against C *or* against D*.*

Remark 4. (*linear terms*) We shall borrow notation from the theory of linear types (see, e.g., [13]) and use the term calculus informally to denote *evidence for* and *evidence against* hypothetical expressions of **MLAH** and the basic operations to combine and transform evidence.

- We write $c^+ : H^+$ and $c^- : H^-$ to denote evidence c^+ *for* H and evidence c^- *against* H. Thus a hypothetical expression H of **MLAH**, regarded as a type of jusitification values, is in fact a pair (H^+, H^-).

- The co-intuitionistic linear consequence relation $E \vdash D_1, \ldots, D_n$ is *reflexive* and *transitive*; it is formally represented by a sequent $E \to D_1, \ldots, D_n$ with logical axioms ant the rule of *cut*. Writing $D = D_1 \wp \ldots \wp D_n$, such a sequent is interpreted by a pair of functions $\langle f_1, f_2 \rangle$ between types of justification methods, where $f_1 : C^+ \to D^+$ sends evidence *for* C to evidence *for* D, and $f_2 : D^- \to C^-$ sends evidence *against* D to evidence *against* C.
- if $c^+ : C^+$ and $d^- : D^-$, then $c^+ \otimes d^- : (C - D)^+$;
- if $g^+ : C^+ \to D^+$ and $g^- : D^- \to C^-$, then $\langle g^+, g^- \rangle : (C - D)^-$.
- If $h : C^- \to D^+$ and $h' : D^- \to C^+$ then $\langle h, h' \rangle : (C \wp D)^+$;
- if $c^- : C^-$ and $d^- : D^-$, then $c^- \otimes d^- : (C \wp D)^-$.
- if $c^+ : C^+$ and $d^+ : D^+$, then $\langle c^+, c^+ \rangle : (C \& D)^+$;
- if $c^- : C^-$, then $\mathtt{inl}(c) : (C \& D)^-$; if $d^- : D^-$ then $\mathtt{inr}(d) : (C \& D)^-$.

With this definition we can prove the following proposition.

Proposition 5. *Let C, D and E be hypothetical expressions. Then the basic adjunction*

$$\frac{C \vdash D \wp E}{\overline{C - D \vdash E}} \tag{8}$$

is valid in the dialogue semantics.

Proof. (1) Let f_1, f_2 be methods, where $f_1 : C^+ \to (D \wp E)^+$ and $f_2 : (D \wp E)^- \to C^-$. We define methods $g_1 : (C - D)^+ \to E^+$ and $g_2 : E^- \to (C - D)^-$. We know that if $c^+ : C^+$ then $f_1(c^+) : (D \wp E)^+$ is a pair of maps $\langle k, k' \rangle$ with $k : D^- \to E^+$ and $k' : E^- \to D^+$. Moreover we have $f_2(d^- \otimes e^-) : C^-$ if $d^- : D^-$ and $e^- : E^-$.

(i) Now evidence *for* $C - D$ is a pair $\langle c, d \rangle$ where $c^+ : C^+$ and $d^- : D^-$. Hence we may define $g_1(c^+ \otimes d^-) = (f_1(c^+))(d^-)$ where $f_1(c^+) = k : D^- \to E^+$ as above and $k(d^-) : E^+$ is evidence for E as required.

(ii) Moreover let $e^- : E^-$. We need to define $g_2(e^-) : (C - D)^-$ as a pair of maps $\langle m, m' \rangle$ where $m : C^+ \to D^+$ and $m' : D^- \to C^-$. But if $c^+ : C^+$ then $f_1(c^+)$ is a map $k : E^- \to D^+$ as above, so we let $g_2(e^-)(c^+) = f_1(c^+)(e^-) : D^+$. Also if $d^- : D^-$ then $f_2(d^- \otimes e^-) : C^-$, so we define $g_2(e^-)(d^-) = f_2(\langle , e \rangle) : C^-$ as required.

(2) Now given methods $g_1 : (C - D)^+ \to E^+$ and $g_2 : E^- \to (C - D)^-$ we define $f_1 : C^+ \to (D \wp E)^+$ and $f_2 : (D \wp E)^- \to C^-$.

(i) Given $c^+ : C^+$ and $d^- : D^-$ we may let $(f_1(c^+))(d^-) = g_1(c^+ \otimes d^-) : E^+$ and $(f_1(c^+))(e^-) = (g_2(e^-))(c^+) : D^+$, since $g_2(e^-)$ maps C^+ to D^+.

(ii) Finally, given $d^- : D^-$ and $e^- : E^-$ we let $f_2(d^- \otimes e^-) = (g_2(e^-))(d^-) : C^-$, since $g_2(e^-)$ maps D^- to C^-. **qed**

Remark 5. Suppose in clause (3) of definition 6 we replace the *multiplicative* notion of *par* $C \wp D$ with an *additive* notion of disjunction $C \oplus D$; here we let *evidence for* $C \oplus D$ be either a pair $(c, 0)$ where c is is a scintilla of evidence *for* C, or a pair $(d, 1)$ where d is a scintilla of evidence *for* D and also we define *evidence against* $C \oplus D$ as a pair $\langle c^-, d^- \rangle$ where c^- and d^- are conclusive evidence against C, D respectively. We claim that with an additive interpretation the above lemma cannot be proved, at least not in an intuitionistic metatheory.

The argument goes through in case $(1)(i)$ since now $f_1(c) = (d, 0)$ or $(e, 1)$ with $d : D^+$ or $e : E^+$; but we have $d : D^-$ and d is *conclusive evidence against* D; therefore $f_1(c)$ can only be $(e, 1)$ as required. Also in case $(1)(ii)$ given $e : E^-$ and $c : C^+$, we have $f_1(c) : (D \oplus E)^+$, but since e is *conclusive evidence against* E, we have $f_1(c) = (d, 0)$ for some $d : D^+$. The argument goes through also in case $(2)(ii)$ without changes.

But consider case $(2)(i)$: given $c : C^+$, if there is some $d : D^+$ then we can set $f_1(c) = (d, 0)$ and if $d : D^-$ then we take $f_1(c) = (e, 1)$ where $e = g_1(\langle c, d \rangle)$ but it may still be possible that there is no evidence whatsoever for or against D. However, if we take positive or negative evidence for D to be represented in *classical* **S4** as $w \Vdash \Diamond D^M$ or $w \Vdash \Box\neg D^M$ in some possible world w belonging to a Kripke model $\mathcal{M} = (W, \leq, \Vdash)$ then the absence of any evidence for D in w entails the existence of negative evidence for D.

The above remark shows that we need to take hypothetical disjunction as the multiplicative *par* in order to have our dialogue semantics for co-intuitionism: in fact we have a dialogue semantics only for the linear fragment of co-intuitionism.

The "dialogue interpretation" of intuitionistic linear expressions, is the exact dual of definition 6.

Definition 7. (dialogue semantics for linear intuitionistic logic)

1. *Evidence for* $\vdash p$ *is conclusive evidence that* p *is true;*
   - evidence against $\mathcal{H}\, p$ *is a scintilla of evidence that* p *may not be*
2. *Evidence* for $A \multimap B$ *is a method that transforms evidence* for A *into evidence* for B *and also evidence* against B *into evidence* against A.
   - *evidence* against an implication $A \multimap B$ *is given by evidence* for A *together with evidence* against B*;*
3. *Evidence* for *a conjunction* $A \otimes B$ *is evidence* for A *and* for B.
   - *evidence* against a conjunction $A \otimes B$ *is a method that transforms evidence* for A *into evidence* against B *and also evidence* for B *into evidence* against A*;*
4. *Evidence* for a disjunction $A \oplus B$ *is evidence* for A *or* for B*;*
   evidence against $A \oplus B$ *is evidence* against S *and* against B.

The proof of the following proposition is completely analogous to that of the proposition 8.

Proposition 6. *Let A, B and C be assertive expressions. Then the basic adjunction*

$$\frac{\frac{A \otimes B \vdash C}{}}{A \vdash B \multimap C} \tag{9}$$

is valid in the dialogue semantics.

5.3 Dialogue semantics and Chu's construction.

It remains to be shown how the dialogue interpretation can be mathematically formalized by a modification of Chu's construction in [8], which is briefly recalled in Appendix, Section 8. Within linear polarized bi-intuitionistic logic **MLAH**, we consider the *assertive* fragment **MLA** and *hypothetical* fragment **MLH** which we map to *intuitionistic linear logic with products* (**IMALL**) given in definition 14 and Table 11, Section 8 of the Appendix. Let $\mathcal{A}$ be a categorical model of **MLA**, namely, a symmetric monoidal closed category with bifunctor $\otimes$ and its right adjoint $\multimap$, free on a set of objects modelling elementary formulas. Similarly let $\mathcal{H}$ be a categorical model of **MLH**, namely, a free symmetric monoidal category with bifunctor $\wp$ and its left adjoint $-$.

It is evident from Definitions (7), (6) and from the remark (4) that the dialogue semantics of both linear intuitionistic and linear co-intuitionistic logic can be expressed with terms typed within multiplicative linear intuitionistic logic with products (**IMALL**), namely, with the *tensor* type ($\otimes$), *linear implication* ($\multimap$) and *product* type (&). In fact the dialogue semantics of **MLA** and **MLH** can be modelled by a pair of functors $F : \mathcal{A} \to \mathcal{A}_d$ and $G : \mathcal{H} \to \mathcal{H}_d$ where the categories $\mathcal{A}_d$ and $\mathcal{H}_d$ are built on pairs $(\mathcal{C}, \mathcal{C})^{op}$ and $(\mathcal{D}, \mathcal{D})^{op}$, respectively, where $\mathcal{C}$ and $\mathcal{D}$ free symmetric monoidal closed categories with products; in a similar way a model of classical multiplicative linear logic **CMLL** is given by a functor $F : \mathcal{A} \to (\mathcal{C}, \mathcal{C}^{op})$, where $\mathcal{A}$ is a free *-autonomous category and $\mathcal{C}$ is a symmetric monoidal closed category with products.

We define a translation $A \mapsto (A^+, A^-)$ of formulas A of *multiplicative linear assertive* logic (**MLA**) into pairs of formulas (A^+, A^-) of **IMALL**, where A^+ is regarded as a type of expressions representing *conclusive evidence for* A and A^- the type of expressions representing *a scintilla of evidence against* A. The translation of formulas is given in Table 9, Part A.

Similarly, the translation $C \mapsto (C^+, C^-)$ of formulas A of *multiplicative linear hypothetical* logic (**MLA**) maps C to pairs of formulas (C^+, C^-) of **IMALL**, where C^+ is the type of expressions representing *a scintilla of evidence for* C and A^- the type of expressions representing *conclusive evidence against* C. The translation of formulas is given in Table 9, Part B.

Remark 6. Special attention is needed to the interpretation of the constant **1** of **MLA** and $\perp$ of **MLH**. Clearly, in (10) we can set $\mathbf{1}^+ = \mathbf{1}$ and $\perp^- = \mathbf{1}$:

$$\mathbf{1} \mapsto \quad (\mathbf{1}^+, \mathbf{1}^-) \quad \text{and} \quad \perp \mapsto (\perp^+, \perp^-) \tag{10}$$

Part A: multiplicative linear assertive **MLA**

$$
\begin{aligned}
{\vdash}p &\mapsto ({\vdash}p^{+}, {\vdash}p^{-});\\
A \otimes B &\mapsto (A^{+} \otimes B^{+}, [A^{+} \multimap B^{-}]\&[B^{+} \multimap A^{-}]);\\
A \multimap B &\mapsto ([A^{+} \multimap B^{+}]\&[B^{-} \multimap A^{-}], A^{+} \otimes B^{-}).
\end{aligned}
$$

Part B: multiplicative linear hypothetical **MLH**

$$
\begin{aligned}
\mathcal{H}p &\mapsto (\mathcal{H}p^{+}, \mathcal{H}p^{-});\\
C \wp D &\mapsto ([C^{-} \multimap D^{+}]\&[D^{-} \multimap C^{+}], C^{-} \otimes D^{-})\\
C - D &\mapsto (C^{+} \otimes D^{-}, [C^{+} \multimap D^{+}]\&[D^{-} \multimap C^{-}]).
\end{aligned}
$$

Table 9. Dialogue interpretation, formally.

But what are $\mathbf{1}^{-}$ and $\bot^{+}$? Although perhaps unintuitive from the viewpoint of our pragmatic interpretation, the natural solution is to let $\mathbf{1}^{-} = \top = \bot^{+}$, where $\top$ is the additive identity for &.

We shall not consider here the axioms $\vdash / \mathcal{H}$ left *and* right *and the role of the special constants* $\mathbf{u}$ *and* $\mathbf{j}$ *in them.*

Example: The following derivations in the sequent calculus of Table 11 are part of the proofs of $A^{-} \equiv (\mathbf{1} \otimes A)^{-}$ and $C^{+} \equiv (\bot \wp C)^{+}$.

$$
\dfrac{\dfrac{\dfrac{A^{-} \vdash A^{-}}{A^{-}, \mathbf{1}^{+} \vdash A^{-}}\ \mathbf{1}\ \mathrm{L}}{A^{-} \vdash \mathbf{1}^{+} \multimap A^{-}}\ {\multimap}\ \mathrm{R} \qquad \dfrac{\overset{\top\ \textit{axiom:}}{A^{-}, A^{+} \vdash \mathbf{1}^{-}}}{A^{-} \vdash A^{+} \multimap \mathbf{1}^{-}}\ {\multimap}\ \mathrm{R}}{A^{-} \vdash [\mathbf{1}^{+} \multimap A^{-}]\&[A^{+} \multimap \mathbf{1}^{-}]}\ \&\ \mathrm{R}
$$

$$
\dfrac{\dfrac{\dfrac{C^{+} \vdash C^{+}}{C^{+}, \bot^{-} \vdash C^{+}}\ \mathbf{1}\ \mathrm{L}}{C^{+} \vdash \bot^{-} \multimap C^{+}}\ {\multimap}\ \mathrm{R} \qquad \dfrac{\overset{\top\ \textit{axiom:}}{C^{-}, C^{+} \vdash \bot^{+}}}{C^{+} \vdash C^{-} \multimap \bot^{+}}\ {\multimap}\ \mathrm{R}}{C^{+} \vdash [\bot^{-} \multimap C^{+}]\&[C^{-} \multimap \bot^{+}]}\ \&\ \mathrm{R}
$$

Definition 8. *We assume that the set* **Prop** $= \{p_0, p_1, p_2 \ldots\}$ *of propositional letters is given together with an* involution without fixed point $(\ )^{\perp} : \mathbf{Prop} \rightarrow \mathbf{Prop}$ (so that $p^{\perp} \neq p$ and $p^{\perp\perp} = p$). *The duality* $(\ )^{\perp} : \mathbf{MLA} \rightarrow \mathbf{MLH}$ *and* $(\ )^{\perp} : \mathbf{MLH} \rightarrow \mathbf{MLA}$ *is defined as follows*[7].

[7] We agree with P. Schroeder-Heister's remark that $A \setminus B$ is dual to $B \rightarrow A$, not to $A \rightarrow B$, cfr.[9, 7], and thus $A - B$ is dual to $B \multimap A$.

$$\begin{array}{cc}
(\vdash p)^{\perp} = \mathcal{H}\, p^{\perp} & (\mathcal{H}\, p)^{\perp} = \vdash p^{\perp}. \\
(\vdash p^{\perp})^{\perp} = \mathcal{H}\, p & (\mathcal{H}\, p^{\perp})^{\perp} = \vdash p. \\
(\mathbf{1})^{\perp} = \perp & (\perp)^{\perp} = \mathbf{1}. \\
(\mathbf{u})^{\perp} = \mathbf{j} & (\mathbf{j})^{\perp} = \mathbf{u}. \\
(A \otimes B)^{\perp} = A^{\perp} \wp B^{\perp} & (C \wp D)^{\perp} = C^{\perp} \otimes D^{\perp}. \\
(A \multimap B)^{\perp} = B^{\perp} - A^{\perp} & (C - D)^{\perp} = D^{\perp} \multimap C^{\perp}.
\end{array} \tag{11}$$

The duality is extended to sequent derivations, sending a derivation d in the purely assertive *part* **MLA-G1** *of* **MLAH-G1** *to a $d^{\perp}$ in the* purely hypothetical *parts* **MLH-G1***, and conversely. The map $(\ )^{\perp}$ acts on sequents and inference rules in the obvious way:*

MLA-G1	$\mapsto$	**MLH-G1**	**(and conversely):**
$\Gamma; \Rightarrow A;$	$\mapsto$	$; A^{\perp} \Rightarrow ; \Gamma^{\perp}$	
logical axiom	$\mapsto$	*logical axiom*	
$\mathbf{1}$*-axiom*	$\mapsto$	$\perp$*-axiom*	
cut	$\mapsto$	*cut*	
$\otimes$*-right,* $\otimes$*-left*	$\mapsto$	$\wp$*-left,* $\wp$*-right, resp.*	
$\multimap$*-right,* $\multimap$*-left*	$\mapsto$	$-$*-left,* $-$*-right, resp.*	

Proposition 7. *Let's make the following assumptions:*

$$\begin{array}{ll}
(\vdash p)^{+} = (\mathcal{H}\, p^{\perp})^{-}, & (\vdash p)^{-} = (\mathcal{H}\, p)^{\perp -}, \\
(\mathcal{H}\, p)^{+} = (\vdash p^{\perp})^{-}, & (\mathcal{H}\, p^{\perp})^{-} = (\vdash p^{\perp})^{-}, \\
\mathbf{u}^{+} = \mathbf{j}^{-}, & \mathbf{u}^{-} = \mathbf{j}^{+}.
\end{array} \tag{12}$$

Then for any formulas A of **MLA** *and C of* **MLH***, $A^{+} = A^{\perp -}$, $A^{-} = A^{\perp +}$ and $C^{+} = C^{\perp -}$, $C^{-} = C^{\perp +}$.*

Proof. The proof is by induction on the complexity of the formulas.

- $(C \wp D)^{+} = [C^{-} \multimap D^{+}] \& [D^{-} \multimap C^{+}] = [C^{\perp +} \multimap D^{\perp -}] \& [D^{\perp +} \multimap C^{\perp -}] = (C^{\perp} \otimes D^{\perp})^{-}$, using the inductive hypothesis;
- $(C \wp D)^{-} = C^{-} \otimes D^{-} = C^{\perp +} \otimes D^{\perp +} = (C^{\perp} \otimes D^{\perp})^{+}$
- $(C - D)^{+} = C^{+} \otimes D^{-} = D^{\perp +} \otimes C^{\perp -} = (D^{\perp} \multimap C^{\perp})^{-}$ using commutativity of $\otimes$ and the inductive hypothesis;
- $(C - D)^{-} = [C^{+} \multimap D^{+}] \& [D^{-} \multimap C^{-}] = [D^{\perp +} \multimap C^{\perp +}] \& [C^{\perp -} \multimap D^{\perp -}] = (D^{\perp} \multimap C^{\perp})^{+}$, using commutativity of & and the inductive hypothesis.
- We treat the other cases dually.

Remark 7. (i) One could may set $p^{\perp} = \neg p$, where "$\neg$" is classical negation, in the spirit of Dalla Pozza and Garola's compatibilism [21]. Then there is an implicit use of the double negation law $\neg\neg p = p$ for p atomic in the clauses $(\mathcal{H}\, p^{\perp})^{\perp} = \vdash p$ and $(\vdash p^{\perp})^{\perp} = \mathcal{H}\, p$; thus in an intuitionistic metatheory one may need to assume that the propositions in **Prop** are decidable.

(*ii*) In the reading of $p^{\perp}$ as $\neg p$, the equations (12) are very plausible. If p is decidable, conclusive evidence for *asserting* p is conclusive evidence against *making the hypothesis* $\neg p$ and a scintilla of evidence for *making the hypothesis* p is evidence against *asserting* $\neg p$.
(*iii*) In the framework of a *game-theoretic semantics*, the "positive / negative polarities" in equations 12 and in Proposition 7 may be read in terms of *player's moves / opponent's moves.* We cannot expand this remark here.

Table 9 only gives the *object component* of the functors $F : \mathcal{A} \to \mathcal{A}_d$ and $G : \mathcal{H} \to \mathcal{H}_d$. and we have not defined their action on morphisms. Here morphisms $f : A \to B$ in $\mathcal{A}$ are representable by [equivalence classes of] sequent derivations of $A; \Rightarrow B;$ in the purely assertive **MLA-G1**, which satisfy the appropriate $\beta - \eta$ equations as cut-elimination and η-expansion hold for **MLA-G1**. Similarly, morphisms $g : C \to D$ in $\mathcal{H}$ are represented by [equivalence classes of] sequent derivations $; C \to; D$ in the purely hypothetical part **MLH-G1**. Thus morphisms $F(f) : F(A) \to F(B)$ are represented by *pairs* [of equivalence classes] $(f^+ : A^+ \vdash B^+, f^- : B^- \vdash A^-)$ of **IMALL** derivations, and similarly for $F(g) : F(C) \to F(D)$. Since **IMALL** enjoys the cut-elimination property (see, e.g., Bierman's thesis [15]) if f is cut-free, we may choose the representatives f^+ and f^- to be cut-free as well.

Proposition 8. *The translations of* **MLA** *and* **MLH** *formulas into* **IMALL** *formulas given in Table 9 can be extended to proofs, namely:*

1. *Let f be a derivation of $\Gamma; \Rightarrow A;$ in the purely assertive part of* **MLAH-G1**. *By induction on f we construct* **IMALL** *derivations $f^+ : \otimes(\Gamma)^+ \vdash A^+$ and $f^- : A^- \vdash \otimes(\Gamma)^-$; if f is cut-free, then we may take f^+ and f^- to be cut-free.*
2. *Let g be a derivation of $; C \Rightarrow; \Delta$ in the purely hypothetical part of* **MLAH-G1**. *By induction on g we construct* **IMALL** *derivations $g^+ : C^+ \vdash \wp(\Gamma)^+$ and $g^- : \wp(\Gamma)^- \vdash C^-$; if g is cut-free, then we may take g^+ and g^- to be cut-free.*
3. *Let $(\Gamma; \Rightarrow A;\ )^{\perp} =; C \Rightarrow; \Delta$. Under the assumptions of equations 12 we may identify $f^+ = g^-$ and $f^- = g^+$.*

The proof of proposition 8 is inspired by Table 13 in Section 8, but inspection of all cases is quite long. Here consider only the case of linear implication. Let the proof f end with $\multimap$-*right*:

$$\dfrac{\begin{array}{c} f \\ \Gamma, A; \Rightarrow B; \end{array}}{\Gamma; \Rightarrow A \multimap B;}$$

By inductive hypothesis we may assume we have **IMALL** derivations $f_0^+ : \Gamma^+, A^+ \vdash B^+$ and also $f_1^- : B^-, \Gamma^+ \vdash A^-$, $f_2^- : B^-, A^+ \vdash \otimes(\Gamma)^-$ such that $f^- : B^- \vdash ((\otimes\Gamma) \otimes A)^-$ is obtained from f_1^- and f_2^-. The derivations f^+ and f^- are obtained as follows:

$$
\dfrac{\dfrac{\begin{array}{c} f_0^+ \\ \Gamma^+, A^+ \vdash B^+ \end{array}}{\Gamma^+ \vdash A^+ \multimap B^+} \qquad \dfrac{\begin{array}{c} f_1^- \\ \Gamma^+, B^- \vdash A^- \end{array}}{\Gamma^+ \vdash B^- \multimap A^-}}{\Gamma^+ \vdash [A^+ \multimap B^+]\&[B^- \multimap A^-]} \qquad \dfrac{\begin{array}{c} f_2^- \\ A^+, B^- \vdash (\otimes\Gamma)^- \end{array}}{(A^+ \otimes B^-) \vdash (\otimes\Gamma)^-}
$$

Now let the proof d end with $\multimap$-*left*:

$$
\dfrac{\begin{array}{c} f \\ \Gamma; \Rightarrow A; \end{array} \qquad \begin{array}{c} g \\ B, \Delta; \Rightarrow E; \end{array}}{\Gamma, A \multimap B, \Delta; \Rightarrow E;}
$$

By inductive hypothesis we may assume we have **IMALL** derivations $f^+ : \Gamma^+ \vdash A^+$ and $f^- : A^- \vdash (\otimes\Gamma)^-$ and also $g^+ : B^+, \Delta^+ \vdash E^+$, $g_0^- : E^-, B^+ \vdash (\otimes\Delta)^-$ and $g_1^- : E^-, \Delta^+ \vdash B^-$ such that g^- is obtained from g_0^- and g_1^- by applications of $\multimap$-R and of &-R. The derivation d^+ is obtained as follows.

$$
\dfrac{\dfrac{\begin{array}{c} f^+ \\ \Gamma^+ \vdash A^+ \end{array} \qquad \begin{array}{c} g^+ \\ B^+, \Delta^+ \vdash E^+ \end{array}}{\Gamma^+, A^+ \multimap B^+ \Delta^+ \vdash E^+}}{\Gamma^+, (A \multimap B)^+ \Delta^+ \vdash E^+}
$$

Finally, d^- results from the following pair of **IMALL** derivations.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} f^+ \\ \Gamma^+ \vdash A^+ \end{array}}{\otimes(\Gamma)^+ \vdash A^+} \qquad \dfrac{\begin{array}{c} g_1^- \\ E^-, \Delta^+ \vdash B^- \end{array}}{E^-, \otimes(\Delta)^+ \vdash B^-}}{E^-, \otimes(\Gamma)^+, \otimes(\Delta)^+ \vdash A^+ \otimes B^-}}{E^-, (\otimes(\Gamma) \otimes (\otimes\Delta))^+ \vdash (A \multimap B)^-}}{E^- \vdash (\otimes(\Gamma) \otimes (\otimes\Delta))^+ \multimap (A \multimap B)^-}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} f^+ \\ \Gamma^+ \vdash A^+ \end{array} \qquad \begin{array}{c} g_0^- \\ E^-, B^+ \vdash \otimes(\Delta)^- \end{array}}{E^-, A^+ \multimap B^+, \Gamma^+ \vdash \otimes(\Delta)^-}}{E^-, (A \multimap B)^+, \otimes(\Gamma)^+ \vdash \otimes(\Delta)^-}}{E^-, (A \multimap B)^+ \vdash \otimes(\Gamma)^+ \multimap \otimes(\Delta)^-} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c} g_1^- \\ E^-, \Delta^+ \vdash B^- \end{array} \qquad \begin{array}{c} f^- \\ A^- \vdash \otimes(\Gamma)^- \end{array}}{E^-, B^- \multimap A^-, \Delta^+ \vdash \otimes(\Gamma)^-}}{E^-, (A \multimap B)^+, \otimes(\Delta)^+ \vdash \otimes(\Gamma)^-}}{E^-, (A \multimap B)^+ \vdash \otimes(\Delta)^+ \multimap \otimes(\Gamma)^-}}{E^-, (A \multimap B)^+ \vdash (\otimes(\Gamma) \otimes (\otimes\Delta))^-}}{E^- \vdash (A \multimap B)^+ \multimap (\otimes(\Gamma) \otimes (\otimes\Delta))^-}
$$

Remark 8. In Chu's construction (Section 8) we have the following involutory operation:

$$(X_{\mathbf{O}}, X_{\mathbf{I}})^\perp = (X_{\mathbf{I}}, X_{\mathbf{O}})$$

This is used to define the orthogonality operation of a $*$-autonomous category on the pair $(\mathcal{C}, \mathcal{C}^{op})$ of symmetric monoidal closed categories; in this way we model the involutory negation of classical linear logic from models of intuitionistic linear logic. Assuming equation 12 and using Proposition 7, we define an operation $(\ ,\ )^{\perp}$ that may be used to model the connectives "$\neg$" and "$\ulcorner$" of **MLAH**:

$$\begin{array}{l} (C^{+}, C^{-})^{\perp} \mapsto (C^{-}, C^{+}) = (C^{\perp +}, C^{\perp -}) \\ (A^{+}, A^{-})^{\perp} \mapsto (A^{-}, A^{+}) = (A^{\perp +}, A^{\perp -}). \end{array} \tag{13}$$

In our setting the operations defined in (13) model the duality between the intuitionistic side (**MLA**) and the co-intuitionistic side (**MLH**) within multiplicative linear bi-intuitionistic logic **MLAH**, *but the two sides remain separable*. A proof d of the sequent $\Gamma;\ \Rightarrow A\ ;\ \Delta$ in **MLAH-G1** may be transformed into a proof d' of $\neg\ \Delta, \Gamma; \rightarrow\ A$; in the intuitionistic side (**MLA**), modelled by a morphism in $\mathcal{A}$. But d can also be transformed into a proof d'' of $; A^{\perp} \Rightarrow\ ; \Delta, \ulcorner\ \Gamma$ in the co-intuitionistic side (**MLH**), a morphism in $\mathcal{H}$. See also the example in Section 8.2.

6 Conclusions.

The mathematical test case of this paper is *bi-intuitionistic logic* in the light of recent mathematical results, in particular by T. Crolard [19] and by P. A. Melliès [42, 43]. In C. Rauszer's original approach [48, 49] this logic is not a conservative extension of first order intuitionistic logic; moreover all categorical models of propositional bi-intuitionistic logic are isomorphic to a partial order by Crolard's theorem. We propose a *polarized* version of bi-intuitionism (see [9, 7, 10, 11]) where the dual intuitionistic and the co-intuitionistic sides of bi-intuitionism are separated and related by two negations that express their duality within the system. P. A. Melliès proposed the notion of *chirality* as an adjunction $L \dashv R$ between monoidal functors $L : \mathcal{A} \rightarrow \mathcal{B}$ and $R : \mathcal{B} \rightarrow \mathcal{A}$, where $\mathcal{A} = (\mathcal{A}, \wedge, \mathbf{true})$ and $\mathcal{B} = (\mathcal{B}, \vee, \mathbf{false})$, together with a monoidal functor $(\_)^{*} : \mathcal{A} \rightarrow \mathcal{B}^{op}$ that allows to give a "De Morgan representation of implication" in $\mathcal{A}$ through disjunction of $\mathcal{B}$. The notion of *chirality* "relaxes" the notion of a duality $(\mathcal{A}, \mathcal{A}^{op})$ and appears as the right mathematical representation of the mirror symmetry between the intuitionistic and co-intuitionistc sides of polarized bi-intuitionism.

In this paper we have considered Dalla Pozza and Garola's pragmatic interpretation of intuitionistic logic [21], and its extension to co-intuitionism and bi-intuitionism, as a framework for logical analysis from the viewpoint of an intuitionistic philosophy. This means that we made sure that such investigations can be performed within an intuitionistic meta-theory and thus, for instance, that any reference to Kripke semantics for classical **S4** is not taken as the foundation of the concepts to be investigated. Dalla Pozza and Garola interpreted intuitionistic connectives according to the Brouwer-Heyting-Kolmogorov, but retained a strict interpretation of Frege's notion of a proposition as an entity capable of being true or false in the classical sense. Thus non-elementary sentences of intuitionistic logic cannot be propositions: in fact only through the

S4 translation they can be given a truth-functional interpretation in Kripke models. We claimed that some conceptual refinements suffice to make the "pragmatic interpretation" a *bona fide* representation of intuitionism: namely, we regard sentences as types of their justification values; we make sure that when an atomic proposition p is asserted ($\vdash p$) or the content of a hypothesis ($\mathcal{H}\, p$), then p and its negation $\neg p$ are intuitionistically meaningful and, finally, that if the law of double negation $\neg\neg p = p$ is applied to such a proposition, then p is decidable. For co-intuitionism we sketch a *meaning-as-use* interpretation that appears as able to fulfill the requirements of Dummett and Prawitz's justificationist approach. We extend the Brouwer-Heyting-Kolmogorov interpretation of intuitionism by regarding co-intuitionistic formulas as types of the evidence for them.

We have given an "intended interpretation" of co-intuitionistic logic as a logic of hypotheses and of their justifications: evidence against a hypothesis is taken as as conclusive evidence for its dual assertion, but evidence for a hypothesis is a "scintilla of evidence", a notion coming from the analysis of legal discourse. Next, assuming a notion of duality between *assertions* and *hypotheses*, we give a "dialogic interpretation" of *multiplicative linear polarized bi-intuitionistic logic* which can be regarded as a translation into *intuitionistic multiplicative linear logic with products*. Mathematically, the interplay between *evidence for* and *evidence against* assertions and hypotheses is inspired by Chu's construction [8], usually regarded as an abstract form of the "game semantics" for linear logic.

References

1. S. Abramsky, R. Jagadeesan and P. Malacaria. Full abstraction for **PCF** (extended abstract), in *Theoretical Aspects of Computer Software: TACS'94, Sendai, Japan* pp. 1-15, Springer Lecture Notes in Computer Science 789, 1994.
2. S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic, *Journal of Symbolic Logic* **59**, 1994, pp. 543-574.
3. J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3), pp.297-347, 1992.
4. I. Angelelli. The Techniques of Disputation in the History of Logic. *The Journal of Philosophy* 67:20, pp. 800-815, 1970.
5. M. Barr. $*$-autonomous categories, *Lecture Notes in Mathematics* 752, Springer Verlag, 1979.
6. M. Barr. $*$-autonomous categories and linear logic, *Mathematical Structures in Computer Science* 1, pp. 159-178, 1991.
7. G. Bellin. Assertions, hypotheses, conjectures: Rough-sets semantics and proof-theory, to appear in *Proceedings of the Natural Deduction Conference Rio 2001, Dag Prawitz Festschrift* (revised 2010).
8. G. Bellin. Chu's Construction: A Proof-theoretic Approach. in Ruy J.G.B. de Queiroz editor, *Logic for Concurrency and Synchronisation*, Kluwer Trends in Logic n.18, 2003, pp. 93-114.

9. G. Bellin and C. Biasi. Towards a logic for pragmatics. Assertions and conjectures. In: *Journal of Logic and Computation*, Volume **14**, Number 4, 2004, pp. 473-506.
10. G. Bellin and A. Menti. On the π-calculus and co-intuitionistic logic. Notes on logic for concurrency and λP systems. To appear, *Fundamenta Informaticae* 203.
11. G. Bellin. Categorical Proof Theory of Co-Intuitionistic Linear Logic, submitted to *Lomecs* 2012.
12. G. Bellin and C. Dalla Pozza. A pragmatic interpretation of substructural logics. In *Reflection on the Foundations of Mathematics (Stanford, CA, 1998)*, Essays in honor of Solomon Feferman, W. Sieg, R. Sommer and C. Talcott eds., Association for Symbolic Logic, Urbana, IL, Lecture Notes in Logic, Volume 15, 2002, pp. 139-163.
13. P. N. Benton, G. M. Bierman, J. M. E. Hyland and V. C. V. dePaiva. A term calculus for Intuitionistic Linear Logic. In: *Typed Lambda Calculi and Applications*, Lecture Notes in Computer Science, Volume 664, 1993, pp.75-90.
14. P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models, *Computer Science Logic*, Lecture Notes in Computer Science, 1995, Volume 933, 1995, pp.121-135.
15. G. M. Bierman. *On Intuitionistic Linear Logic*, PhD Thesis, Wolfson College, University of Cambridge, UK, 1994.
16. A. Blass. Questions and Answers – A Category Arising in Linear Logic, Complexity Theory, and Set Theory, In *Advances in Linear Logic* (ed. J.-Y. Girard, Y. Lafont, and L. Regnier) London Math. Soc. Lecture Notes 222, 1995, pp. 61-81.
17. C. Biasi and F. Aschieri. A Term Assignment for Polarized Bi-intuitionistic Logic and its Strong Normalization. In *Fundamenta Informaticae*, Special issue on Logic for Pragmatics, **84**, 2, pp.185-205, 2008
18. G. Brewka and T. Gordon. Carneades and Abstract Dialectical Frameworks: A Reconstruction. In: P. Baroni, M. Giacomin and G. Simari *Computational Models of Argument*, Proceedings of COMMA 2010, IOS Press, 2010.
19. T. Crolard. Subtractive logic, in *Theoretical Computer Science* 254,1-2, 2001, pp. 151-185.
20. T. Crolard. A formulae-as-Types Interpretation of SubtrActive Logic. *Journal of Logic and Computation*, vol. 14(4). 2004, pp. 529-570.
21. C. Dalla Pozza and C. Garola. A pragmatic interpretation of intuitionistic propositional logic, *Erkenntnis* **43**. 1995, pp.81-109.
22. J. Czemark. A remark on Gentzen's calculus of sequents, *Notre Dame Journal of Formal Logic* 18(3). pp. 471-474. 1977.
23. V. Danos, J-B. Joinet and H. Schellinx. LKQ and LKT: Sequent calculi for second order logic based upon dual linear decompositions of classical implication.In J-Y. Girard, Y. Lafont, L. Regnier eds., *Advances in linear logic*, London Mathematical Society Lecture Note Series 222, Cambridge University Press, Cambridge, UK, New York, NY, USA, 1995, pp. 211–224.
24. V. C. V. dePaiva. A Dialectica-like Model of Linear Logic. In *Proceedings of Category Theory and Computer Science, Manchester, UK, September 1989,*, D.Pitt, D.Rydeheard, P.Dybjer, A.Pitt and A.Poign'e eds. Springer Lecture Notes in Computer Science 389, 1989.
25. M. Dummett. *The Logical Basis of Metaphysics* Cambridge, Mass.: Cambridge University Press, 1991.

26. D. Gabbay *Semantical Investigations in Heyting Intuitionistic Logic*, Reidel, Dodrecht, 1981.
27. J-Y. Girard. Linear Logic, *Theoretical Computer Science*, **50**, 1, 1987.
28. J-Y. Girard. A new constructive logic: classical logic. *Mathematical Structures in Computer Science*, 1, 1991 pp. 255-296.
29. J-Y. Girard. On the unity of logic. *Annals of Pure and Applied Logic* 59, 1993, pp.201-217.
30. J-Y. Girard. *Le Point Aveugle Cours de Logique. Tome I : Vers la Perfection, Tome II : Vers l'imperfection.*, Hermann Éditeurs Paris, 2006 et 2007.
31. N. Goodman. The Logic of Contradiction. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 27, 1981 pp.119-126.
32. T. F. Gordon and D. Walton. Proof burdens and standards. in: I. Rahwan and G. Simari eds., *Argumentations in Artificial Intelligence*, pp.239-258, 2009.
33. R. Goré. Dual intuitionistic logic revisited, in R. Dyckhoff, *Tableaux00: Automated Reasoning with Analytic Tableaux and Related Methods*, Springer, 2000.
34. J. M. E. Hyland and C. H. L. Ong, On Full Abstraction of **PCF**: I, II, and III. *Information and Computation* **163**, pp. 285-408, (2000)
35. O. Laurent. Polarized proof-nets and $\lambda\mu$-calculus, *Theoretical Computer Science* 290, pp. 161-188, 2003.
36. F. W. Lawvere, Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes. In A. Carboni, M.C. Pedicchio and G. Rosolini (eds.), *Category Theory (Como 1990)*, Lecture Notes in Mathematics 1488, Springer-Verlag 1991, pp. 279-297.
37. A. Lecomte. *Meaning, Logic and Ludics*, Imperial College Press, London, 2011
38. P. Lincoln, J. Mitchell and A. Scedrov. Stochastic Interaction and Linear Logic. In J-Y. Girard, Y. Lafont, L. Regnier eds., *Advances in linear logic*, London Mathematical Society Lecture Note Series 222, Cambridge University Press, Cambridge, UK, New York, NY, USA, 1995, pp.147-166.
39. E. G. K. López-Escobar. On Intuitionistic Sentential Connectives I, *Revista Colombiana de Matemàticas* XIX, 1985, pp. 117-130
40. M. Makkai and G. E. Reyes. Completeness results for intuitionistic and modal logic in a categorical setting, *Annals of Pure and Applied Logic*, 72, 1995, pp.25-101.
41. J.C.C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting, *Journal of Symbolic Logic* **13**, 1948, pp. 1-15,
42. P-A. Melliès. Game semantics in string diagrams. *Proceedings of the Annual ACM/IEEE Symposium in Logic in Computer Science* LICS 2012.
43. P-A. Melliès. A micrological study of negation. Manuscript, available at the author's web page.
44. D. Nelson. Constructible falsity, *The Journal of Symbolic Logic*, 14, pp. 16-26, 1949
45. D. Prawitz. *Natural deduction. A proof-theoretic study.* Almquist and Wikksell, Stockholm, 1965.
46. D. Prawitz. Meaning and Proofs: On the Conflict Between Classical and Intuitionistic Logic, *Theoria* 43 (1), 1977, pp. 2-40.
47. D. Prawitz. Is there a general notion of proof? *Proceedings of the XIV Congress of Logic, Methodology and Philosophy of Science, Nancy 2011* forthcoming.
48. C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, in *Fundamenta Mathematicae*, 83, 1974, pp. 219-249.

49. C. Rauszer. Applications of Kripke Models to Heyting-Brouwer Logic, in *Studia Logica* 36, 1977, pp. 61-71.
50. G. Reyes and H. Zolfaghari, Bi-Heyting algebras, Toposes and Modalities, in *Journal of Philosophical Logic*, 25, 1996, pp. 25-43.
51. Dual Intuitionistic Logic and a Variety of Negations: The Logic of Scientific Reseach, *Studia Logica* **80**, 2005, pp. 347-367.
52. P. Schroeder-Heister. Schluß und Umkehrschluß: ein Beitrag zur Definitionstheorie, in C. F. Gethmann (ed.) *Deutsches Jahrbuch Philosophie 02. Lebenswelt und Wissenschaft, Felix Meiner Verlag, Hamburg 2009, pp. 1065-1092.*
53. A. M. Tan. *Full Completeness for Models of Linear Logic*. PhD Thesis, King's College, University of Cambridge, 1987
54. L. Tranchini. Natural Deduction for Dual-intuitionistic Logic, *Studia Logica*, published online June 2012.
55. I. Urbas. Dual Intuitionistic Logic, *Nore Dame Journal of Formal Logic*. 37, pp 440-451, 1996.
56. T. Uustalu. A note on anti-intuitionistic logic. Abstract presented at the Nordic Workshop of Programmong Theory (NWPT'97), Tallin Estonia, 1997.

7 APPENDIX

7.1 Categorical models of bi-intuitionism.

Definition 9. *A categorical model of* **MLA** *is built on a symmetric monoidal closed category. A* symmetric monoidal category *is a category $\mathcal{A}$ equipped with a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and an object* **1** *(the identity of $\otimes$) together with natural isomorphisms*

1. $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$;
2. $\lambda_A : 1 \bullet A \xrightarrow{\sim} A$
3. $\rho_A : A \otimes 1 \xrightarrow{\sim} A$
4. $\gamma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$.

which satisfy the following coherence diagrams.

$$
\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & (((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}
$$

$$\begin{tikzcd}
A \otimes (1 \otimes B) \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "id_A \otimes \lambda_B"'] & & (A \otimes 1) \otimes B \arrow[dl, "\rho_A \otimes id_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, "id_{A \otimes B}"'] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes 1 \arrow[rr, "\gamma_{A,1}"] \arrow[dr, "\rho_a"'] & & 1 \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}$$

The following equality is also required to hold: $\lambda_1 = \rho_1 : \perp \bullet \perp \to 1$.

Definition 10. *A* symmetric monoidal closed category *is a symmetric monoidal category* $(\mathcal{A}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ *such that for every object* B *of* $\mathcal{A}$ *the functor* $\_ \otimes B : \mathcal{A} \to \mathcal{A}$ *has a right adjoint* $B \multimap \_ : \mathcal{A} \to \mathcal{A}$. *Thus for every* $A, C \in \mathcal{A}$ *there is an object* $B \multimap C$ *and a natural bijection*

$$\mathcal{A}(A \otimes B, C) \to (A, B \multimap C).$$

The *exponent* of B and C is an object $B \multimap C$ together with an arrow $\in_{B,C}$: $(B \multimap C) \otimes C \to C$ such that for any arrow $f : A \otimes B \to C$ there exists a unique $f^* : A \to (B \multimap C)$ making the following diagram commute:

$$\begin{tikzcd}
A \otimes B \arrow[r, "f"] \arrow[d, "f^* \otimes id_B"'] & C \\
(B \multimap C) \otimes B \arrow[ur, "\in_{B,C}"'] &
\end{tikzcd}$$

In particular a *cartesian closed category* (with finite products) is a symmetric monoidal closed category where the *categorical product* $\times$ is the monoidal functor $\otimes$. A main example of cartesian closed category is **Set**, where product is the ordinary Cartesian product and exponents are defined from sets of functions.

Definition 11. *A categorical model of* **MLH** *is a symmetric monoidal category* $(\mathcal{H}, \wp, \perp, \alpha, \lambda, \rho, \gamma)$, *such that for every* $D \in \mathcal{H}$ *the functor* $(D \wp \_) : \mathcal{H} \to \mathcal{H}$ *has a* left *adjoint* $(\_ - D) : \mathcal{H} \to \mathcal{H}$. *Thus for every* $C, E \in \mathcal{H}$ *there is an object* $C - D$ *and a natural bijection*

$$\mathcal{H}(C, D \wp E) \to (C - D, E).$$

The co-exponent of C and D is an object $C - D$ together with an arrow $\ni_{D,C}: C \to (C - D) \wp D$ such that for any arrow $f : C \to D \wp E$ there exists a unique $f_* : (C - D) \to E$ making the following diagram commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\ni_{D,C}"'] & E \wp D \\ & (C - D) \wp D \arrow[u, "f_* \wp id_D"'] \end{tikzcd}$$

Lemma. (Crolard [19]) *In the category* **Set** *the co-exponent $C - D$ of two sets C and D is defined if and only if $C = \emptyset$ or $D = \emptyset$.*

Proof: In **Set** coproducts are *disjoint unions*, write $E \oplus D$ for the disjoint union of E and D. If $C \neq \emptyset \neq D$ then the functions f and $\ni_{D,C}$ for every $c \in C$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \oplus 1_D$ leaves the side unchanged. Hence, if we take a nonempty set E and f with the property that for some c different sides are chosen by f and $\ni_{D,C}$, then the diagram does not commute. It is clear that such a failure does occur in *any category* where coproducts involves a choice between the arguments: in logic this is the case of an *additive* disjunction such as the intuitionistic one $(C \cup D)$ or the linear *plus* $(C \oplus D)$.

7.2 Dialogue chiralities.

The concept of *chirality* (see Melliès [43]) is useful to study a pair of structures $(\mathcal{A}, \mathcal{B})$, where one of the two structures cannot be defined simply as the opposite of the other and the duality has to be somehow "relaxed". The case of models of bi-intuitionism is to the point: here we have two monoidal categories, where the "intuitionistic" structure $\mathcal{A}$ is cartesian closed, but by Crolard's theorem the "co-intuitionistic" one cannot be just $\mathcal{A}^{op}$.

Definition 12. *A dialogue chirality on the left is a pair of monoidal categories $(\mathcal{A}, \wedge, true)$ and $(\mathcal{B}, \vee, false)$ equipped with an adjunction*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

whose unit and counit are denoted as

$$\eta : Id \quad \to \quad R \circ L \qquad \epsilon : L \circ R \to Id$$

together with a monoidal functor[8]

$$(-)^* \quad ; \quad \mathcal{A} \quad \to \quad \mathcal{B}^{op(0,1)}$$

and a family of bijections

[8] In the context of 2-categories, the notation $\mathbf{B}^{op(0,1)}$ means that the *op* operation applies to 0-cells and 1-cells.

$$\chi_{m,a,b} \quad : \quad \langle m \wedge a | b \rangle \quad \to \quad \langle a | m^* \vee b \rangle$$

natural in m, a, b (curryfication). *Here the bracket $\langle a|b \rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathcal{A}$:*

$$\langle a|b \rangle \quad = \quad \mathcal{A}(a, R(b)).$$

The family χ is moreover required to make the diagram

$$\begin{tikzcd}
\langle (m \wedge n) \wedge a \mid b \rangle \arrow[rr, "\chi_{m \wedge n}"'] \arrow[d, "assoc."'] & & \langle a \mid (m \wedge n)^* \vee b \rangle \\
 & = & \\
\langle m \wedge (n \wedge a) \mid b \rangle \arrow[r, "\chi_m"] & \langle n \wedge a \mid m^* \vee b \rangle \arrow[r, "\chi_n"] & \langle a \mid n^* \vee (m^* \vee b) \rangle \arrow[uu, "assoc.\ monoid.\ of\ (-)^*"']
\end{tikzcd}$$

commute for all objects a, m, n, and all morphisms $f : m \to n$ of the category $\mathcal{A}$ and all objects b of the category $\mathcal{B}$.

We sketch the construction of a chirality "from the syntax" of the bi-intuitionistic calculus. Let $(\mathcal{A}, \wedge, \mathbf{true})$ be the monoidal category, free on objects $\{\vdash p_1, \vdash p_2, \ldots\}$, where $\wedge$ is $\cap$ and **true** is the constant $\curlyvee$, whose objects are *intuitionistic assertive* formulas and whose morphisms $f : A \to B$ are equivalence classes of **AH-G1** sequent derivations (modulo permissible permutations of inferences). Similarly, we let $(\mathcal{B}, \vee, \mathbf{false})$ be the monoidal category, free on objects $\{\mathcal{H}\, p_1, \mathcal{H}\, p_2, \ldots\}$, where $\vee$ is $\curlyvee$ and **false** is the constant $\curlywedge$, whose objects are *co-intuitionistic hypothetical* formulas and whose morphisms $f : C \to D$ are equivalence classes of **AH-G1** sequent derivations (modulo permissible permutations of inferences). Consider Proposition 2 in Section 4: it gives the basic proof theoretic ingredients of the construction.

- The operations $\Diamond : \mathcal{A} \to \mathcal{B}$ and $\boxdot : \mathcal{B} \to \mathcal{A}$ are adjoint functors between the cartesian category $(\mathcal{A}, \cap, \curlyvee)$ and the monoidal category $(\mathcal{B}, \curlyvee, \curlywedge)$.
- The proofs of Proposition 2 (*ii*) correspond to the construction of the unit and the co-unit of the adjunction.
- We use the "internal" co-intuitionistic negation "$_j \frown$" to define the contravatiant monoidal functor $(\ \_\ )^*$ as $({}_j\!\frown\!\Diamond\ \_\ ) : \mathcal{A} \to \mathcal{B}^{op}$.
- We let $\langle A|C \rangle$ be the set of (equivalence classes of) derivations of $A \Rightarrow \boxdot C$.
- The cartesian category $(\mathcal{A}, \cap, \curlyvee)$ is in fact *cartesian closed*, i.e., exponents $A \supset B$ can be defined so that there is a natural bijection between $\mathcal{A}(M \wedge A, \boxdot B)$ and $\mathcal{A}(A, M \supset \boxdot B)$.
- The provable equivalences in Proposition 2 (*iii*) provide a "De Morgan" definition of intuitionistic implication in polarized bi-intuitionistic logic, i.e., a natural bijection between $\mathcal{A}(A, M \supset \boxdot B)$ and $\mathcal{A}(A, \boxdot(({}_j\!\frown\!\Diamond M) \curlyvee B)$.
- By composing, we obtain the family of natural bijections

$$\chi_{M,A,B} : \langle M \wedge A | B \rangle \to \langle A | M^* \vee B \rangle.$$

8 Appendix II. Chu's construction proof-theoretically

Here we give the main definitions of Chu's construction, in the proof-theoretic version of [8].

8.1 Classical and Intuitionistic multiplicative linear logic

Definition 13. (classical multiplicative linear logic **CMLL**) *The language of* **CMLL** *is built from the constants* $\mathbf{1}$ *and* $\bot$ *and sentential letters using multiplicative conjunction* ($\otimes$) *and disjunction* ($\wp$) *according to the following grammar:*

- *an infinite sequence of atomic sentences* $p_1, p_2, \ldots$;
  $A, B \;:= p \mid \mathbf{1} \mid \bot \mid A \otimes B \mid A \wp B$
- *linear negation* $(\ )^\bot$ *is defined by De Morgan laws;*
  $\mathbf{1}^\bot =_{df} \bot$; $\bot^\bot =_{df} \mathbf{1}$; $(A \otimes B)^\bot =_{df} A^\bot \wp B^\bot$; $(A \wp B)^\bot =_{df} A^\bot \otimes B^\bot$;
- *linear implication is defined as* $A \multimap B =_{df} A^\bot \wp B$

The sequent calculus of **CMLL** *is given in Table 10.*

logical axiom:

$$\vdash A, A^\bot$$

cut rule

$$\frac{\vdash \Gamma, A \qquad \vdash A^\bot, \Delta}{\vdash \Gamma, \Delta}$$

exchange

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$

$\mathbf{1}$ *axiom:*

$$\vdash \mathbf{1}$$

$\bot$ *rule*

$$\frac{\vdash \Gamma,}{\vdash \Gamma, \mathbf{1}}$$

$\otimes$ *rule*

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}$$

$\wp$ *rule:*

$$\frac{\vdash \Gamma, A_0, A_1}{\vdash \Gamma, A_0 \wp A_1}$$

Table 10. Sequent Calculus for CMLL.

Definition 14. *(intuitionistic multiplicative linear logic* **IMLL** *with product types) The language of* **IMLL** *with product types is built from the constants* $\mathbf{1}$, $\top$ *and* $\mathbf{0}$ *and sentential letters using multiplicative conjunction* ($\otimes$)*, linear implication* ($\multimap$)*, and additive conjunction* (&) *according to the following grammar:*

- *an infinite sequence of atomic sentences* $p_1, p_2, \ldots$;
  $A, B \ := p \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid A \otimes B \mid A \multimap B \mid A \& B.$

The sequent calculus of **IMLL** *with products is given in Table 11.*

logical axiom:

$$A \vdash A$$

cut rule

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma \Delta \vdash B}$$

exchange

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

1 *axiom:*

$$\vdash \mathbf{1}$$

1 *L:*

$$\frac{\Gamma \vdash A}{\mathbf{1}, \Gamma \vdash A}$$

$\otimes$ *R:*

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$\otimes$ *L:*

$$\frac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C}$$

$\multimap$ *R:*

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$\multimap$ *L:*

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}$$

$\top$ *axiom:*

$$\Gamma \vdash \top$$

$\&$ *R:*

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$$

$\&_i$ *L:*

$$\frac{A_i, \Gamma \vdash C}{A_0 \& A_1, \Gamma \vdash C}$$

Table 11. Sequent Calculus for IMLL with product types.

Remark 9. As discussed at the beginning of Section 5, classical and intuitionistic linear logic are considered here as being about abstract deductive schemes, without specifying an "intended interpretation" for the the language of **CMLL** and **IMALL**. On the contrary, in Section 5.1 the language **MLAH** and the sequent calculus **MLAH-G1** are within the "pragmatic interpretation" of polarized bi-intuitionism, elementary formulas are types of illocutionary acts and molecular expressions do have assertive or hypothetical mood. However, the deductive methods available will be restricted to *linear inference rules* and as a consequence, it will be possible to interpret the formulas of the language **MLAH** in terms of our "dialogue semantics". Therefore the logic of linear polarized bi-intuitionism is a mathematical construction that appears to have an interpretation in common sense reasoning. If the "dialogic interpretation" can be modified and extended to polarized bi-intuitionism without the linear constraint then our understanding of the relations between linear and non-linear polarized bi-intuitionism will be considerably improved.

$(P^{\perp})_{\mathbf{O}} = P_{\mathbf{I}}$ (P atomic)	$(P^{\perp})_{\mathbf{I}} = P_{\mathbf{O}}$;
$\mathbf{1}_{\mathbf{O}} = \mathbf{1}, \quad \mathbf{1}_{\mathbf{I}} = \top$	$\perp_{\mathbf{I}} = \mathbf{1} \quad \perp_{\mathbf{O}} = \top$;
$(A \otimes B)_{\mathbf{O}} = A_{\mathbf{O}} \otimes B_{\mathbf{O}}$	$(A \wp B)_{\mathbf{I}} = A_{\mathbf{I}} \otimes B_{\mathbf{I}}$;
$(A \otimes B)_{\mathbf{I}} = (A_{\mathbf{O}} \multimap B_{\mathbf{I}}) \& (B_{\mathbf{O}} \multimap A_{\mathbf{I}})$	$(A \wp B)_{\mathbf{O}} = (A_{\mathbf{I}} \multimap B_{\mathbf{O}}) \& (B_{\mathbf{I}} \multimap A_{\mathbf{O}})$

Table 12. Functorial trip translation, the propositions.

Theorem 2. ([8], section 3) *Let $\mathcal{A}$ be the free $*$-autonomous category on a set of objects $\{P, P', \ldots\}$ and let $\mathcal{C}$ be the symmetric monoidal closed category with products, free on the set of objects $\{P_{\mathbf{O}}, P_{\mathbf{I}}, P'_{\mathbf{O}}, P'_{\mathbf{I}}, \ldots\}$ (a pair $P_{\mathbf{O}}, P_{\mathbf{I}}$ in $\mathcal{C}$ for each P in $\mathcal{A}$).*
We can give $\mathcal{C} \times \mathcal{C}^{op}$ the structure of a $*$-autonomous category thus:

$$(X_{\mathbf{O}}, X_{\mathbf{I}}) \otimes (Y_{\mathbf{O}}, Y_{\mathbf{I}}) =_{df} \big(X_{\mathbf{O}} \otimes Y_{\mathbf{O}}, (X_{\mathbf{O}} \multimap Y_{\mathbf{I}}) \times (Y_{\mathbf{O}} \multimap X_{\mathbf{I}})\big)$$
with unit $(\mathbf{1}, \top)$ and involution $(X_{\mathbf{O}}, X_{\mathbf{I}})^{\perp} = (X_{\mathbf{I}}, X_{\mathbf{O}})$

where $\mathbf{1}$ is the unit of $\otimes$ and $\top$ the terminal object of $\mathcal{C}$.

Therefore there is a functor F from $\mathcal{A}$ to $\mathcal{C} \times \mathcal{C}^{op}$ sending an object P to $(P_{\mathbf{O}}, P_{\mathbf{I}})$.
If $\pi : I \to \wp(\Gamma)$ is a morphism of $\mathcal{A}$ represented as a proof-net $\mathcal{R}$ with conclusions Γ, then the morphism $(\mathbf{1}, \top) \to \big(\wp(\Gamma)_{\mathbf{O}}, \wp(\Gamma)_{\mathbf{I}}\big)$ encodes all Girard's trips (in a sense specified in [8]).

We are interested in the proof theoretic interpretation of this result, determined by the form of the functor $F : \mathcal{A} \to \mathcal{C} \times \mathcal{C}^{op}$. We have the following data:

- a language $\mathcal{L}^C$ of classical **CMLL** on an infinite list of atoms p_1, p_2, …;
- a language $\mathcal{L}^I_{\mathbf{IO}}$ of intuitionistic **IMALL** on an infinite list of atoms $p_{1\mathbf{O}}$, $p_{1\mathbf{I}}$, $p_{2\mathbf{O}}$, $p_{2\mathbf{I}}$ … (a pair $p_{\mathbf{O}}$, $p_{\mathbf{I}}$ in $\mathcal{L}^I_{\mathbf{IO}}$ for each p in $\mathcal{L}^C$);
- translations $(\ )_{\mathbf{O}}$ and $(\ )_{\mathbf{I}}$ of the formulas of $\mathcal{L}^C$ into formulas of $\mathcal{L}^I_{\mathbf{IO}}$ by induction on the construction of $\mathcal{L}^C$ formulas according to Table 12;
- derivations in the sequent calculus **CMLL** with *pointed sequents*, i.e., where each sequent has a selected formula (written in boldface). The

"Pointed" CMLL sequents		**IMALL sequents**
$\vdash P^{\perp}, \mathbf{P}$	$\Rightarrow$	$P_{\mathbf{O}} \vdash P_{\mathbf{O}}$
$\vdash \mathbf{P}^{\perp}, P$	$\Rightarrow$	$P_{\mathbf{I}} \vdash P_{\mathbf{I}}$
$cut\ \dfrac{\vdash \Gamma, \mathbf{A} \quad \vdash A^{\perp}, \Delta, \mathbf{C}}{\vdash \Gamma, \Delta, \mathbf{C}}$	$\Rightarrow$	$\dfrac{\Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}} \quad A_{\mathbf{I}}^{\perp}, \Delta_{\mathbf{I}} \vdash C_{\mathbf{O}}}{\Gamma_{\mathbf{I}}, \Delta_{\mathbf{I}} \vdash C_{\mathbf{O}}}\ cut$
$\otimes_{\mathbf{O}}\ \dfrac{\vdash \Gamma, \mathbf{A} \quad \vdash \Delta, \mathbf{B}}{\vdash \Gamma, \Delta, \mathbf{A} \otimes \mathbf{B}}$	$\Rightarrow$	$\dfrac{\Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}} \quad \Delta_{\mathbf{I}} \vdash B_{\mathbf{O}}}{\Gamma_{\mathbf{I}}, \Delta_{\mathbf{I}} \vdash A_{\mathbf{O}} \otimes B_{\mathbf{O}}}\ \otimes\text{-R}$
$\otimes_{\mathbf{I}}\ \dfrac{\vdash \Gamma, \mathbf{A} \quad \vdash B, \Delta, \mathbf{C}}{\vdash \Gamma, \Delta, A \otimes B, \mathbf{C}}$	$\Rightarrow$	$\dfrac{\dfrac{\Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}} \quad B_{\mathcal{I}}, \Delta_{\mathbf{I}} \vdash C_{\mathbf{O}}}{\Gamma_{\mathbf{I}}, A_{\mathbf{O}} \multimap B_{\mathbf{I}}, \Delta_{\mathbf{I}} \vdash C_{\mathbf{O}}}\ \multimap\text{-L}}{\Gamma_{\mathbf{I}}, (A_{\mathbf{O}} \multimap B_{\mathbf{I}}) \& (B_{\mathbf{O}} \multimap A_{\mathbf{I}}), \Delta_{\mathbf{I}} \vdash C_{\mathbf{O}}}$
$\wp_{\mathbf{O}}\ \dfrac{\vdash \Gamma, A, \mathbf{B}}{\vdash \Gamma, \mathbf{A} \wp \mathbf{B}}$ and $\dfrac{\vdash \Gamma, \mathbf{A}, B}{\vdash \Gamma, \mathbf{A} \wp \mathbf{B}}$	$\Rightarrow$	$\dfrac{\dfrac{\Gamma_{\mathbf{I}}, A_{\mathbf{I}} \vdash B_{\mathbf{O}}}{\Gamma_{\mathbf{I}} \vdash A_{\mathbf{I}} \multimap B_{\mathbf{O}}} \quad \dfrac{\Gamma_{\mathbf{I}}, B_{\mathbf{I}} \vdash A_{\mathbf{O}}}{\Gamma_{\mathbf{I}} \vdash B_{\mathbf{I}} \multimap A_{\mathbf{O}}}\ \multimap\text{-R}}{\Gamma_{\mathbf{I}} \vdash (A_{\mathbf{I}} \multimap B_{\mathbf{O}}) \& (B_{\mathbf{I}} \multimap A_{\mathbf{O}})}$
$\wp_{\mathbf{I}}\ \dfrac{\vdash A, B, \Gamma, \mathbf{C}}{\vdash A \wp B, \Gamma, \mathbf{C}}$	$\Rightarrow$	$\dfrac{A_{\mathbf{I}}, B_{\mathbf{I}}, \Gamma_{\mathbf{I}} \vdash C_{\mathbf{O}}}{A_{\mathbf{I}} \otimes B_{\mathbf{I}}, \Gamma_{\mathbf{I}} \vdash C_{\mathbf{O}}}\ \otimes\text{-L}$
$\vdash \mathbf{1} \quad \Rightarrow \quad \vdash \mathbf{1}$		$\perp_{\mathbf{O}}\ \dfrac{\vdash \Gamma}{\vdash \Gamma, \perp} \quad \Rightarrow \quad \Gamma_{\mathbf{I}} \vdash \top$
$\perp_{\mathbf{I}}\ \dfrac{\vdash \Gamma, \mathbf{A}}{\vdash \perp, \Gamma, \mathbf{A}}$	$\Rightarrow$	$\dfrac{\Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}}}{\mathbf{1}, \Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}}}\ \mathbf{1}\text{-L}$

Table 13. Functorial trip translation, the proofs.

"pointing" must respect the constraints on the sequent calculus rules exhibited on the left column of Table 13: thus the selection of the "pointed formula" in the sequent premises is uniquely determined by the selection for the conclusion, *except in the case of a* par ($\wp_{\mathbf{O}}$) *rule*, where two selections are possible.

- Given a sequent derivation d in **CMLL**, a *switching* of d is a selection of a formula in the endsequent and, for each *par* inference introducing a formula $A \wp B$ in the sequent-concluson a selection of one of the formulas A of B in the sequent premise.
- Finally we have translations of classical **CMLL** derivations *with a switching* into intuitionistic **IMALL** derivations according to Table 13.

In the statement of the theorem, Girard's *switchings* in a proof net are mentioned, which are tools to verify the correctness of the proof net representation of proofs. Such switchings correspond to ways of systematically selecting a formula in each sequent of classical **CMLL** derivation (*pointed sequents* $\vdash \Gamma, \mathbf{A}$): see Table 13. The key insight of [8] is that the orientation of the subformulas X as *input* ($X_{\mathbf{I}}$) or *output* ($X_{\mathbf{O}}$) resulting from such "switchings" suffices to recover derivations in intuitionistic linear logic with sequents of the form $\Gamma_{\mathbf{I}} \vdash A_{\mathbf{O}}$.

8.2 An Example.

Let $\vdash \Gamma$ be the sequent $\vdash A^{\perp}, B^{\perp} \wp (A \otimes B)$ and let d be (the only) cut-fee derivation of $\vdash \Gamma$:

$$\dfrac{\dfrac{\vdash A^{\perp}, A \qquad \vdash B^{\perp}, B}{\vdash A^{\perp}, B^{\perp}, A \otimes B}}{\vdash A^{\perp}, B^{\perp} \wp (A \otimes B)}$$

The derivation d, regarded as a morphism $\pi : I \to \wp(\Gamma)$ in a free *-autonomous category, is mapped by Chu's functor to the morphism $(\mathbf{1}, \top) \to (\wp(\Gamma)_{\mathbf{O}}, \wp(\Gamma)_{\mathbf{I}})$. The right contravariant component is just the axiom $\wp(\Gamma)_{\mathbf{I}} \vdash \top$. By the translation in Table 13 the left component is given by cut free **IMALL** derivations f_1, f_2 and g, which are given by the three switchings, selecting the formulas $A^{\perp}$, $B^{\perp}$ and $A \otimes B$, respectively.

1. $f_1 : A_{\mathbf{O}} \vdash [B_{\mathbf{O}} \multimap (A_{\mathbf{O}} \otimes B_{\mathbf{O}})]$, $f_2 : A_{\mathbf{O}} \vdash \Big[[A_{\mathbf{O}} \multimap B_{\mathbf{I}}] \& [B_{\mathbf{O}} \multimap A_{\mathbf{I}}]\Big] \multimap B_{\mathbf{I}}$;
2. $g : B_{\mathbf{O}} \otimes ([B_{\mathbf{O}} \multimap A_{\mathbf{I}}] \& [A_{\mathbf{O}} \multimap B_{\mathbf{I}}] \vdash A_{\mathbf{I}}$.

Next consider the cut-free **MLAH-G1** derivations d of $A; \Rightarrow B \multimap (A \otimes B)$; and $d^{\perp}$ of $; (B^{\perp} \wp A^{\perp}) - B^{\perp} \Rightarrow ; A^{\perp}$, where we let $A = \vdash p$ and $B = \vdash q$; here the Definition 8 of the duality for *formulas* is extended to *sequent derivations* in an obvious way.

$$\dfrac{\dfrac{A; \Rightarrow A; \qquad B; \Rightarrow B;}{A, B; \Rightarrow A \otimes B;}}{A; \Rightarrow B \multimap (A \otimes B);} \qquad \dfrac{\dfrac{; B^{\perp} \Rightarrow ; B^{\perp} \qquad ; A^{\perp} \Rightarrow ; A^{\perp}}{; B^{\perp} \wp A^{\perp} \Rightarrow ; B^{\perp}, A^{\perp}}}{; (B^{\perp} \wp A^{\perp}) - B^{\perp} \Rightarrow ; A^{\perp}}$$

The definition of Table 9, formalizing the dialogue semantics of definitions 6 and 7, maps **MLA** and **MLH** formulas to **IMALL** formulas; the translation in proposition 8 maps the derivation d of $A; \Rightarrow B \multimap (A \otimes B)$; to the following pair of derivations $(h_1 \times h_2, k)$:

$$h_1: \quad \dfrac{\dfrac{A^+ \vdash A^+ \quad B^+ \vdash B^+}{A^+, B^+ \vdash A^+ \otimes B^+}}{A^+ \vdash B^+ \multimap (A^+ \otimes B^+)}$$

$$h_2: \quad \dfrac{\dfrac{\dfrac{A^+ \vdash A^+ \quad B^- \vdash B^-}{A^+, A^+ \multimap B^- \vdash B^-}}{A^+, [A^+ \multimap B^-]\&[B^+ \multimap (A^-] \vdash B^-}}{A^+ \vdash \Big[[A^+ \multimap B^-]\&[B^+ \multimap (A^-]\Big] \multimap B^-}$$

$$\dfrac{h_1 \qquad h_2}{A^+ \vdash [B^+ \multimap (A^+ \otimes B^+)]\&\Big[[A^+ \multimap B^-]\&[B^+ \multimap A^-] \multimap B^-\Big]}$$

$$k: \quad \dfrac{\dfrac{\dfrac{B^+ \vdash B^+ \quad A^- \vdash A^-}{B^+, B^+ \multimap A^- \vdash A^-}}{B^+, [B^+ \multimap A^-]\&[A^+ \multimap B^-] \vdash A^-}}{B^+ \otimes ([B^+ \multimap A^-]\&[A^+ \multimap B^-] \vdash A^-.}$$

Under the assumptions in (7) and given Proposition 12, it is not difficult to see that the derivation $d^\perp$ is mapped to a pair of derivations that can be identified with $(k, h_1 \times h_2)$.